Electronic Journal of Qualitative Theory of Differential Equations
2008, No. **25**, 1-10; **http://www.math.u-szeged.hu/ejqtde/**

Integrable and continuous solutions of a nonlinear quadratic integral equation

El-Sayed A.M.A and Hashem H.H.G
Faculty of Science, Alexandria University, Alexandria, Egypt
e.mails : amasayed@hotmail.com, amasayed5@yahoo.com

Abstract

We are concerned here with a nonlinear quadratic integral equation of Volterra type. The existence of at least one L_1- positive solution will be proved under the Carathèodory condition. Secondly we will make a link between Peano condition and Carathèodory condition to prove the existence of at least one positive continuous solution. Finally the existence of the maximal and minimal solutions will be proved.

Keywords: Quadratic integral equation; Positive integrable solution; Continuous solution; Maximal and minimal solutions.

1 Introduction

Quadratic integral equations are often applicable in the theory of radiative transfer, kinetic theory of gases, in the theory of neutron transport and in the traffic theory. Especially, the so-called quadratic integral equation of Chandraskher type can be very often encountered in many applications (see[1]-[4]).

Let $I = [0, T]$, $L_1 = L_1[0, T]$ be the space of Lebesgue integrable functions on I and $C[0, T]$ be the space of continuous functions defined on I.

Consider now the following integral equation

$$x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s))\, ds. \tag{1}$$

It is proved that ([5], Caratheodory Theorem) if f is measurable in $t \in [t_0, T]$ for each fixed $x \in B \subset R$ and continuous in $x \in B$ for each fixed $t \in [t_0, T]$ and there exists a function $m \in L_1$ such that $|f(t, x)| \leq m(t)$ then, there exists a solution $x \in C[t_0, T]$ of the integral equation (1) and the equivalent initial value problem

$$\frac{d}{dt}x(t) = f(t, x(t)), \quad t > t_0; \quad x(t_0) = x_0.$$

This result has been generalized (by the authors [7]) for the fractional-order integral equation

$$x(t) = x(t_0) + \int_{t_0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, f(s, x(s))\, ds. \tag{2}$$

In this work, we generalize this result for the nonlinear quadratic integral equation

$$x(t) \ = \ a(t) \ + \ g(t, x(t)) \int_0^t \ k(t,s) \ f(s, x(s)) \ ds, \quad t \ \in \ [0,1]. \tag{3}$$

Firstly we prove the existence of at least one positive solution $x \in L_1$ of the quadratic integral equation (3) where the functions f and g satisfy Carathèodory condition.
Secondly we prove the existence of at least one positive continuous solution for equation (3) where g satisfies Peano condition and f satisfies Carathèodory condition. Also the existence of the maximal and minimal (continuous) solutions will be proved

2 Preliminaries

The following theorems will be needed in our work (see [5], [6] and [9]).
Tychonov's Theorem. Suppose B is a complete, locally convex linear space and S is a closed convex subset of B. Let the mapping $T: B \to B$ be continuous and $T(S) \subset S$. If the closure of $T(S)$ is compact, then T has a fixed-point in S.

Schauder fixed-point Theorem. Let S be a convex subset of a Banach space B, let the mapping $T: S \to S$ be compact, continuous. Then T has at least one fixed-point in S.

Arzelà-Ascoli Theorem. Let E be a compact metric space and $C(E)$ the Banach space of real or complex valued continuous functions normed by

$$\| f(t) \| \ = \ \max_{t \ \in \ E} \ | \ f(t) \ |.$$

If $A = \{f_n\}$ is a sequence in $C(E)$ such that f_n is uniformly bounded and equi-continuous. Then the closure of A is compact.

Lusin Theorem. Let $m : [0,1] \to R$ be a measurable function. For any $\epsilon > 0$ there exists a closed subset A_ϵ of $[0,1]$, meas.$(A_\epsilon^c) < \epsilon$, such that m restricted to A_ϵ is continuous.

Scorza Dragoni Theorem. Let $k : [0,1] \times [0,1] \to R$ be a function satisfying Carathèodory condition (i.e. it is measurable in t for all $x \in R$ and continuous in x for all $t \in [0,1]$). For any $\epsilon > 0$ there exists a closed subset A_ϵ of $[0,1]$, meas.$(A_\epsilon^c) < \epsilon$, such that k restricted to $A_\epsilon \times [0,1]$ is continuous.

3 L_1-positive solution

Let $I = [0,1]$, and consider the assumptions:

(i) $a : I \to R_+ = [0, +\infty)$ is integrable on I;

(ii) $f, g : I \times R_+ \rightarrow R_+$ satisfy Caratheodory condition (i.e. measurable in t for all $x \in R_+$ and continuous in x for all $t \in [0,1]$) and there exist two functions $m_1,\ m_2 \in L_1$ such that

$$g(t,x) \leq m_1(t), \quad f(t,x) \leq m_2(t) \quad \text{for all } (t,x) \in I \times R_+;$$

(iii) $k : [0,1] \times [0,1] \rightarrow R$ satisfies Carathèodory condition (i.e. it is measurable in t for all s and continuous in s for all t).

For the existence of at least one L_1-positive solution of the quadratic integral equation (3) we have the following theorem.

Theorem 3.1 Let the assumptions (i)-(iii) be satisfied. Then the nonlinear quadratic integral equation (3) has at least one L_1-positive solution.
Proof. Consider the set $Q \subset L_1$ such that

$$Q = \{ x \in L_1, |\, x(t) \,| \leq x_0(t) \ a.e. \}$$

where

$$x_0(t) = a(t) + m_1(t) \int_0^t k(t,s)\, m_2(s)\, ds. \tag{4}$$

The set Q can be shown to be nonempty, bounded, closed and convex in L_1.
Let H be the operator defined by the formula

$$(Hx)(t) = a(t) + g(t, x(t)) \int_0^t k(t,s)\, f(s, x(s))\, ds, \quad t \in I. \tag{5}$$

We shall prove that $H : Q \rightarrow Q$. For that let $x \in Q$, then

$$(Hx)(t) \leq |\, a(t) \,| + m_1(t) \int_0^t k(t,s)\, m_2(s)\, ds$$

$$\leq |\, a(t) \,| + m_1(t) \int_0^t k(t,s)\, m_2(s)\, ds = x_0(t),$$

so $Hx \in Q$ and hence $HQ \subset Q$.
To apply Schauder fixed-point Theorem, we shall prove that HQ is relatively compact in L_1.
By using Lusin Theorem and Scorza Dragoni Theorem , we can find a closed subset A_n of $[0,1]$, with meas.$(A_n^c) < \frac{1}{n}$ such that $a|_{A_n},\ m_1|_{A_n},\ k|_{A_n \times [0,1]}$ and $g|_{A_n \times Q}$ are uniformly continuous.
Assume that x_h is any sequence in Q, then for $t_1, t_2 \in A_n$, we have

$$(Hx_h)(t_2) - (Hx_h)(t_1) = a(t_2) - a(t_1)$$

$$+ g(t_2, x_h(t_2)) \int_0^{t_2} k(t_2,s)\, f(s, x_h(s))\, ds - g(t_1, x_h(t_1)) \int_0^{t_1} k(t_1,s)\, f(s, x_h(s))\, ds$$

$$= a(t_2) - a(t_1) + g(t_2, x_h(t_2)) \int_0^{t_2} k(t_2, s)\ f(s, x_h(s))\ ds$$

$$- g(t_1, x_h(t_1)) \int_0^{t_1} k(t_1, s)\ f(s, x_h(s))\ ds + g(t_2, x_h(t_2)) \int_0^{t_1} k(t_1, s)\ f(s, x_h(s))\ ds$$

$$- g(t_2, x_h(t_2)) \int_0^{t_1} k(t_1, s)\ f(s, x_h(s))\ ds$$

$$= a(t_2) - a(t_1) + g(t_2, x_h(t_2)) \int_0^{t_1} k(t_2, s)\ f(s, x_h(s))ds$$

$$+ g(t_2, x_h(t_2)) \int_{t_1}^{t_2} k(t_2, s)\ f(s, x_h(s))ds - g(t_2, x_h(t_2)) \int_0^{t_1} k(t_1, s)\ f(s, x_h(s))ds$$

$$+ g(t_2, x_h(t_2)) \int_0^{t_1} k(t_1, s)\ f(s, x_h(s))ds - g(t_1, x_h(t_1)) \int_0^{t_1} k(t_1, s)\ f(s, x_h(s))ds$$

$$= a(t_2) - a(t_1) + g(t_2, x_h(t_2)) \int_0^{t_1} [\ k(t_2, s) - k(t_1, s)\ ]\ f(s, x_h(s))\ ds$$

$$+ g(t_2, x_h(t_2)) \int_{t_1}^{t_2} k(t_2, s)\ f(s, x_h(s))\ ds$$

$$+ [\ g(t_2, x_h(t_2)) - g(t_1, x_h(t_1))\ ] \int_0^{t_1} k(t_1, s)\ f(s, x_h(s))\ ds$$

Then we get

$$|\ (Hx_h)(t_2) - (Hx_h)(t_1)\ | \leq |\ a(t_2) - a(t_1)\ | + m_1(t_2) \int_{t_1}^{t_2} k(t_2, s)\ m_2(s)\ ds$$

$$+ m_1(t_2) \int_0^{t_1} |\ k(t_2, s) - k(t_1, s)\ |\ m_2(s)\ ds$$

$$+ |\ g(t_2, x_h(t_2)) - g(t_1, x_h(t_1))\ | \int_0^{t_1} k(t_1, s)\ m_2(s)\ ds$$

This means that the sequence $\{Hx_h\}$ is sequence of equi-continuous functions on A_n and we can prove that this sequence is uniformly bounded.
Now

$$|\ (Hx_h)(t)\ | = |\ a(t)\ | + m_1(t) \int_0^t k(t, s)\ m_2(s)\ ds$$

$$\leq M_1 + M_2\ K \int_0^t m_2(s)\ ds$$

$$\leq M_1 + M_2\ K \int_0^1 m_2(s)\ ds = M$$

where $|\ a(t)\ |_{A_n} \leq M_1$, $|\ m_1(t)\ |_{A_n} \leq M_2$ and $k|_{A_n \times [0,1]} \leq K$.
Hence by Arzelà-Ascoli Theorem Hx_h is relatively compact subset of $C(A_n)$ and this can be done for each $n \in N$. This implies the existence of convergent subsequence $\{x_{h_j}\}$

of $\{x_h\}$ in each $C(A_n)$. Given $\epsilon > 0$ and choose $n_1 \in N$ so that $meas(A_{n_1}) < \epsilon$, then

$$\int_0^1 \mid Hx_{h_j} - Hx_{h_l} \mid dt = \int_{A^c_{n_1}} \mid Hx_{h_j} - Hx_{h_l} \mid dt$$
$$+ \int_{A_{n_1}} \mid Hx_{h_j} - Hx_{h_l} \mid dt.$$

Since $C(A_n)$ is complete metric space, hence this subsequence is a Cauchy sequence in each $C(A_n)$, $n = 1, 2, 3, ...$
That is for given $\epsilon > 0$ and j, l are arbitrary large we have

$$\| Hx_{h_j} - Hx_{h_l} \|_{C(A_n)} < \epsilon. \tag{6}$$

But we want to prove that the set HQ is relatively compact in L_1, that is $\overline{HQ}$ is compact in L_1.
To do this, we will prove that the sequence $\{ Hx_h \}$ is convergent in L_1, since L_1 is complete metric space, then it is sufficient to prove that the subsequence $\{ Hx_{h_j} \}$ is a Cauchy sequence in L_1.
i.e. $\forall \, \eta > 0$, $\exists \, N(\eta)$ and $\int_{A_n} x_0(t) \, dt < \eta/4$ such that

$$\| Hx_{h_j} - Hx_{h_l} \|_{L_1} < \eta, \quad j, l > N(\eta).$$

Now from (4) and (5) we have

$$\int_0^1 \mid Hx_{h_j} - Hx_{h_l} \mid dt = \int_{A^c_n} \mid Hx_{h_j} - Hx_{h_l} \mid dt$$
$$+ \int_{A_n} \mid Hx_{h_j} - Hx_{h_l} \mid dt.$$
$$\leq \int_{A^c_n} \{ \mid Hx_{h_j} \mid + \mid Hx_{h_l} \mid \} \, dt + \| Hx_{h_j} - Hx_{h_l} \|_{C(A_n)}$$
$$\leq \eta/4 + \eta/4 + \| Hx_{h_j} - Hx_{h_l} \|_{C(A_n)}.$$

Choose N such that $l, j > N$, then (6) implies that $\| Hx_{h_j} - Hx_{h_l} \|_{C(A_n)} \leq \eta/2$.
This means that the subsequence $\{ Hx_{h_j} \}$ is a Cauchy sequence in L_1 which implies that HQ is relatively compact in L_1. Then H has at least one fixed point. Hence there exists at least one solution $x \in L_1$ of (1).
Since all conditions of Shauder's fixed-point Theorem hold, then H has a fixed point in Q. ■

4 Continuous solutions

Let $I = [0, 1]$, and consider the assumptions:

(i) $a : I \to R_+ = [0, +\infty)$ is continuous on I;

(ii) $f:\ I \times R_+ \rightarrow R_+$ satisfies Carathéodory condition (i.e. measurable in t for all $x \in R_+$ and continuous in x for all $t \in [0,1]$) and there exists function $m \in L_1$ such that

$$f(t,x(t)) \leq m(t) \qquad \forall\ (t,x) \in I \times R_+;$$

(iii) $g:\ I \times R_+ \rightarrow R_+$ is continuous in t, x and $|\ g(t,x)\ | \leq M$;

(iv) $k:\ [0,1]\times[0,1] \rightarrow R$ satisfies Carathèodory condition (i.e. measurable in t for all s and continuous in s for all t.)

Now for the existence of at least one positive continuous solution of the nonlinear quadratic integral equation (3) we have the following theorem.

Theorem 4.1 Let the assumptions (i)-(iv) be satisfied. Then the equation (3) has at least one positive solution $x \in C(I)$.

Proof. We shall use Tychonov's fixed point Theorem to prove this theorem

It can be verified that [6] C is complete locally convex linear space. Define a subset S of C by

$$S = \{\ x \in C:\ |\ x(t)\ | \leq M_2\ \}, \qquad t \in [0,1],$$

where M_2 is a positive constant. It is clear that the set S is closed and convex.

Let H be the operator defined by the formula

$$(Hx)(t) = a(t) + g(t,x(t)) \int_0^t k(t,s)\ f(s,x(s))\ ds, \quad \forall\ x \in S.$$

Assumptions (ii) and (iii) imply that $H:\ S \rightarrow C$ is continuous operator in x . We shall prove that $HS \subset S$.

For every $x \in S$ we have

$$\begin{aligned} |\ (Hx)(t)| &\leq |\ a(t)\ | + M \int_0^t k(t,s)\ m(s)\ ds \qquad t \in [0,1] \\ &\leq |\ a(t)\ | + M \int_0^1 k(t,s)\ m(s)\ ds \\ &= M_2, \end{aligned}$$

where $|\ a(t)\ | \leq M_1$. Then, $Hx \in S$ and hence $HS \subset S$.

Also for $x \in S$ and t_1 and $t_2 \in [0,1]$ we can have

$$(Hx)(t_2) - (Hx)(t_1) = a(t_2) - a(t_1)$$

$$+ g(t_2,x(t_2)) \int_0^{t_2} k(t_2,s)\ f(s,x(s))\ ds - g(t_1,x(t_1)) \int_0^{t_1} k(t_1,s)\ f(s,x(s))\ ds.$$

$$+ g(t_2,x(t_2)) \int_0^{t_1} k(t_1,s)\ f(s,x(s))\ ds - g(t_2,x(t_2)) \int_0^{t_1} k(t_1,s)\ f(s,x(s))\ ds$$

$$= a(t_2) - a(t_1) + g(t_2,x(t_2)) \int_0^{t_1} k(t_2,s)\ f(s,x(s))\ ds$$

$$+\ g(t_2,x(t_2)) \int_{t_1}^{t_2} k(t_2,s)\ f(s,x(s))\ ds\ -\ g(t_1,x(t_1)) \int_0^{t_1} k(t_1,s)\ f(s,x(s))\ ds.$$

$$+\ g(t_2,x(t_2)) \int_0^{t_1} k(t_1,s)\ f(s,x(s))\ ds\ -\ g(t_2,x(t_2)) \int_0^{t_1} k(t_1,s)\ f(s,x(s))\ ds$$

$$=\ a(t_2)\ -\ a(t_1)\ +\ g(t_2,x(t_2)) \int_0^{t_1} [k(t_2,s)\ -\ k(t_1,s)]\ f(s,x(s))\ ds$$

$$+\,g(t_2,x(t_2)) \int_{t_1}^{t_2} k(t_2,s)\, f(s,x(s))\, ds + [g(t_2,x(t_2)) - g(t_1,x(t_1))] \int_0^{t_1} k(t_1,s)\, f(s,x(s))\, ds.$$

Then we have

$$|\ (Hx)(t_2)\ -\ (Hx)(t_1)\ |\ \leq\ |\ a(t_2)\ -\ a(t_1)\ |\ +\ M \int_{t_1}^{t_2} k(t_2,s)\ m(s)\ ds$$

$$+\ M \int_0^{t_1} |\ k(t_2,s) - k(t_1,s)\ |\ m(s)\ ds + |g(t_2,x(t_2)) - g(t_1,x(t_1))| \int_0^{t_1} k(t_1,s)\ m(s)\ ds.$$

This means that the functions of HS are equi-continuous on $[0,1]$, then by Arzelá-Ascoli Theorem the closure of HS is compact .
Hence, all conditions of Tychonov fixed-point Theorem hold, then H has a fixed point in S. ■

5 Maximal and minimal solutions

Definition 5.1 [8] Let $q(t)$ be a solution of the nonlinear quadratic integral equation (3). Then $q(t)$ is said to be a maximal solution of (3) if every solution of (3) satisfies the inequality $x(t)\ <\ q(t),\ \forall t \in I$. A minimal solution $s(t)$ can be defined by similar way by reversing the above inequality i.e. $x(t)\ >\ s(t),\ \forall t \in I$.
We shall use the following lemma to prove the existence of the maximal and minimal solutions.
Lemma 5.1 Let $a(t)$ be a continuous function on I and $k(t,s)$ satisfying the assumption (iv) of Theorem 4.1. Let $f(t,x),\ g(t,x)\ \in\ L_1$ and $x(t),\ y(t)$ be continuous functions on $[0,1]$ satisfying

$$\begin{aligned} x(t) &\leq a(t)\ +\ g(t,x(t)) \int_0^t k(t,s)\ f(s,x(s))\ ds, \\ y(t) &\geq a(t)\ +\ g(t,y(t)) \int_0^t k(t,s)\ f(s,y(s))\ ds \end{aligned}$$

and one of them is strict. If $f(t,x),\ g(t,x)$ are monotonic nondecreasing in x , then

$$x(t)\ <\ y(t),\quad t\ >\ 0. \tag{7}$$

Proof. Let the conclusion (7) be false, then there exists t_1 such that

$$x(t_1)\ =\ y(t_1),\quad t_1\ >\ 0$$

and

$$x(t) \ < \ y(t), \quad 0 \ < \ t \ < \ t_1.$$

From the monotonicity of $f(t,x),\ g(t,x)$ in x , we get

$$\begin{aligned}
x(t_1) \quad &\leq \quad a(t_1) \ + \ g(t_1, x(t_1)) \int_0^{t_1} k(t_1,s) \ f(s,x(s)) \ ds, \\
&< \quad a(t_1) \ + \ g(t_1, y(t_1)) \int_0^{t_1} k(t_1,s) \ f(s,y(s)) \ ds \\
&< \quad y(t_1),
\end{aligned}$$

which contradicts the fact that $x(t_1) \ = \ y(t_1)$, then $x(t) \ < \ y(t)$. ■

Theorem 5.1 Let the assumptions of Theorem 4.1 be satisfied and $f(t,x),\ g(t,x)$ are nondecreasing in x on I . Then there exist maximal and minimal solutions of equation (3).

Proof. Firstly we shall prove the existence of the maximal solution of (3). Let $\epsilon \ > \ 0$ be given. Now consider the quadratic integral equation

$$x_\epsilon \ (t) \ = \ a(t) \ + \ g_\epsilon(t, x_\epsilon(t)) \int_0^t k(t,s) \ f_\epsilon(s, x_\epsilon(s)) \ ds \qquad t \ \in \ [0,1], \tag{8}$$

where

$$\begin{aligned}
f_\epsilon(t, x_\epsilon(t)) \ &= \ f(t, x_\epsilon(t)) \ + \ \epsilon, \\
g_\epsilon(t, x_\epsilon(t)) \ &= \ g(t, x_\epsilon(t)) \ + \ \epsilon,
\end{aligned}$$

Clearly the functions $f_\epsilon(t, x_\epsilon)$ and $g_\epsilon(t, x_\epsilon)$ satisfy assumptions (ii) and (iii) and therefore equation (8) has at least a positive solution $x_\epsilon(t) \ \in \ C(I)$. Let ϵ_1 and ϵ_2 be such that $0 \ < \ \epsilon_2 \ < \ \epsilon_1 \ < \ \epsilon$. Then

$$x_{\epsilon_2} \ (t) \ = \ a(t) \ + \ g_{\epsilon_2}(t, x_{\epsilon_2}(t)) \int_0^t k(t,s) \ f_{\epsilon_2}(s, x_{\epsilon_2}(s)) \ ds$$

$$x_{\epsilon_2} \ (t) \ = \ a(t) \ + \ g(t, x_{\epsilon_2}(t)) \ + \ \epsilon_2) \int_0^t k(t,s) \ (f(s, x_{\epsilon_2}(s)) \ + \ \epsilon_2) \ ds \tag{9}$$

$$\begin{aligned}
x_{\epsilon_1} \ (t) \ &= \ a(t) \ + \ g_{\epsilon_1}(t, x_{\epsilon_1}(t)) \int_0^t k(t,s) \ f_{\epsilon_1}(s, x_{\epsilon_1}(s)) \ ds \\
&= \ a(t) \ + \ (g(t, x_{\epsilon_1}(t)) \ + \ \epsilon_1) \int_0^t k(t,s) \ (f(s, x_{\epsilon_1}(s)) \ + \ \epsilon_1) \ ds
\end{aligned}$$

$$x_{\epsilon_1} \ (t) \ > \ a(t) \ + \ g(t, x_{\epsilon_1}(t)) \ + \ \epsilon_2) \int_0^t k(t,s) \ (f(s, x_{\epsilon_1}(s)) \ + \ \epsilon_2) \ ds. \tag{10}$$

Applying Lemma 5.1 on (9) and (10), we have

$$x_{\epsilon_2}(t) \ < \ x_{\epsilon_1}(t) \qquad for \ t \ \in \ I.$$

As shown before the family of functions $x_\epsilon(t)$ is equi-continuous and uniformly bounded. Hence by Arzelá-Ascoli Theorem, there exists a decreasing sequence ϵ_n such that $\epsilon \ \to \ 0$

as $n \rightarrow \infty$ and $\lim_{n\rightarrow\infty} x_{\epsilon_n}(t)$ exists uniformly in I and denote this limit by $q(t)$. From the continuity of the functions $f_\epsilon(t, x_\epsilon)$ and $g_\epsilon(t, x_\epsilon)$ in the second argument, we get

$$q(t) = \lim_{n\rightarrow\infty} x_{\epsilon_n}(t) = a(t) + g(t, q(t)) \int_0^t k(t,s)\, f(s, q(s))\, ds$$

which implies that $q(t)$ as a solution of (3).
Finally, we shall show that $q(t)$ is the maximal solution of (3). To do this, let $x(t)$ be any solution of (3). Then

$$\begin{aligned} x_\epsilon\,(t) &= a(t) + g_\epsilon(t, x_\epsilon(t)) \int_0^t k(t,s)\, f_\epsilon(s, x_\epsilon(s))\, ds \\ &= a(t) + (g(t, x_\epsilon(t)) + \epsilon) \int_0^t k(t,s)\, (f(s, x_\epsilon(s)) + \epsilon)\, ds \\ &> a(t) + g(t, x_\epsilon(t)) \int_0^t k(t,s)\, f(s, x_\epsilon(s))\, ds. \end{aligned}$$

Also

$$x(t) = a(t) + g(t, x(t)) \int_0^t k(t,s)\, f(s, x(s))\, ds$$

implies that

$$x(t) < x_\epsilon(t) \qquad for\ t \in I$$

from the uniqueness of the maximal solution (see [8]), it is clear that $x_\epsilon(t)$ tends to $q(t)$ uniformly in $t \in I\ \ as\ \epsilon \rightarrow 0$.
By similar way as done above we can prove the existence of the minimal solution.■

References

[1] J. Banas, A. Martininon, Monotonic solutions of a quadratic integral equation of Volterra type, *Comput. Math. Appl.* 47 (2004), 271 - 279.

[2] J.Banas, M. Lecko, W. G. El-Sayed, Existence theorems of some quadratic integral equation, *J.Math. Anal. Appl.* 227 (1998), 276 - 279.

[3] J.Banas, K. Goebel, Measure of noncompactness in Banach space, *Lecture Note in Pure and Appl. Math.* , vol. 60. Dekker, New York, 1980.

[4] J.Banas, B. Rzepka Monotonic solutions of a quadratic integral equations of fractional order *J.Math. Anal. Appl.* 332 (2007), 1371 -1378.

[5] R. F. Curtain and A. J. Pritchard, Functional Analysis in Modern Applied Mathematics *Academic Press*, 1977.

[6] K. Deimling, Nonlinear Functional Analysis, *Springer - Verlag, Berlin,* 1985.

[7] A.M.A. El-Sayed, F.M. Gaafar and H.H.G Hashim, On the maximal and minimal solutions of arbitrary-orders nonlinear functional integral and differential equations, *Math. Sci. Res. J.* 8(11) (2004), 336-348.

[8] V.Lakshmikantham and S. Leela. Differential and Integral Inequalities, Vol. 1, *NewYork- London*, 1969.

[9] Scorza Dragoni G., Un teorema sulle funzioni continue rispetto ad una e misurabili rispetto ad un´altra variabile, Rend. Sem. Mat. Univ. Padova 17 (1948), 102-106.

(Received April 30, 2008)