Electronic Journal of Qualitative Theory of Differential Equations
2009, No. **52**, 1-13; **http://www.math.u-szeged.hu/ejqtde/**

APPROXIMATION OF SOLUTIONS OF NONLINEAR HEAT TRANSFER PROBLEMS

RAHMAT ALI KHAN

Centre for Advanced Mathematics and Physics, National University of Sciences and Technology(NUST), Campus of College of Electrical and Mechanical Engineering, Peshawar Road, Rawalpindi, Pakistan,
rahmat_alipk@yahoo.com

Abstract. We develop a generalized approximation method (GAM) to obtain solution of a steady state one-dimensional nonlinear convective-radiative-conduction equation. The GAM generates a bounded monotone sequence of solutions of linear problems. The sequence of approximants converges monotonically and rapidly to a solution of the original problem. We present some numerical simulation to illustrate and confirm our results.

Keywords: heat transfer equations, upper and lower solutions, generalized approximation method

1. Introduction

Most metallic materials have variable thermal properties, usually depending on temperature. The governing equations describing the temperature distribution along such surfaces are nonlinear. In consequence, exact analytic solutions of such nonlinear problems are not available in general. Scientists use some approximation techniques for example, perturbation method [7], [25], homotopy perturbation method [1], [3], [5], [12], [13], [14] , [24], to approximate solutions of the nonlinear problems. However, these methods have the drawback that the series solutions may not always converge to a solution of the problem and, in some cases, produce inaccurate and meaningless results.

In this paper, we develop the generalized approximation method (GAM), [2], [6], [15], [16], [17], to approximate solutions of nonlinear problems. This method produces excellent result and is independent of the choice of a small parameter. It generates a bounded monotone sequence of solutions of linear problems which converges uniformly and rapidly to a solution

Acknowledgement: The author is thankful to the referee for valuable comments and suggestions that improve the manuscript.
Research is supported by HEC, Pakistan.

of the original problem. Hence it can be applied to a much larger class of nonlinear boundary value problems. Moreover, we show that our results are consistent and accurately represent the actual solution of the problem for any value of the parameter. For the numerical simulation, we use the computer programme, Mathematica. For computational purposes, the linear iteration is important. The generalized approximation method which uses linear problems is a particular version of the well studied quasilinearization method [8, 9, 10, 18, 19, 20, 21]. At each iteration, we are dealing with linear problems and obtain a monotone sequence of solutions of linear problems which converges to a solution of the original nonlinear problem.

2. HEAT TRANSFER PROBLEM: INTEGRAL FORMULATION

Consider a straight fin of length L made of materials with temperature dependent thermal conductivity $k = k(T)$. The fin is attached to a base surface of temperature T_b extended into a fluid of temperature T_a with $T_b > T_a$ and its tip is insolated. Assume that the thermal conductivity k vary linearly with temperature, that is,

$$k(T) = k_a[1 + \eta(T - T_a)], \tag{2.1}$$

where η is constant and k_a is the thermal conductivity at temperature T_a. Choose the tip of the fin as origin $x = 0$ and the base of the fin at position $x = L$. The fin surface transfers heat through convection, conduction and radiation. Assume that the emissivity coefficient of the surface E_g is constant and the convective heat transfer coefficient h depends on the temperature. The convective heat transfer coefficient h usually varies as a power law of the type,

$$h = h(T) = h_b(\frac{T - T_a}{T_b - T_a})^n, \tag{2.2}$$

see [23], where h_b is the heat transfer coefficient at the base temperature and the number n depends on the heat transfer mode. For example, for laminar film boiling or condensation $n = -1/4$, for laminar natural convection $n = 1/4$, for turbulent natural convection $n = 1/3$, for nucleate boiling $n = 2$ and for radiation $n = 3$. Here, we restrict our study to the case $n > 1$ and $T \geq T_a$. Our results are also valid for the case $n \leq 1$ but with possibly different upper and lower solutions.

The energy equation describing one dimensional steady state temperature distribution is given by

$$\tag{2.3} \begin{aligned} &\frac{d}{dx}\big[k(T)\frac{dT}{dx}\big] - \frac{ph}{A}(T-T_a) - \frac{E_g\sigma}{A}(T-T_a)^4 = 0,\ x \in [0,L],\\ &\frac{dT}{dx}(0) = 0, \quad T(L) = T_b, \end{aligned}$$

see [11] and [22], where A is the cross-sectional area and p is a parameter of the fin. In view of (2.1) and (2.2), the boundary value problem (2.3) can be rewritten as follows

$$\begin{aligned} &\frac{d}{dx}\big[(1+\eta(T-T_a))\frac{dT}{dx}\big] - \frac{ph_b}{Ak_a}\frac{(T-T_a)^{n+1}}{(T_b-T_a)^n} - \frac{E_g\sigma}{Ak_a}(T-T_a)^4 = 0,\ x \in [0,L],\\ &\frac{dT}{dx}(0) = 0, \quad T(L) = T_b, \end{aligned}$$

Introducing the dimensionless quantities $\theta = \frac{T-T_a}{T_b-T_a}$, $y = \frac{x}{L}$, we obtain

$$\tag{2.4} \begin{aligned} &\frac{d}{dy}\big[(1+\epsilon_1\theta)\frac{d\theta}{dy}\big] - N\theta^{n+1} - \epsilon_2\theta^4 = 0,\ y \in [0,1],\\ &\frac{d\theta}{dy}(0) = 0, \quad \theta(1) = 1, \end{aligned}$$

where, $\epsilon_1 = \eta(T_b - T_a)$, $N = \frac{h_b p L^2}{k_a A}$ and $\epsilon_2 = \frac{L^2 E_g\sigma(T_b-T_a)^3}{k_a A}$. From the definition of θ, we have $\theta \geq 0$. From the differential equation in (2.4), we obtain $\frac{d}{dy}[(1+\epsilon_1\theta)\frac{d\theta}{dy}] \geq 0$, which implies that the function $(1+\epsilon_1\theta)\frac{d\theta}{dy}$ in nondecreasing on $[0,1]$. Hence using the boundary condition at 0, it follows that $\frac{d\theta}{dy} \geq 0$, that is, the function θ is monotonically increasing on $[0,1]$. Hence, $0 \leq \theta(y) \leq \theta(1) = 1$, $y \in [0,1]$ and these provide bounds for the possible solutions of the BVP (2.4). Moreover, from (2.4), we obtain $\frac{d^2\theta}{dy^2} \geq -\frac{\epsilon_1(\frac{d\theta}{dy})^2}{1+\epsilon_1\theta}$, which implies that the function $\frac{d\theta}{dy}$ may not be monotone on $[0,1]$.

Now, for simplicity, we write the problem (2.4) as follows

$$\tag{2.5} \begin{aligned} &-\frac{d^2\theta}{dy^2} = \frac{\epsilon_1(\frac{d\theta}{dy})^2 - N\theta^{n+1} - \epsilon_2\theta^4}{(1+\epsilon_1\theta)},\ y \in [0,1] = I,\\ &\theta'(0) = 0, \quad \theta(1) = 1, \end{aligned}$$

which can be written as an equivalent integral equation

$$\tag{2.6} \theta(y) = 1 + \int_0^1 G(y,s)\big[\frac{\epsilon_1(\frac{d\theta}{dy})^2 - N\theta^{n+1} - \epsilon_2\theta^4}{(1+\epsilon_1\theta)}\big]ds = 1 + \int_0^1 G(y,s)f(\theta,\theta'),$$

where $f(\theta, \theta') = \frac{\epsilon_1(\frac{d\theta}{dy})^2 - N\theta^{n+1} - \epsilon_2\theta^4}{(1+\epsilon_1\theta)}$ and

$$G(y, s) = \begin{cases} 1 - s, & 0 \leq y < s \leq 1, \\ 1 - y, & 0 \leq s < y \leq 1, \end{cases}$$

is the Green's function. Clearly, $G(y, s) > 0$ on $(0, 1) \times (0, 1)$.

Recall the concept of lower and upper solutions corresponding to the BVP (2.5).

Definition 2.1. A function $\alpha \in C^1(I)$ is called a lower solution of the BVP (2.5), if it satisfies the following inequalities,

$$\begin{aligned} -\alpha''(y) &\leq f(\alpha(y), \alpha'(y)), \quad y \in (0, 1) \\ &\alpha'(0) \geq 0,\ \alpha(1) \leq 1. \end{aligned}$$

An upper solution $\beta \in C^1(I)$ of the BVP (2.5) is defined similarly by reversing the inequalities.

For example, $\alpha = 0$ and $\beta = 1$ are lower and upper solutions of the BVP (2.5) respectively as they satisfy the inequalities:

$$\begin{aligned} &\alpha''(y) + f(\alpha, \alpha') = 0,\ y \in I,\ \alpha'(0) = 0,\ \alpha(1) < 1 \\ &\beta''(y) + f(\beta, \beta') = -\frac{(N + \epsilon_2)}{1 + \epsilon_1} < 0,\ y \in I,\ \beta'(0) = 0,\ \beta(1) = 1. \end{aligned}$$

For broad variety of nonlinear boundary value problems, it is possible to find a solution between the lower and the upper solutions. To give an estimate of the derivative u' of a possible solution, we recall the concept of Nagumo function.

Definition 2.2. A continuous function $\omega : (0, \infty) \to (0, \infty)$ is called a Nagumo function if

$$\int_\lambda^\infty \frac{sds}{\omega(s)} = \infty,$$

for $\lambda = \max\{|\alpha(0) - \beta(1)|, |\alpha(1) - \beta(0)|\}$. We say that $f \in C[\mathbb{R} \times \mathbb{R}]$ satisfies a Nagumo condition relative to α, β if for $y \in [\min \alpha, \max \beta]$, there exists a Nagumo function ω such that $|f(y, y')| \leq \omega(|y'|)$.

For $\theta \in [0, 1] = [\min \alpha, \max \beta]$, we have

$$|f(\theta, \theta')| \leq \epsilon_1 |\theta'|^2 + N + \epsilon_2 = \omega(|\theta'|)$$

and

$$\int_1^\infty \frac{sds}{\omega(s)} = \int_1^\infty \frac{sds}{\epsilon_1 s^2 + N + \epsilon_2} = \infty,$$

which implies that f satisfies a Nagumo condition with $\omega(s) = \epsilon_1 s^2 + N + \epsilon_2$ as a Nagumo function. Hence by Theorem 1.4.1 of [4] (page 14) , there exists a constant $C > \lambda$ such that any solution θ of the BVP (2.4) which satisfies $\alpha \leq \theta \leq \beta$ on I, must satisfies $|\theta'| \leq C$ on I.

Using the relation $\int_1^C \frac{sds}{\omega(s)} \geq \max\beta - \min\alpha = 1$, we obtain $C \geq [e^{2\epsilon_1} + (e^{2\epsilon_1} - 1)\frac{N+\epsilon_2}{\epsilon_1}]^{\frac{1}{2}}$. In particular, we may choose $C = [e^{2\epsilon_1} + (e^{2\epsilon_1} - 1)\frac{N+\epsilon_2}{\epsilon_1}]^{\frac{1}{2}}$. Hence, any solution θ of the BVP (2.4) such that $0 \leq \theta \leq 1$ satisfies $|\theta'| \leq C$ and this provide estimate for the derivative of a solution θ.

The following result is known [4] (Theorem 1.5.1, Page 31).

Theorem 2.3. *Assume that $\alpha, \beta \in C^1(I)$ are lower and upper solutions of the BVP (2.4) such that $\alpha \leq \beta$ on I. Assume that $f : \mathbb{R} \times \mathbb{R} \to (0, \infty)$ is continuous and satisfies a Nagumo's condition on I relative to α, β. Then the BVP (2.4) has a solution $\theta \in C^1(I)$ such that $\alpha \leq \theta \leq \beta$ and $|\theta'| \leq C$ on I, where C depends only on α, β and h.*

We note that the BVP (2.4) satisfies the conditions of Theorem 2.3 with $\alpha = 0$ and $\beta = 1$ as lower and upper solutions.

3. GENERALIZED APPROXIMATION METHOD (GAM)

Notice that $f_\theta(\theta, \theta') = -\frac{(N(n+1)n\theta^n + 4\epsilon_2\theta^3 + Nn\epsilon_1\theta^{n+1} + 3\epsilon_1\epsilon_2\theta^4 + \epsilon_1\theta'^2)}{(1+\epsilon_1\theta)^2} < 0$, $f_{\theta'}(\theta, \theta') = \frac{2\epsilon_1\theta'}{(1+\epsilon_1\theta)}$,

$$\begin{aligned} f_{\theta\theta}(\theta, \theta') &= \frac{2\epsilon_1(N^2 + \epsilon_1^2\theta'^2) - 2(6 + 8\epsilon_1\theta + 3\epsilon_1^2\theta^2)\epsilon_2\theta^2}{(1+\epsilon_1\theta)^3}, \\ f_{\theta'\theta'}(\theta, \theta') &= \frac{2\epsilon_1}{1+\epsilon_1\theta} \text{ and } f_{\theta\theta'}(\theta, \theta') = \frac{-2\epsilon_1^2\theta'}{(1+\epsilon_1\theta)^2}. \end{aligned} \tag{3.1}$$

Hence, the quadratic form

(3.2)

$$\begin{aligned} v^T H(f) v &= (\theta - z)^2 f_{\theta\theta}(z, z') + 2(\theta - z)(\theta' - z') f_{\theta\theta'}(z, z') + (\theta' - z')^2 f_{\theta'\theta'}(z, z') \\ &= \left((\theta - z)\sqrt{\frac{2\epsilon^3 z'^2}{(1+\epsilon z)^3}} - (\theta' - z')\sqrt{\frac{2\epsilon_1}{(1+\epsilon_1 z)}}\right)^2 + \frac{(\theta - z)^2}{(1+\epsilon_1 z)^3}[N\epsilon_1(n\epsilon_1(n - \epsilon_1) + 2)z^n] \\ &- \frac{(\theta - z)^2}{(1+\epsilon_1 z)^3} R(z), \end{aligned}$$

where $R(z) = [2\epsilon_2 z^2(6+8\epsilon_1 z+3\epsilon_1^2 z^2)+(n(n+1)+2n^2 z)Nz^{n-1}]$, $H(f) = \begin{pmatrix} f_{\theta\theta} & f_{\theta\theta'} \\ f_{\theta\theta'} & f_{\theta'\theta'} \end{pmatrix}$ is the Hessian matrix and $v = \begin{pmatrix} \theta - z \\ \theta' - z' \end{pmatrix}$. If the quadratic form $v^T H(f) v \ngeq 0$ on $[\min\alpha, \max\beta] \times$

$[-C,C]$, then we need to choose an auxiliary function ϕ such that $v^T H(F)v \geq 0$ on $[\min\alpha, \max\beta] \times [-C,C]$, where $F = f + \phi$. In our case, we choose $\phi(\theta) = \frac{m(z)}{2}\theta^2$, where $m(z) = 2\epsilon_2 z^2(6 + 8\epsilon_1 z + 3\epsilon_1^2 z^2) + n(3n+1)N \geq R(z)$. Hence,

$$v^T H(F)v \geq 0, \text{ on } [\min\alpha, \max\beta] \times [-C,C], \tag{3.3}$$

which implies that

$$F(\theta,\theta') \geq F(z,z') + F_\theta(z,z')(\theta - z) + F_{\theta'}(z,z')(\theta' - z'), \tag{3.4}$$

where z, $\theta \in [\min\alpha,\ \max\beta] = [0,1]$, z', $\theta' \in [-C,C]$, which further implies that

$$f(\theta,\theta') \geq f(z,z') + F_\theta(z,z')(\theta - z) + F_{\theta'}(z,z')(\theta' - z') - (\phi(\theta) - \phi(z)). \tag{3.5}$$

Using the relation $\phi(\theta) - \phi(z) = \frac{m(z)}{2}(\theta + z)(\theta - z) \leq m(z)(\theta - z)$ for $\theta \geq z$, we obtain

$$f(\theta,\theta') \geq f(z,z') + (F_\theta(z,z') - m(z))(\theta - z) + f_{\theta'}(z,z')(\theta' - z'), \text{ for } \theta \geq z.$$

Now,

$$\begin{aligned} F_\theta(z,z') - m(z) &= -\Big[\frac{(N(n+1)nz^n + 4\epsilon_2 z^3 + Nn\epsilon_1 z^{n+1} + 3\epsilon_1\epsilon_2 z^4 + \epsilon_1 z'^2)}{(1+\epsilon_1 z)^2} + (1-z)m(z)\Big] \\ &\geq -m_1, \end{aligned}$$

where,

$$\begin{aligned} m_1 = \max\{&\frac{(N(n+1)nz^n + 4\epsilon_2 z^3 + Nn\epsilon_1 z^{n+1} + 3\epsilon_1\epsilon_2 z^4 + \epsilon_1 z'^2)}{(1+\epsilon_1 z)^2} + (1-z)m(z) : \\ &\qquad z \in [0,1],\ z' \in [-C,C]\}. \end{aligned}$$

Hence,

$$f(\theta,\theta') \geq f(z,z') - m_1(\theta - z) + f_{\theta'}(z,z')(\theta' - z'), \text{ for } \theta \geq z. \tag{3.6}$$

Define $g : \mathbb{R}^4 \to \mathbb{R}$ by

$$g(\theta,\theta';\, z,z') = f(z,z') - m_1(\theta - z) + f_{\theta'}(z,z')(\theta' - z') = q(z) - m_1\theta + p(z)\theta', \tag{3.7}$$

where $p(z) = \frac{2\epsilon_1 z'}{1+\epsilon_1 z}$, $q(z) = m_1 z - \frac{\epsilon_1 z'^2 + Nz^{n+1} + \epsilon_2 z^4}{1+\epsilon z}$.

Clearly, g is continuous and satisfies the following relations

$$\begin{cases} f(\theta,\theta') \geq g(\theta,\theta';\, z,z') \text{ for } \theta \geq z, \\ f(\theta,\theta') = g(\theta,\theta';\, \theta,\theta'), \end{cases} \tag{3.8}$$

where θ, $z \in [0,1]$, θ', $z' \in \times[-C,C]$. We note that for every θ, $z \in [0,\, 1]$ and $z' \in$ some compact subset of $\mathbb{R}$, g satisfies a Nagumo condition relative to α, β. Hence, there exists a constant C_1 such that any solution θ of the linear BVP

$$
\begin{aligned}
&-\theta''(y) = g(\theta,\theta';\, z, z') = q(z) - m_1\theta + p(z)\theta',\ y \in I,\\
&\theta'(0) = 0, \quad \theta(1) = 1,
\end{aligned}
\tag{3.9}
$$

with the property that $\alpha \le \theta \le \beta$ on I, must satisfies $|\theta'| < C_1$ on I. We note that the linear problem (3.9) can be solved analytically.
To develop the iterative scheme, we choose $w_0(y) = \alpha(y) = 0$ as an initial approximation and consider the following linear BVP

$$
\begin{aligned}
-\theta''(y) = g(\theta,\theta';\, w_0, w_0') = -m_1\theta,\ y \in I,\\
\theta'(0) = 0, \quad \theta(1) = 0,
\end{aligned}
\tag{3.10}
$$

whose solution is $w_1(y) = \frac{\cosh(\sqrt{m_1}y)}{\cosh(\sqrt{m_1})}$.

In general, using (3.8) and the definition of lower and upper solutions, we obtain

$$
\begin{aligned}
g(w_0, w_0';\, w_0, w_0') = f(w_0, w_0') \ge -w_0'',\\
g(\beta, \beta';\, w_0, w_0') \le f(\beta, \beta') \le -\beta'',\ \text{on } I,
\end{aligned}
$$

which imply that w_0 and β are lower and upper solutions of (3.10). Hence, by Theorem 2.3, there exists a solution w_1 of (3.10) such that $w_0 \le w_1 \le \beta$, $|w_1'| < C_1$ on I. Using (3.8) and the fact that w_1 is a solution of (3.10), we obtain

$$
-w_1''(y) = g(w_1, w_1';\, w_0, w_0') \le f(w_1, w_1')
\tag{3.11}
$$

which implies that w_1 is a lower solution of (2.4). Similarly, we can show that w_1 and β are lower and upper solutions of

$$
\begin{aligned}
-\theta''(y) = g(\theta,\theta';\, w_1, w_1'),\ y \in I,\\
\theta'(0) = 0, \quad \theta(1) = 1.
\end{aligned}
\tag{3.12}
$$

Hence, there exists a solution w_2 of (3.12) such that $w_1 \le w_2 \le \beta$, $|w_2'| < C_1$ on I.
Continuing this process we obtain a monotone sequence $\{w_n\}$ of solutions satisfying

$$
\alpha = w_0 \le w_1 \le w_2 \le w_3 \le \ldots \le w_{n-1} \le w_n \le \beta,\ |w_n'| < C_1 \text{ on } I,
$$

where w_n is a solution of the linear problem

$$-\theta''(y) = g(\theta, \theta'; w_{n-1}, w'_{n-1}), \; y \in I$$
$$\theta'(0) = 0, \; \theta(1) = 1$$

and is given by

$$w_n(y) = 1 + \int_0^1 G(y,s) g(w_n(s), w'_n(s); w_{n-1}(s), w'_{n-1}(s)) ds, \; y \in I. \tag{3.13}$$

The sequence of functions w_n is is uniformly bounded and equicontinuous. The monotonicity and uniform boundedness of the sequence $\{w_n\}$ implies the existence of a pointwise limit w on I. From the boundary conditions, we have

$$0 = w'_n(0) \to w'(0) \text{ and } 1 = w_n(1) \to w(1).$$

Hence w satisfy the boundary conditions. Moreover, by the dominated convergence theorem, for any $y \in I$,

$$\int_0^1 G(y,s) g(w_n(s), w'_n(s); w_{n-1}(s), w'_{n-1}(s)) ds \to \int_0^1 G(y,s) f(w(s), w'(s)) ds.$$

Passing to the limit as $n \to \infty$, we obtain

$$w(y) = 1 + \int_0^1 G(y,s) f(w(s), w'(s)) ds, \; y \in I,$$

that is, w is a solution of (2.4).

Hence, the sequence of approximants $\{w_n\}$ converges to the unique solution of the nonlinear BVP (2.4). Moreover, the convergence is quadratic, see [16], [17]. The fact that the sequence converges rapidly to the solution of the problem can also be seen from the numerical experiment.

4. NUMERICAL RESULTS FOR GAM, HPM AND VIM

Starting with the initial approximation $w_0 = 0$ and set $N = 0.5$, $n = 1.1$, results obtained via GAM for ($\epsilon_1 = 0.2\epsilon_2 = 0.3$), $\epsilon_1 = 0.5\epsilon_2 = 0.5$ and $\epsilon_1 = 1\epsilon_2 = 1$ are shown in Tables (Table 1, Table 2 Table 3 respectively) and also graphically in Fig.1, Fig.2 and Fig.3. Form the tables and graphs, it is clear that with only few iterations it is possible to obtain good approximations of the exact solution. Moreover, the convergence is very fast. Even for larger values of N, n, the GAM produces excellent results and fast convergence, see for example, Fig.5, Fig 6 and Fig.7. In fact, the GAM accurately approximate the actual solution of the problem independent of the choice of the parameters ϵ_1 and ϵ_2 involved, Fig. 4 and Fig. 8.

y	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
w_1	0.415591	0.430306	0.455217	0.490916	0.538247	0.598333	0.672598	0.762801	0.87108	1
w_2	0.678627	0.68872	0.705478	0.728805	0.758576	0.794641	0.836853	0.885104	0.939402	1
w_3	0.771179	0.777893	0.789096	0.804813	0.82509	0.850008	0.879701	0.914372	0.954327	1
w_4	0.792837	0.798656	0.808407	0.822175	0.840082	0.862301	0.88906	0.920655	0.957472	1
w_5	0.797117	0.802753	0.81221	0.825584	0.843017	0.864699	0.890878	0.921871	0.958078	1
w_6	0.797921	0.803523	0.812924	0.826224	0.843567	0.865148	0.891218	0.922098	0.958191	1
w_7	0.798071	0.803667	0.813057	0.826343	0.84367	0.865231	0.891281	0.92214	0.958212	1

Table 1; GAM for $N = 0.5$, $n = 1.1$, $\epsilon_1 = 0.2$, $\epsilon_2 = 0.3$

y	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
w_1	0.415591	0.430306	0.455217	0.490916	0.538247	0.598333	0.672598	0.762801	0.87108	1
w_1	0.633881	0.645693	0.665265	0.692429	0.726952	0.768547	0.816897	0.871696	0.932727	1
w_1	0.633881	0.645693	0.665265	0.692429	0.726952	0.768547	0.816897	0.871696	0.932727	1
w_1	0.781477	0.787865	0.798528	0.8135	0.832833	0.856616	0.884984	0.918133	0.956343	1
w_1	0.795044	0.800863	0.810606	0.824344	0.842181	0.864264	0.890789	0.922014	0.958274	1
w_1	0.799332	0.80497	0.814421	0.827767	0.845129	0.866673	0.892615	0.923233	0.95888	1
w_1	0.800653	0.806235	0.815595	0.82882	0.846036	0.867414	0.893176	0.923608	0.959066	1

Table 2; GAM for $N = 0.5$, $n = 1.1$, $\epsilon_1 = 0.5$, $\epsilon_2 = 0.5$

y	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
w_1	0.415591	0.430306	0.455217	0.490916	0.538247	0.598333	0.672598	0.762801	0.87108	1
w_1	0.563954	0.578328	0.602123	0.635093	0.676886	0.727048	0.785031	0.850229	0.922041	1
w_1	0.665224	0.676386	0.694797	0.720182	0.7522	0.790471	0.834626	0.884357	0.939478	1
w_1	0.724421	0.733153	0.747614	0.767676	0.793195	0.82403	0.860085	0.901337	0.947887	1
w_1	0.756648	0.763985	0.776192	0.793248	0.81514	0.841885	0.873548	0.910269	0.952292	1
w_1	0.773581	0.78018	0.791197	0.806666	0.826649	0.851243	0.880601	0.914947	0.9546	1
w_1	0.782336	0.788554	0.798958	0.813609	0.832606	0.85609	0.884256	0.917373	0.955797	1
w_1	0.786807	0.792832	0.802923	0.817158	0.835652	0.858569	0.886127	0.918615	0.956409	1

Table 3; GAM for $N = 0.5$, $n = 1.1$, $\epsilon_1 = 1$, $\epsilon_2 = 1$

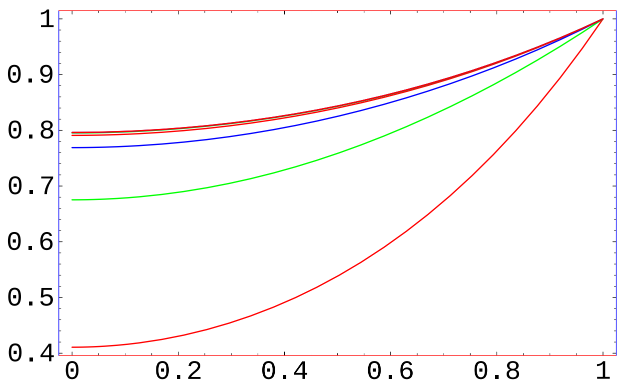

Fig.1, $[N = 0.5,\ n = 1.1,]$, results via GAM for $\epsilon_1 = 0.2, \epsilon_2 = 0.3$

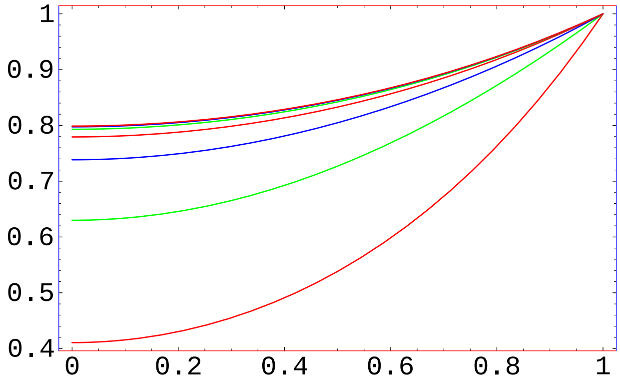

Fig.2, $[N = 0.5,\ n = 1.1]$, results via GAM for $\epsilon_1 = 0.5,\ \epsilon_2 = 0.5$

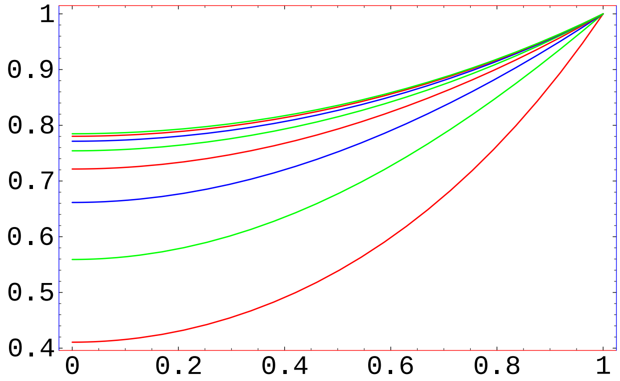

Fig.3, $[N = 0.5,\ n = 1.1]$, results via GAM for $\epsilon_1 = 1,\ \epsilon_2 = 1$

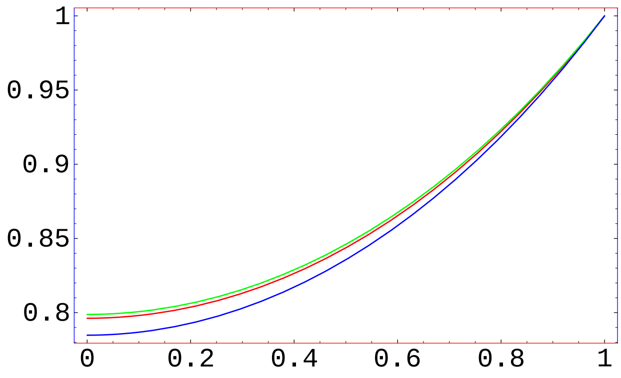

Fig.4, $N = 0.5,\ n = 1.1]$,GAM for $\epsilon_1 = 0.2, \epsilon_2 = 0.3$ (Red), $\epsilon_1 = 0.5,\ \epsilon_2 = 0.5$ (Green) and $\epsilon_1 = 1,\ \epsilon_2 = 1$ (Blue)

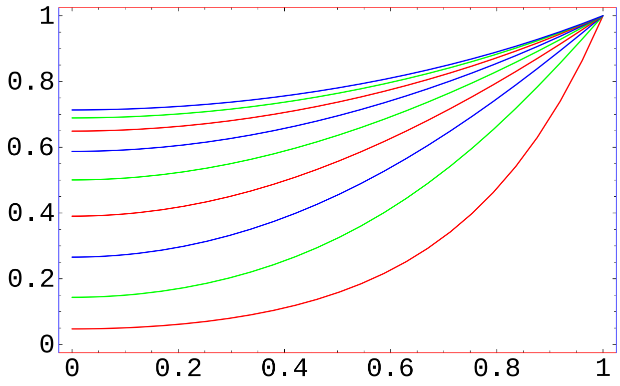

Fig.5, $[N = 1,\, n = 2]$, results via GAM for $\epsilon_1 = 0.2$, $\epsilon_2 = 0.3$

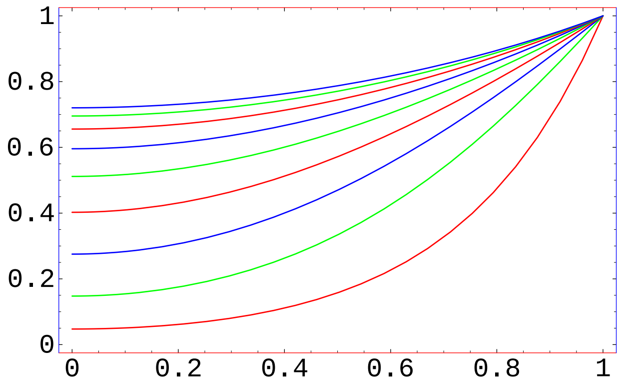

Fig.6, $[N = 1,\, n = 2]$, results via GAM for $\epsilon_1 = 0.5$, $\epsilon_2 = 0.5$

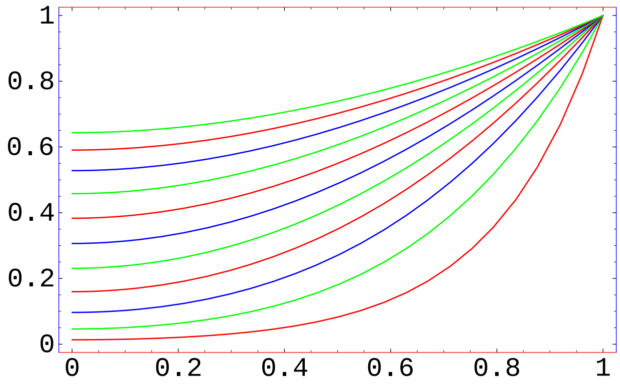

Fig.7, $[N = 1,\, n = 2]$, results via GAM for $\epsilon_1 = 1$, $\epsilon_2 = 1$

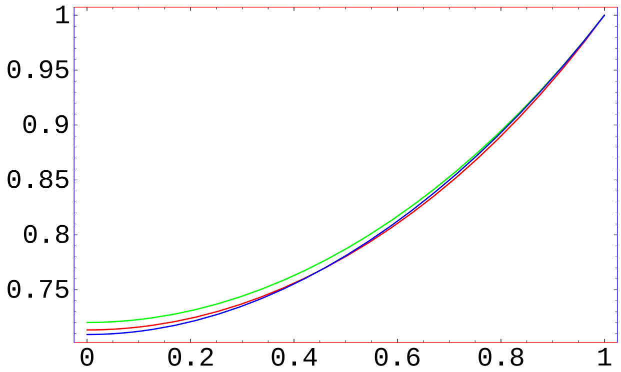

Fig.8, $N = 1$, $n = 2$],GAM for $\epsilon_1 = 0.2, \epsilon_2 = 0.3$ (Red), $\epsilon_1 = 0.5$, $\epsilon_2 = 0.5$ (Green) and $\epsilon_1 = 1$, $\epsilon_2 = 1$ (Blue)

5. Conclusion

In this paper, the GAM is developed for the heat flow problems. The GAM generates a bounded monotone sequence of solutions of linear problems that converges monotonically and rapidly to a solution of the original problem. It also ensure existence of solution with lower and upper solutions as estimates for the exact solution. It does not require the existence of small or large parameter as most of the perturbation type methods do. The results obtained via GAM are accurate for any value of the parameters involved. Hence it is a powerful tool for solutions of nonlinear problems.

References

[1] O. Abdulaziz, I. Hashim and S. Momani, Application of homotopy-perturbation method to fractional IVPs Journal of *Computational and Applied Mathematics*, **216** (2008), 574-584.

[2] B. Ahmad and J. J. Nieto, Existence and approximation of solutions for a class of nonlinear impulsive functional differential equations with anti-periodic boundary conditions, *Nonlinear Analysis: Theory, Methods and Applications*, doi:10.1016/j.na.2007.09.018

[3] A. Belendez, C. Pascual, A. Marquez and D.I. Mendez, Application of He's Homotopy Perturbation Method to the Relativistic (An)harmonic Oscillator. I: Comparison between Approximate and Exact Frequencies, *International Journal of Nonlinear Sciences and Numerical Simulation* **8**(2007), 483-492.

[4] Stephen R. Bernfeld and V. Lakshmikantham, An introduction to nonlinear boundar value problems, Academic Press, Inc. New Yark and Landa (1974).

[5] A. Belendez, C. Pascual, D.I. Mendez, M.L. Alvarez and C. Neipp , Application of He's Homotopy Perturbation Method to the Relativistic (An)harmonic Oscillator. II: A More Accurate Approximate Solution, *International Journal of Nonlinear Sciences and Numerical Simulation* **8** (2007), 493-504.

[6] A. Cabada and J. J. Nieto, Quasilinearization and rate of convergence for higher-order nonlinear periodic boundary-value problems, *Journal of Optimization Theory and Applications* **108** (2001), 97-107

[7] M. Van Dyke, Perturbation Methods in Fluid Mechanics, Annotated Edition, Parabolic Press, Standford, CA, (1975).

[8] P. W. Eloe and Yang Gao, The method of quasilinearization and a three-point boundary value problem, *J. Korean Math. Soc.,* **39** (2002), 319–330.

[9] P. W. Eloe, The quasilinearization method on an unbounded domain, *Proc. Amer. Math. Soc.,* **131**(2002)5 1481-1488.

[10] P. W. Eloe and Y. Zhang, A quadratic monotone iteration scheme for two point boundary value problems for ordinary differential equations, *Nonlinear Anal.,* **33**(1998) 443-453.

[11] D. D. Ganji and A. Rajabi, Assessment of homotopy perturbation method in heat radiation equations, *Int. Commun. heat and Mass transfer* **33**(2006),391–400. doi:10.1016/j.cnsns.2007.09.007.

[12] J. H. He, Homotopy perturbation technique, *Comput.Methods Appl. Mech. Eng.,* **178**(1999)(257), 3–4.

[13] J. H. He, A coupling method of a homotopy technique and a perturbation technique for non-linear problems, *Int. J. Nonlinear Mech.,* **35**(2000),37–43 .

[14] B. Keramati, An approach to the solution of linear system of equations by He's homotopy perturbation method, *Chaos, Solitons and Fractals*, In Press, doi:10.1016/j.chaos.2007.11.020.

[15] R. A. Khan, Generalized quasilinearization for periodic problems, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.* **14** (2007), 497-507.

[16] R. A. Khan, Generalized approximations method for heat radiation equations, *Appl. Math. Comput.*, Doi:10.1016/j.amc.2009.02.028.

[17] R. A. Khan, The Generalized approximations and nonlinear heat transfer equations, *Elec. J. Qualitative theory of Diff. Equations*, **2** (2009), 1–15.

[18] V. Lakshmikantham and A.S. Vatsala, Generalized quasilinearization for nonlinear problems, Kluwer Academic Publishers, Boston (1998).

[19] V. Lakshmikanthama and A.S. Vatsala, Generalized quasilinearization versus Newtons method, *Appl. Math. Comput.* **164** (2005), 523-530.

[20] V. Lakshmikantham, An extension of the method of quasilinearization, *J. Optim. Theory Appl.,* **82** (1994), 315–321.

[21] V. Lakshmikantham, Further improvement of generalized quasilinearization, *Nonlinear Anal.,* **27** (1996), 315–321.

[22] Mo. Miansari, D. D. Ganji and Me.Miansari, Application of He's variational iteration method to nonlinear heat transfer equations, *Physics Letter A,* **372**(2008), 779-785.

[23] S. Kim and C-H. Huang, A series solution of the non-linear fin problem with temperature-dependent thermal conductivity and heat transfer coefficient, *J. Phys. D: Appl. Phys.*, **40**(2007) 2979-2987.

[24] Xiaoyan Ma, Liping Wei and Zhongjin Guo, He's homotopy perturbation method to periodic solutions of nonlinear Jerk equations, *Journal of Sound and Vibration* **314** (2008), 217-227.

[25] A. H. Nayfeh, Perturbation Methods, Wiley, New York, (1973).

(Received April 21, 2009)