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Existence of solutions to neutral differential equations with deviated argument

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Abstract: In this paper we shall study a neutral differential equation with deviated argument in an arbitrary Banach space X. With the help of the analytic semigroups theory and fixed point method we establish the existence and uniqueness of solutions of the given problem. Finally, we give examples to illustrate the applications of the abstract results.

Keywords: Neutral differential equation with deviated argument, Banach fixed point theorem, Analytic semigroup.

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1 Introduction

We consider the following neutral differential equation with deviated argument in a Banach space X:

$$\left\{ \begin{aligned} &\tfrac{d}{dt}[u(t) + g(t, u(a(t)))] + A[u(t) + g(t, u(a(t)))] \\ &\qquad\qquad = f(t, u(t), u[h(u(t), t)]), \quad 0 < t \leq T < \infty, \\ &u(0) = u_0, \end{aligned} \right. \tag{1.1}$$

where $-A$ is the infinitesimal generators of an analytic semigroup. f, g, h and a are suitably defined functions satisfying certain conditions to be specified later.

Initial results related to the differential equations with deviated arguments can be found in some research papers of the last decade but still a complete theory seems to be missing . For the initial works on existence, uniqueness and stability of various types of solutions of different kind of differential equations, we refer to [1]-[10] and the references cited in these papers.

Adimy *et al* [1] have studies the existence and stability of solutions of the following general class of nonlinear partial neutral functional differential equations:

$$\begin{aligned} \frac{d}{dt}(u(t) - g(t, u_t)) &= A(u(t) - g(t, u_t)) + f(t, u_t), \quad t \geq 0, \\ u_0 &= \varphi \in C_0, \end{aligned} \tag{1.2}$$

where the operator A is the Hille-Yosida operator not necessarily densely defined on the Banach space B. The functions g and f are continuous from $[0, \infty) \times C_0$ into B.

In this paper, we use the Banach fixed point theorem and analytic semigroup theory to prove existence and uniqueness of different kind of solutions to the given problem (1.1). The plan of the paper is as follows. In Section 3, we prove the existence and uniqueness of local solutions and in Section 4, the existence of global solution for the problem (1.1) is given. In the last section, we have given an example.

The results presented in this paper easily can be apply to the same problem (1.1) with nonlocal condition under some modified assumptions on the function f and operator A.

2 Preliminaries and Assumptions

We note that if $-A$ is the infinitesimal generator of an analytic semigroup then for $c > 0$ large enough, $-(A + cI)$ is invertible and generates a bounded analytic semigroup. This allows us to reduce the general case in which $-A$ is the infinitesimal generator of an analytic semigroup to the case in which the semigroup is bounded and the generator is invertible. Hence without loss of generality we suppose that

$$\|S(t)\| \leq M \quad \text{for} \quad t \geq 0$$

and

$$0 \in \rho(-A),$$

where $\rho(-A)$ is the resolvent set of $-A$. It follows that for $0 \leq \alpha \leq 1$, A^α can be defined as a closed linear invertible operator with domain $D(A^\alpha)$ being dense in X. We have $X_\kappa \hookrightarrow X_\alpha$ for $0 < \alpha < \kappa$ and the embedding is continuous. For more details on the fractional powers of closed linear operators we refer to Pazy [11]. It can be proved easily that $X_\alpha := D(A^\alpha)$ is a Banach space with norm $\|x\|_\alpha = \|A^\alpha x\|$ and it is equivalent to the graph norm of A^α. Also, for each $\alpha > 0$, we define $X_{-\alpha} = (X_\alpha)^*$, the dual space of X_α is a Banach space endowed with the norm $\|x\|_{-\alpha} = \|A^{-\alpha} x\|$.

It can be seen easily that $\mathcal{C}_t^\alpha = C([0, t]; X_\alpha)$, for all $t \in [0, T]$, is a Banach space endowed with the supremum norm,

$$\|\psi\|_{t,\alpha} := \sup_{0 \leq \eta \leq t} \|\psi(\eta)\|_\alpha, \quad \psi \in \mathcal{C}_t^\alpha.$$

We set,

$$\mathcal{C}_T^{\alpha-1} = C([0,T]; X_{\alpha-1}) = \{y \in \mathcal{C}_T^{\alpha} : \|y(t) - y(s)\|_{\alpha-1} \le L|t-s|, \forall\ t,s \in [0,T]\},$$

where L is a suitable positive constant to be specified later and $0 \le \alpha < 1$.

We assume the following conditions:

(**A1**): $0 \in \rho(-A)$ and $-A$ is the infinitesimal generator of an analytic semigroup $\{S(t) : t \ge 0\}$.

(**A2**): Let $U_1 \subset \mathrm{Dom(f)}$ be an open subset of $\mathbb{R}_+ \times X_\alpha \times X_{\alpha-1}$ and for each $(t,u,v) \in U_1$ there is a neighborhood $V_1 \subset U_1$ of (t,u,v). The nonlinear map $f : \mathbb{R}_+ \times X_\alpha \times X_{\alpha-1} \to X$ satisfies the following condition,

$$\|f(t,x,\psi) - f(s,y,\tilde{\psi})\| \le L_f[|t-s|^{\theta_1} + \|x-y\|_\alpha + \|\psi - \tilde{\psi}\|_{\alpha-1}],$$

where $0 < \theta_1 \le 1$, $0 \le \alpha < 1$, $L_f > 0$ is a positive constant, $(t,x,\psi) \in V_1$, and $(s,y,\tilde{\psi}) \in V_1$.

(A3): Let $U_2 \subset \mathrm{Dom(h)}$ be an open subset of $X_\alpha \times \mathbb{R}_+$ and for each $(x,t) \in U_2$ there is a neighborhood $V_2 \subset U_2$ of (x,t). The map $h : X_\alpha \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the following condition,

$$|h(x,t) - h(y,s)| \le L_h[\|x-y\|_\alpha + |t-s|^{\theta_2}],$$

where $0 < \theta_2 \le 1$, $0 \le \alpha < 1$, $L_h > 0$ is a positive constant, $(x,t),\ (y,s) \in V_2$ and $h(.,0) = 0$.

(A4): Let $U_3 \subset \mathrm{Dom(g)}$ be an open subset of $[0,T] \times X_{\alpha-1}$ and for each $(t,x) \in U_3$ there is a neighborhood $V_3 \subset U_3$ of (x,t). The function $g : [0,T] \times X_{\alpha-1} \to X_\alpha$ is continuous for $(t,u) \in [0,T] \times X_{\alpha-1}$ such that

$$\|A^\alpha g(t,x) - A^\alpha g(s,y)\| \le L_g\{|t-s| + \|x-y\|_{\alpha-1}\},$$

where $0 \le \alpha < 1$, $L_g > 0$ is a positive constant and $(x,t),\ (y,s) \in V_3$.

(**A5**): The function $a : [0,T] \to [0,T]$ satisfies the following two conditions:

(i) a satisfies the delay property $a(t) \le t$ for all $t \in [0,T]$;

(ii) The function a is Lipschitz continuous; that is, there exist a positive constant L_a such that

$$|a(t) - a(s)| \le L_a|t-s|, \text{ for all } t,s \in [0,T] \quad \text{and} \quad 1 > \|A^{-1}\|L_a.$$

Definition 2.1 *A continuous function $u \in C_T^{\alpha-1} \cap C_T^{\alpha}$ is said to be a mild solution of equation (1.1) if u is the solution of the following integral equation*

$$\begin{aligned} u(t) \quad &= \quad S(t)[u(0) + g(0, u_0)] - g(t, u(a(t))) \\ &+ \int_0^t S(t-s) f(s, u(s), u[h(u(s), s)])ds, \ t \in [0, T] \end{aligned} \tag{2.1}$$

and satisfies the initial condition $u(0) = u_0$.

Definition 2.2 *By a solution of the problem (1.1), we mean a function $u : [0, T] \to X_\alpha$ satisfying the following four conditions:*

(i) $u(.) + g(., u(a(.))) \in C_T^{\alpha-1} \cap C^1((0, T), X) \cap C([0, T], X)$,

(ii) $u(t) \in D(A)$, *and* $(t, u(t), u[h(u(t), t)]) \in U_1$,

(iii) $\frac{d}{dt}[u(t) + g(t, u(a(t)))] + A[u(t) + g(t, u(a(t)))] = f(t, u(t), u[h(u(t), t)])$ *for all* $t \in (0, T]$,

(iv) $u(0) = u_0$.

3 Existence of Local Solutions

We can prove that assumptions **(A2)–(A3)**, for $0 \le \alpha < 1$, $0 < T_0 \le T$, and $u \in \mathcal{C}_{T_0}^{\alpha}$ imply that $f(s, u(s), u[h(u(s), s)])$ is continuous on $[0, T_0]$. Therefore, we can show that there exists a positive constant N such that

$$\|f(s, u(s), u[h(u(s), s)])\| \le N = L_f[{T_0}^{\theta_1} + \delta(1 + LL_h) + LL_h T_0^{\theta_2}] + N_0,$$

where $N_0 = \|f(0, u_0, u_0)\|$. Similarly with the help of the assumptions **(A4)–(A5)**, we can show easily that $\|A^\alpha g(t, u(a(t)))\| \le L_g[T_0 + \delta] + \|g(0, u_0)\|_\alpha = N_1$. Also, we denote $\|A^{-1}\| = M_2$ and $\|A^{-\alpha}\| = M_3$.

Theorem 3.1 *Let us assume that the assumptions (A1)-(A5) are hold and $u_0 \in D(A^\alpha)$ for $0 \le \alpha < 1$. Then, the differential equation (1.1) has a unique local mild solution if*

$$\left(L_g + C_\alpha L_f [2 + LL_h] \frac{T_0^{1-\alpha}}{1-\alpha} \right) < 1. \tag{3.1}$$

Proof. Now for a fixed $\delta > 0$, we choose $0 < T_0 \le T$ such that

$$\|(S(t) - I)A^\alpha[u_0 + g(0, u_0)]\| + L_g[T_0 + \delta] \le \frac{\delta}{2}, \text{ for all } t \in [0, T_0] \tag{3.2}$$

and

$$C_\alpha N \frac{T_0^{1-\alpha}}{1-\alpha} \le \frac{\delta}{2}. \tag{3.3}$$

We set

$$\mathcal{W} = \{u \in C_{T_0}^{\alpha} \cap C_{T_0}^{\alpha-1} : u(0) = u_0, \ \ \|u - u_0\|_{T_0,\alpha} \leq \delta\}.$$

Clearly, $\mathcal{W}$ is a closed and bounded subset of $C_T^{\alpha-1}$.

We define a map $\mathcal{F} : \mathcal{W} \to \mathcal{W}$ given by

$$\begin{aligned}(\mathcal{F}u)(t) \quad &= \quad S(t)[u_0 + g(0, u_0)] - g(t, u(a(t))) \\ &+ \quad \int_0^t S(t-s) f(s, u(s), u[h(u(s), s)]) ds, \quad t \in [0, T].\end{aligned} \tag{3.4}$$

In order to proved this theorem first we need to show that $\mathcal{F}u \in \mathcal{C}_{T_0}^{\alpha-1}$ for any $u \in \mathcal{C}_{T_0}^{\alpha-1}$. Clearly, $\mathcal{F} : \mathcal{C}_T^{\alpha} \to \mathcal{C}_T^{\alpha}$.

If $u \in \mathcal{C}_{T_0}^{\alpha-1}$, $\ \ T > t_2 > t_1 > 0$, and $0 \leq \alpha < 1$, then we get

$$\begin{aligned}&\|(\mathcal{F}u)(t_2) - (\mathcal{F}u)(t_1)\|_{\alpha-1} \\ &\quad \leq \|(S(t_2) - S(t_1))(u_0 + g(0, u_0))\|_{\alpha-1} \\ &\qquad + \|A^{-1}\| \|A^{\alpha} g(t_2, u(a(t_2))) - A^{\alpha} g(t_1, u(a(t_1)))\| \\ &\qquad + \int_0^{t_1} \|(S(t_2 - s) - S(t_1 - s)) A^{\alpha-1}\| \|f(s, u(s), u[h(u(s), s)])\| ds \\ &\qquad + \int_{t_1}^{t_2} \|S(t_2 - s) A^{\alpha-1}\| \|f(s, u(s), u[h(u(s), s)])\| ds.\end{aligned} \tag{3.5}$$

We have,

$$\begin{aligned}\|(S(t_2) - S(t_1))(u_0 + g(0, u_0))\|_{\alpha-1} \quad &\leq \quad \int_{t_1}^{t_2} \|A^{\alpha-1} S'(s)(u_0 + g(0, u_0))\| ds \\ &= \quad \int_{t_1}^{t_2} \|A^{\alpha} S(s)(u_0 + g(0, u_0))\| ds \\ &\leq \quad \int_{t_1}^{t_2} \|S(s)\| [\|u_0\|_{\alpha} + \|g(0, u_0)\|_{\alpha}] ds \\ &\leq \quad C_1 (t_2 - t_1),\end{aligned} \tag{3.6}$$

where $C_1 = [\|u_0\|_{\alpha} + \|g(0, u_0)\|_{\alpha}] M$.

Also, we can see that

$$\begin{aligned}&\|A^{\alpha-1} g(t_2, u(a(t_2))) - A^{\alpha-1} g(t_1, u(a(t_1)))\| \\ &\leq \|A^{-1}\| \|A^{\alpha} g(t_2, u(a(t_2))) - A^{\alpha} g(t_1, u(a(t_1)))\| \\ &\leq \|A^{-1}\| L_g [(t_2 - t_1) + \|u(a(t_2)) - u(a(t_1))\|_{\alpha-1}] \\ &\leq \|A^{-1}\| [L_g + L L_a](t_2 - t_1).\end{aligned} \tag{3.7}$$

We observe that,

$$\|(S(t_2 - s) - S(t_1 - s))\|_{\alpha-1} \quad \leq \quad \int_0^{t_2 - t_1} \|A^{\alpha-1} S'(l) S(t_1 - s)\| dl$$

$$
\begin{aligned}
&\leq \int_0^{t_2-t_1} \|S(l)A^\alpha S(t_1-s)\| dl \\
&\leq MC_\alpha (t_2-t_1)(t_1-s)^{-\alpha}.
\end{aligned} \tag{3.8}
$$

Now we use the inequality (3.8) to get the inequality given below,

$$
\int_0^{t_1} \|(S(t_2-s)-S(t_1-s))A^{\alpha-1}\| \|f(s,u(s),u[h(u(s),s)])\| ds \leq C_2(t_2-t_1), \tag{3.9}
$$

where $C_2 = NMC_\alpha \frac{T_0^{1-\alpha}}{1-\alpha}$.

Similarly,

$$
\int_{t_1}^{t_2} \|S(t_2-s)A^{\alpha-1}\| \|f(s,u(s),u[h(u(s),s)])\| ds \leq C_3(t_2-t_1), \tag{3.10}
$$

where $C_3 = \|A^{\alpha-1}\|MN$.

We use the inequalities (3.6) (3.7) (3.9) and (3.10) in inequality (3.5) and get the following inequality,

$$
\|(\mathcal{F}u)(t_2)-(\mathcal{F}u)(t_1)\|_{\alpha-1} \leq L|t_2-t_1|, \tag{3.11}
$$

where, $L = \frac{C_1+C_2+C_3+\|A^{-1}\|L_g}{1-\|A^{-1}\|L_a}$. Hence, $\mathcal{F} : \mathcal{C}_{T_0}^{\alpha-1} \to \mathcal{C}_{T_0}^{\alpha-1}$.

Our next task is to show that $\mathcal{F} : \mathcal{W} \to \mathcal{W}$. Now, for $t \in (0, T_0]$ and $u \in \mathcal{W}$, we have

$$
\begin{aligned}
&\|(\mathcal{F}u)(t)-u_0\|_\alpha \\
&\leq \|(S(t)-I)A^\alpha[u_0+g(0,u_0)]\| \\
&\quad + \|A^\alpha g(s,u(a(s))) - A^\alpha g(0,u(a(0)))\| \\
&\quad + \int_0^t \|S(t-s)A^\alpha\| \|f(s,u(s),u[h(u(s),s)])\| ds \\
&\leq \|(S(t)-I)A^\alpha[u_0+g(0,u_0)]\| + L_g[T_0+\delta] + C_\alpha N \frac{T_0^{1-\alpha}}{1-\alpha}.
\end{aligned}
$$

Hence, from inequalities (3.2) and (3.3), we get

$$
\|\mathcal{F}u-u_0\|_{T_0,\alpha} \leq \delta.
$$

Therefore, $\mathcal{F} : \mathcal{W} \to \mathcal{W}$.

Now, if $t \in (0, T_0]$ and $u, v \in \mathcal{W}$, then

$$
\begin{aligned}
&\|(\mathcal{F}u)(t)-(\mathcal{F}v)(t)\|_\alpha \\
&\leq \|A^\alpha g(t,u(a(t))) - A^\alpha g(t,v(a(t)))\| \\
&\quad + \int_0^t \|S(t-s)A^\alpha\| \|f(s,u(s),u[h(u(s),s)]) - f(s,v(s),v[h(u(s),s)])\| ds.
\end{aligned} \tag{3.12}
$$

We have the following inequalities,

$$\|A^\alpha g(t,u(a(t))) - A^\alpha g(t,v(a(t)))\| \leq L_g\|A^{-1}\|\|u-v\|_{T_0,\alpha}, \tag{3.13}$$

$$\|f(s,u(s),u[h(u(s),s)]) - f(s,v(s),v[h(v(s),s)])\| \leq L_f[2+LL_h]\|u-v\|_{T_0,\alpha}. \tag{3.14}$$

We use the inequalities (3.13) and (3.14) in the inequality (3.12) and get

$$\|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\|_\alpha \leq \left(L_g\|A^{-1}\| + C_\alpha L_f[2+LL_h]\frac{T_0^{1-\alpha}}{1-\alpha}\right)\|u-v\|_{T_0,\alpha}. \tag{3.15}$$

Hence from inequality (3.1), we get the following inequality given below

$$\|\mathcal{F}u - \mathcal{F}v\|_{T_0,\alpha} < \|u-v\|_{T_0,\alpha}.$$

Therefore, the map $\mathcal{F}$ has a unique fixed point $u \in \mathcal{W}$ which is given by,

$$\begin{aligned} u(t) \quad &= \quad S(t)[u_0 + g(0,u_0)] - g(t,u(a(t))) \\ &+ \int_0^t S(t-s)f(s,u(s),u[h(u(s),s)])ds, \quad t \in [0,T_0]. \end{aligned} \tag{3.16}$$

Hence, the mild solution u of equation (1.1) is given by the equation (3.16) and belong to $C_{T_0}^\alpha \cap C_{T_0}^{\alpha-1}$.

Theorem 3.2 *Let us assume that the assumptions (A1)-(A5) are hold and $u_0 \in D(A^\alpha)$ for $0 \leq \alpha < 1$. Then, the differential equation (1.1) has a unique local solution in the sense of the Definition 2.2.*

Proof. In order to prove this theorem, we first need to prove that the mild solution u is Hölder continuous on $(0,T_0]$. From Theorem 2.6.13 in Pazy [11], it follows that for every $0 < \beta < 1-\alpha$, $t > s > 0$ and every $0 < h < 1$, we have

$$\begin{aligned} \|(S(h)-I)A^\alpha S(t-s)\| &\leq C_\beta h^\beta \|A^{\alpha+\beta}S(t-s)\| \\ &\leq Ch^\beta (t-s)^{-(\alpha+\beta)}, \end{aligned} \tag{3.17}$$

where $C = C_\beta C_{\alpha+\beta}$.

For $0 < t < t+h \leq T_0$, we have

$$\begin{aligned} &\|u(t+h) - u(t)\|_\alpha \\ &\leq \|((S(h)-I)S(t)(u_0 + g(0,u_0))\|_\alpha \\ &\quad + \|A^\alpha g(t_2,u(a(t_2))) - A^\alpha g(t_1,u(a(t_1)))\| \\ &\quad + \int_0^t \|(S(h)-I)A^\alpha S(t-s)\|\|f(s,u(s),u[h(u(s),s)])\|ds \\ &\quad + \int_t^{t+h} \|S(t+h-s)A^\alpha\|\|f(s,u(s),u[h(u(s),s)])\|ds. \end{aligned} \tag{3.18}$$

We calculate the first term of the above inequality (3.18) as follows;

$$\begin{aligned}\|(S(h)-I)S(t)A^{\alpha}(u_0+g(0,u_0))\| &\leq Ct^{-(\alpha+\beta)}\{\|u_0\|+\|g(0,u_0)\|\}h^{\beta}\\ &\leq M_1h^{\beta}, \end{aligned} \tag{3.19}$$

where $M_1 = Ct^{-(\alpha+\beta)}\{\|u_0\|+\|g(0,u_0)\|\}$ depends on t and blows up as t decreases to zero.

Second term of the above inequality (3.18) we calculate as follows,

$$\begin{aligned}&\|A^{\alpha}g(t+h,u(a(t+h)))-A^{\alpha}g(t,u(a(t)))\|\\ &\leq \|A^{\alpha}g(t+h,u(a(t+h)))-A^{\alpha}g(t,u(a(t)))\|\\ &\leq L_g[h+\|A\|\|u(a(t+h))-u(a(t))\|_{\alpha-1}]\\ &\leq L_g[h+\|A\|LL_ah]\\ &\leq M_2h, \end{aligned} \tag{3.20}$$

where $M_2 = L_g[1+\|A\|LL_a]$ is a constant independent of t.

Third and the fourth term of the inequality (3.18) can be calculated as follows:

$$\begin{aligned}&\int_0^t \|(S(h)-I)A^{\alpha}S(t-s)\|\|f(s,u(s),u[h(u(s),s)])\|ds\\ &\leq Ch^{\beta}N\int_0^t (t-s)^{-(\alpha+\beta)}ds\\ &\leq M_3h^{\beta}, \end{aligned} \tag{3.21}$$

$$\begin{aligned}\int_t^{t+h} \|A^{\alpha}S(t+h-s)\|\|f(s,u(s),u[h(u(s),s)])\|ds, &\leq C_{\alpha}N\int_t^{t+h}(t+h-s)^{-\alpha}ds\\ &\leq M_4h^{\beta}, \end{aligned} \tag{3.22}$$

where M_3 and M_4 can be chosen to be independent of t.

Therefore,

$$\|u(t+h)-u(t)\|_{\alpha} \leq C^{'}h^{\beta},$$

where $C^{'}$ is a positive constant. Thus, u is locally Hölder continuous on $(0,T_0]$.

Hence,

$$\begin{aligned}&\|f(t,u(t),u[h(u(t),t)])-f(s,u(s),u[h(u(s),s)])\|\\ &\leq L_f\{|t-s|^{\theta_1}+\|u(t)-u(s)\|_{\alpha}+L|h(u(t),t)-h(u(s),s)|\}\\ &\leq L_f\{|t-s|^{\theta_1}+\|u(t)-u(s)\|_{\alpha}+LL_h[|t-s|^{\theta_2}+\|u(t)-u(s)\|_{\alpha}]\}\\ &\leq L_f\{|t-s|^{\theta_1}+C^{'}|t-s|^{\beta}+LL_h[|t-s|^{\theta_2}+C^{'}|t-s|^{\beta}]\}. \end{aligned} \tag{3.23}$$

Hence, the map $t \mapsto f(t,u(t),u[h(u(t),t)])$ is locally Hölder continuous. Therefore,

$$f(t,u(t),u[h(u(t),t)]) \in C([0,T],X)\cap C^{\beta^{'}}((0,T],X),$$

where $0 < \beta^{'} < \min\{\theta_1,\ \beta,\ \theta_2\}$. Similarly, we can prove that $u(.)+g(.,u(a(.)))$ is also Hölder continuous on $(0,T_0]$. Therefore from Theorem 3.1 pp. 110 and Corollary 3.3, pp. 113, Pazy [11], the function $u(.)+g(.,u(a(.))) \in C^{\alpha-1}_{T_0}\cap C^1((0,T_0),X)\cap C([0,T_0],X)$ and $u(.)$ is the unique solution of the problem (1.1) in the sense of Definition 2.2. This completes the proof of the Theorem. □

4 Existence of Global Solutions

Theorem 4.1 *Suppose that $0 \in \rho(-A)$ and the operator $-A$ generates the analytic semigroup $S(t)$ with $\|S(t)\| \leq M$, for $t \geq 0$, the conditions* ***(A1)–(A4)*** *are satisfied and $u_0 \in D(A^\alpha)$. If there are continuous nondecreasing real valued function $k_1(t)$, $k_2(t)$ and $k_3(t)$ such that*

$$\|f(t,x,y)\| \leq k_1(t)(1+\|x\|_\alpha + \|y\|_{\alpha-1}), \text{ for all } t \geq 0, \quad x \in X_\alpha,\ y \in X_{\alpha-1}, \tag{4.1}$$

$$|h(z,t)| \leq k_2(t)(1+\|z\|_\alpha), \text{ for all } t \geq 0, \quad z \in X_\alpha, \tag{4.2}$$

$$\|g(t,v)\|_\alpha \leq k_3(t)(1+\|v\|_{\alpha-1}), \text{ for all } t \geq 0, \quad v \in X_{\alpha-1}, \tag{4.3}$$

then the initial value problem (1.1) has a unique solution which exists for all $t \in [0,T]$.

Proof: By theorem (3.1) we can continue the solution of equation (1.1) as long as $\|u(t)\|_\alpha$ stays bounded. It is therefore sufficient to show that if u exists on $[0,T[$ then $\|u(t)\|_\alpha$ is bounded as $t \uparrow T$.

We have the following inequality,

$$\begin{aligned} \|u[h(u(s),s)]\|_{\alpha-1} &\leq \|u[h(u(s),s)] - u(0)\|_{\alpha-1} + \|u_0\|_{\alpha-1} \\ &\leq L|h(u(s),s)| + \|u_0\|_{\alpha-1} \\ &\leq Lk_2(T) + Lk_2(T)\|u\|_{s,\alpha}] + \|u_0\|_{\alpha-1}. \end{aligned} \tag{4.4}$$

For $t \in [0,T[$, we have

$$\begin{aligned} \|u(t)\|_\alpha \quad &\leq \quad \|S(t)A^\alpha[u_0 + g(0,u_0)]\| + \|g(t,u(a(t)))\|_\alpha \\ &+ \quad \int_0^t \|A^\alpha S(t-s)\|\|f(s,u(s),u[h(u(s),s)])\|ds \\ &\leq \quad M[\|u_0\|_\alpha + k_3(T)\{1+\|u_0\|_\alpha\}] + k_3(T)[1+\|A^{-1}\|\|u\|_{t,\alpha}] \\ &+ \quad C_\alpha \int_0^t (t-s)^{-\alpha} k_1(T)[1+\|u\|_{s,\alpha} + \|u[h(u(s),s)]\|_{\alpha-1}]ds, \\ &\leq \quad M[\|u_0\|_\alpha + k_3(T)\{1+\|u_0\|_\alpha\}] + k_3(T) + k_3(T)\|A^{-1}\|\|u\|_{t,\alpha} \\ &+ \quad k_1(T)C_\alpha \int_0^t (t-s)^{-\alpha} ds + k_1(T)C_\alpha \int_0^t (t-s)^{-\alpha}\|u\|_{s,\alpha} ds \\ &+ \quad Lk_2(T)k_1(T)C_\alpha \int_0^t (t-s)^{-\alpha} ds + \|u_0\|_{\alpha-1} Lk_2(T)k_1(T)C_\alpha \int_0^t (t-s)^{-\alpha} ds \\ &+ \quad Lk_2(T)k_1(T)C_\alpha \int_0^t (t-s)^{-\alpha}\|u\|_{s,\alpha} ds. \end{aligned}$$

Hence,

$$\|u\|_{t,\alpha} \leq C_1 + C_2 \int_0^t (t-s)^{-\alpha}\|u\|_{s,\alpha} ds, \tag{4.5}$$

where $C_1 = \frac{M}{1-k_3(T)}[\|u_0\|_\alpha + k_3(T)\{1+\|u_0\|_\alpha\}] + \frac{k_3(T)}{1-k_3(T)} + \frac{k_1(T)C_\alpha T^{1-\alpha}}{(1-k_3(T))(1-\alpha)} + \frac{Lk_2(T)k_1(T)C_\alpha T^{1-\alpha}}{(1-k_3(T))(1-\alpha)}$ $+ \|u_0\|_{\alpha-1}\frac{Lk_2(T)k_1(T)C_\alpha T^{1-\alpha}}{(1-k_3(T))(1-\alpha)}$ and $C_2 = \frac{k_1(T)C_\alpha[1+Lk_2(T)]}{1-k_3(T)}$. Hence by applying the Gronwall's lemma to the above inequality (4.5), we get the required results. This completes the proof of the theorem. □

5 Examples

Let $X = L^2(0,1)$. We consider the following partial differential equations with deviated argument,

$$\begin{cases} \partial_t[w(t,x) + \partial_x f_1(t, w(a(t),x))] - \partial_x^2[w(t,x) + f_1(t, w(a(t),x))] \\ \quad = f_2(x, w(t,x)) + f_3(t,x,w(t,x)), \quad x \in (0,1),\ t > 0, \\ \quad w(t,0) = w(t,1) = 0,\ t \in [0,T],\ 0 < T < \infty, \\ \quad w(0,x) = u_0,\ x \in (0,1), \end{cases} \tag{5.1}$$

where

$$f_2(x, w(t,x)) = \int_0^x K(x,s) w(s, h(t)(a_1|w(s,t)| + b_1|w_s(s,t)|))ds.$$

The function $f_3 : \mathbb{R}_+ \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is measurable in x, locally Hölder continuous in t, locally Lipschitz continuous in w and uniformly in x. Further we assume that $a_1, b_1 \geq 0, (a_1, b_1) \neq (0,0)$, $h : \mathbb{R}_+ \to \mathbb{R}_+$ is locally Hölder continuous in t with $h(0) = 0$ and $K : [0,1] \times [0,1] \to \mathbb{R}$. The function $f_1 : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is locally Hölder continuous in t, locally Lipschitz continuous in w.

We define an operator A, as follows,

$$Au = -u'' \quad \text{with} \quad u \in D(A) = \{u \in H_0^1(0,1) \cap H^2(0,1) :\ u'' \in X\}. \tag{5.2}$$

Here clearly the operator A is self-adjoint with compact resolvent and is the infinitesimal generator of an analytic semigroup $S(t)$. Now we take $\alpha = 1/2$, $D(A^{1/2}) = H_0^1(0,1)$ is the Banach space endowed with the norm,

$$\|x\|_{1/2} := \|A^{1/2}x\|, \quad x \in D(A^{1/2})$$

and we denote this space by $X_{1/2}$. Also, for $t \in [0,T]$, we denote

$$C_t^{1/2} = C([0,t]; D(A^{1/2})),$$

endowed with the sup norm

$$\|\psi\|_{t,1/2} := \sup_{0 \leq \eta \leq t} \|\psi(\eta)\|_\alpha, \quad \psi \in \mathcal{C}_t^{1/2}.$$

We observe some properties of the operators A and $A^{1/2}$ defined by (5.2). For $u \in D(A)$ and $\lambda \in \mathbb{R}$, with $Au = -u'' = \lambda u$, we have $\langle Au, u\rangle = \langle \lambda u, u\rangle$; that is,

$$\left\langle -u'', u \right\rangle = |u'|_{L^2}^2 = \lambda |u|_{L^2}^2$$

so $\lambda > 0$. A solution u of $Au = \lambda u$ is of the form

$$u(x) = C\cos(\sqrt{\lambda}x) + D\sin(\sqrt{\lambda}x)$$

and the conditions $u(0) = u(1) = 0$ imply that $C = 0$ and $\lambda = \lambda_n = n^2\pi^2$, $n \in \mathbb{N}$. Thus, for each $n \in \mathbb{N}$, the corresponding solution is given by

$$u_n(x) = D\sin(\sqrt{\lambda_n}x).$$

We have $\langle u_n, u_m\rangle = 0$ for $n \neq m$ and $\langle u_n, u_n\rangle = 1$ and hence $D = \sqrt{2}$. For $u \in D(A)$, there exists a sequence of real numbers $\{\alpha_n\}$ such that

$$u(x) = \sum_{n\in\mathbb{N}} \alpha_n u_n(x), \quad \sum_{n\in\mathbb{N}} (\alpha_n)^2 < +\infty \quad \text{and} \quad \sum_{n\in\mathbb{N}} (\lambda_n)^2(\alpha_n)^2 < +\infty.$$

We have

$$A^{1/2}u(x) = \sum_{n\in\mathbb{N}} \sqrt{\lambda_n}\, \alpha_n\, u_n(x)$$

with $u \in D(A^{1/2})$; that is, $\sum_{n\in\mathbb{N}} \lambda_n(\alpha_n)^2 < +\infty$. $X_{-\frac{1}{2}} = H^1(0,1)$ is a Sobolev space of negative index with the equivalent norm $\|.\|_{-\frac{1}{2}} = \sum_{n=1}^{\infty} |\langle ., u_n\rangle|^2$. For more details on the Sobolev space of negative index, we refer to Gal [6].

The equation (5.1) can be reformulated as the following abstract equation in $X = L^2(0,1)$:

$$\begin{aligned} \frac{d}{dt}[u(t) + g(t, u(a(t)))] + A[u(t) + g(t, u(a(t)))] &= f(t, u(t), u[h(u(t), t)]) \quad t > 0, \\ u(0) &= u_0, \end{aligned} \tag{5.3}$$

where $u(t) = w(t, .)$ that is $u(t)(x) = w(t, x)$, $x \in (0,1)$. The function $g : \mathbb{R}_+ \times X_{1/2} \to X$, such that $g(t, u(a(t)))(x) = \partial_x f_1(t, w(a(t), x))$ and the operator A is same as in equation (5.2).

The function $f : \mathbb{R}_+ \times X_{1/2} \times X_{-1/2} \to X$, is given by

$$f(t, \psi, \xi)(x) = f_2(x, \xi) + f_3(t, x, \psi), \tag{5.4}$$

where $f_2 : [0,1] \times X \to H_0^1(0,1)$ is given by

$$f_2(t, \xi) = \int_0^x K(x, y)\xi(y)dy, \tag{5.5}$$

and

$$\|f_3(t, x, \psi)\| \le Q(x, t)(1 + \|\psi\|_{H^2(0,1)}) \tag{5.6}$$

with $Q(., t) \in X$ and Q is continuous in its second argument. We can easily verify that the function f satisfies the assumptions (H1)-(H4). For more details see [6].

For the function a we can take

(i) $a(t) = kt$, where $t \in [0, T]$ and $0 < k \le 1$.

(ii) $a(t) = kt^n$ for $t \in I = [0,1]$ $k \in (0,1]$ and $n \in \mathbb{N}$;

(iii) $a(t) = k \sin t$ for $t \in I = [0, \frac{\pi}{2}]$, and $k \in (0,1]$.

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