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# OSCILLATION CRITERIA FOR A CERTAIN SECOND-ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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Abstract. In this paper, by using the generalized Riccati technique and the integral averaging technique, some new oscillation criteria for certain second order retarded differential equation of the form

$$\left(r(t)\left|u'(t)\right|^{\alpha-1}u'(t)\right)' + p(t)\,f(u(\tau(t))) = 0$$

are established. The results obtained essentially improve known results in the literature and can be applied to the well known half-linear and Emden-Fowler type equations.

## 1. Introduction

This paper is concerned with the problem of oscillatory behavior of the retarded functional differential equation

$$\left(r(t)\left|u'(t)\right|^{\alpha-1}u'(t)\right)' + p(t)\,f(u(\tau(t))) = 0,\ t \geq t_0 \geq 0. \tag{1.1}$$

We suppose throughout the paper that the following conditions hold.

$(H_1)$ $\alpha$ is a positive number,

$(H_2)$ $r \in C([t_0,\ \infty)),\ r(t) > 0,\ R(t) = \int_{t_0}^{t} r^{-1/\alpha}(s)\,ds \to \infty$ as $t \to \infty$,

$(H_3)$ $p \in C([t_0,\ \infty)),\ p(t) > 0$,

$(H_4)$ $\tau \in C^1([t_0,\ \infty)),\ \tau(t) \leq t,\ \tau'(t) > 0,\ \tau(t) \to \infty$ as $t \to \infty$,

$(H_5)$ $f \in C((-\infty,\ \infty)),\ xf(x) > 0$ for $x \neq 0,\ f \in C^1(R_D)$, where $R_D = (-\infty,\ -D) \cup (D,\ \infty),\ D > 0$.

By a solution of (1.1), we mean a function $u \in C^1([T_u,\ \infty)),\ T_u \geq t_0$, which has the property $r(t)\left|u'(t)\right|^{\alpha-1}u'(t) \in C^1([T_u,\ \infty))$ and satisfies (1.1) on $[T_u,\ \infty)$. We consider only those solutions $u(t)$ of (1.1) which satisfy $\sup\{|u(t)| :\ t \geq T_u\} > 0$ for all $T_u \geq t_0$. We assume that (1.1) possesses such a solution. A nontrivial solution of (1.1) is said to be oscillatory if it has a sequence of zeros tending to infinity, otherwise it is called nonoscillatory. An equation is said to be oscillatory if all its solutions are oscillatory.

The oscillatory behavior of functional differential equations with deviating arguments has been the subject of intensive study in the last three decades, see for example the monographs Agarwal et al [2], Dosly and Rehak [9], Gyori and Ladas [13], and Ladde et al [20].

The study of oscillation of second order differential equations is of great interest. Many criteria have been found which involve the behavior of the integral of a

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combination of the coefficients. Recently Agarwal et al [4], Chern et al [8], Elbert [11], Kusano et al [16, 17, 18, 19], and Mirzov [23, 24] have observed some similar properties between the half-linear equation

$$\left(r(t)\left|u'(t)\right|^{\alpha-1}u'(t)\right)' + p(t)\left|u(\tau(t))\right|^{\alpha-1}u(\tau(t)) = 0 \tag{1.2}$$

and the corresponding linear equation

$$\left(r(t)\,u'(t)\right)' + p(t)\,u(\tau(t)) = 0. \tag{1.3}$$

Some of the above results are improved by Dzurina and Stavroulakis [10]. On the other hand Ladde et al [20] presented the following oscillatory criteria for Eq. (1.3)

$$\int^{\infty} R^{1-\epsilon}(\tau(t))\,p(t)\,dt = \infty,\ 0 < \epsilon < 1. \tag{1.4}$$

In this paper, we shall continue in this direction the study of oscillatory properties of (1.1). By using the generalized Riccati technique and the integral averaging technique, we shall establish some new oscillatory criteria. The first our purpose is to improve the above mentioned results. The second aim is to show that many others known criteria are included in the our obtained results. The third intention of paper, is to apply obtained results for investigation of oscillation of the generalized Emden-Fowler equation

$$\left(r(t)\left|u'(t)\right|^{\alpha-1}u'(t)\right)' + p(t)\left|u(\tau(t))\right|^{\beta-1}u(\tau(t)) = 0. \tag{1.5}$$

Note that, in this direction, although there is an extensive literature on the oscillatory behavior of Eq. (1.2) and (1.3), there is not much done for Eq. (1.5). We refer to the reader, see [1, 2, 3, 4, 26, 30] in delay case and [15, 29] in ordinary case.

## 2. Main Results

In this section we prove our main result.

**Theorem 1.** *Let there exist a constant $k > 0$ such that*

$$f'(x)/\left|f(x)\right|^{1-\frac{1}{\alpha}} \geq k \text{ for all } x \in R_D. \tag{2.1}$$

*If*

$$\lim_{t\to\infty}\int_{t_0}^{t}\left(\frac{1}{r(s)}\int_{s}^{\infty}p(z)\,dz\right)^{1/\alpha}ds = \infty \tag{2.2}$$

*and there exists a differentiable function $\rho : [t_0,\ \infty) \to (0,\infty)$ such that*

$$\rho'(s) \geq 0 \text{ and } \limsup_{t\to\infty}\int_{t_0}^{t}\left\{p(s)\,\rho^{\alpha}(s) - \mu\frac{r(\tau(s))\,\rho'^{\alpha+1}(s)}{\tau'^{\alpha}(s)\,\rho(s)}\right\}ds = \infty, \tag{2.3}$$

*where $\mu := \left(\dfrac{\alpha}{\alpha+1}\right)^{\alpha+1}\left(\frac{\alpha}{k}\right)^{\alpha}$, then Eq. (1.1) is oscillatory.*

*Proof.* Assume the theorem false. Let $u(t)$ be a nonoscillatory solution of (1.1). Without loss of generality we may assume that $u(t) > 0$. This implies that

$$\left(r(t)\left|u'(t)\right|^{\alpha-1}u'(t)\right)' = -p(t)\,f(u(\tau(t))) < 0.$$

Hence the function $r(t)\left|u'(t)\right|^{\alpha-1}u'(t)$ is decreasing and therefore there are two cases $u'(t) > 0$ and $u'(t) < 0$. The case $u'(t) < 0$, by the hypothesis $(H_2)$, is impossible and we see that $u'(t) > 0$ on $[t,\ \infty)$ for some $t_1 \geq t_0$. Define

$$w(t) := r(t)\frac{(\rho(t)\,u'(t))^{\alpha}}{f(u(\tau(t)))},\ t \in [t_1,\ \infty). \tag{2.4}$$

Then $w(t) > 0$. Differentiating $w(t)$ and using Eq. (1.1), since $r(t)\,u'^{\alpha}(t)$ is decreasing, we have Riccati type inequality

$$w'(t) \leq \frac{\alpha\rho'(t)}{\rho(t)}w(t) - \rho^{\alpha}(t)\,p(t) - \frac{(w(t))^{1+1/\alpha}\,\tau'(t)\,f'(u(\tau(t)))}{(r(\tau(t)))^{1/\alpha}\,\rho(t)\,f(u(\tau(t)))^{1-1/\alpha}}. \tag{2.5}$$

Let us assume that $u(t)$ is bounded. Then there exist some positive constants $c_1$ and $c_2$ such that for all $t \geq t_0$

$$c_2 \leq u(t) \leq c_1 \text{ and } c_2 \leq u(\tau(t)) \leq c_1.$$

Integrating Eq. (1.1) from $t$ to $\infty$, we obtain

$$r(t)\,u'^{\alpha}(t)\Big|_t^{\infty} = -\int_t^{\infty} p(s)\,f(u(\tau(s)))\,ds.$$

Since $r(t)\,u'^{\alpha}(t)$ is positive and decreasing, we have

$$r(t)\,u'^{\alpha}(t) \geq \int_t^{\infty} p(s)\,f(u(\tau(s)))\,ds.$$

Integrating this inequality again from $t_0$ to $t$, we have

$$u(t) \geq \int_{t_0}^{t}\left(\frac{1}{r(s)}\int_s^{\infty} p(z)\,f(u(z))\,dz\right)^{1/\alpha} ds.$$

Denote $f_0 = \min\limits_{u\in[c_1,\ c_2]} f(u)$. Then from this inequality

$$c_1 \geq u(t) \geq f_0^{1/\alpha}\int_{t_0}^{t}\left(\frac{1}{r(s)}\int_s^{\infty} p(z)\,dz\right)^{1/\alpha} ds. \tag{2.6}$$

Letting $t \to \infty$ the last inequality contradicts to (2.2). Therefore, we conclude $u(t) \to \infty$ as $t \to \infty$. Thus $u(\tau(t)) \in R_D$ for all $t$ large enough. Now it is easy to see that condition $f'(u(\tau(t)))\,/\,(f(u(\tau(t))))^{1-1/\alpha} \geq k$ implies

$$w'(t) \leq \frac{\alpha\rho'(t)}{\rho(t)}w(t) - \rho^{\alpha}(t)\,p(t) - k\frac{(w(t))^{1+1/\alpha}\,\tau'(t)}{(r(\tau(t)))^{1/\alpha}\,\rho(t)},\ t \geq t_1. \tag{2.7}$$

By using the inequality

$$Ax - Bx^{1+1/\alpha} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}A^{\alpha+1}B^{-\alpha},\ B > 0,\ A \geq 0,\ x \geq 0 \tag{2.8}$$

we get

$$w'(t) \leq -\left\{ p(t)\,\rho^{\alpha}(t) - \mu \frac{r(\tau(t))\,{\rho'}^{\alpha+1}(t)}{\rho(t)\,{\tau'}^{\alpha}(t)} \right\},\ t \geq t_1. \tag{2.9}$$

Integrating this inequality from $t_1$ to $t$, we get

$$w(t) \leq w(t_1) - \int_{t_1}^{t} \left\{ \rho^{\alpha}(s)\,p(s) - \mu \frac{r(\tau(s))\,{\rho'}^{\alpha+1}(s)}{\rho(s)\,{\tau'}^{\alpha}(s)} \right\} ds.$$

Letting $\limsup_{t\to\infty}$, we get in view of (2.3) that $w(t) \to -\infty$, which contradicts $w(t) > 0$ and the proof is complete. ∎

The conclusions of Theorem 1 leads to the following.

**Corollary 1.** *Let the condition (2.3) in Theorem 1 be replaced by*

$$\int_{t_0}^{\infty} \rho^{\alpha}(s)\,p(s)\,ds = \infty,\ \text{and}\ \int_{t_0}^{\infty} \frac{r(\tau(s))\,{\rho'}^{\alpha+1}(s)}{\rho(s)\,{\tau'}^{\alpha}(s)} ds < \infty, \tag{2.10}$$

*then the conclusion of Theorem 1 holds.*

Next we state the following result.

**Corollary 2.** *Assume that (2.1) and (2.2) are satisfied. If there exists a differentiable positive function $\rho$ such that*

$$\rho'(t) > 0 \ \text{for all}\ t \geq t_0, \tag{2.11}$$

$$\int_{t_0}^{\infty} \frac{r(\tau(s))\,{\rho'}^{\alpha+1}(s)}{\rho(s)\,{\tau'}^{\alpha}(s)} ds = \infty, \tag{2.12}$$

*and*

$$\liminf_{t\to\infty} \frac{\rho^{\alpha+1}(t)\,p(t)\,(\tau'(t))^{\alpha}}{r(\tau(t))\,(\rho'(t))^{\alpha+1}} > \mu, \tag{2.13}$$

*then Eq. (1.1) is oscillatory.*

*Proof.* It is enough to show that (2.12) and (2.13) together implies (2.3). From (2.13), it follows that there exist $\epsilon > 0$ such that for all large $t$

$$\frac{\rho^{\alpha+1}(t)\,p(t)\,(\tau'(t))^{\alpha}}{r(\tau(t))\,(\rho'(t))^{\alpha+1}} > \mu + \epsilon.$$

This means that

$$\rho^{\alpha}(t)\,p(t) - \mu \frac{r(\tau(t))\,(\rho'(t))^{\alpha+1}}{\rho(t)\,(\tau'(t))^{\alpha}} > \epsilon \frac{r(\tau(t))\,(\rho'(t))^{\alpha+1}}{\rho(t)\,(\tau'(t))^{\alpha}}.$$

Now, it is obvious that from (2.11), this inequality implies (2.3) and the assertion of this corollary follows from Theorem 1. ∎

If we choose $\rho(t) = R(\tau(t))$ in Theorem 1, Corollary 1 and Corollary 2, we have the following oscillation criteria.

**Theorem 2.** *Assume that (2.1) and (2.2) hold. If*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left\{R^{\alpha}\left(\tau\left(s\right)\right)p\left(s\right)-\mu\frac{\tau'\left(s\right)}{R\left(\tau\left(s\right)\right)r^{1/\alpha}\left(\tau\left(s\right)\right)}\right\}ds=\infty, \tag{2.14}$$

*then Eq. (1.1) is oscillatory.*

**Corollary 3.** *Assume that (2.1) and (2.2) hold. If there exists a positive constant $\epsilon$ such that*

$$\int_{t_0}^{\infty}R^{\alpha\epsilon}\left(\tau\left(s\right)\right)p\left(s\right)ds=\infty,\ 0<\epsilon<1, \tag{2.15}$$

*then Eq. (1.1) is oscillatory.*

**Corollary 4.** *Assume that (2.1) and (2.2) hold. If*

$$\liminf_{t\to\infty}\frac{R^{\alpha+1}\left(\tau\left(t\right)\right)r^{1/\alpha}\left(\tau\left(t\right)\right)p\left(t\right)}{\tau'\left(t\right)}>\mu, \tag{2.16}$$

*then Eq. (1.1) is oscillatory.*

If we take $\rho\left(t\right)=\tau\left(t\right)$, the conclusions of Theorem 1, Corollary 1 and Corollary 2 lead to the following oscillation criteria.

**Theorem 3.** *Assume that (2.1) and (2.2) hold. If*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left\{\tau^{\alpha}\left(s\right)p\left(s\right)-\mu\frac{r\left(\tau\left(s\right)\right)\tau'\left(s\right)}{\tau\left(s\right)}\right\}ds=\infty, \tag{2.17}$$

then Eq. (1.1) is oscillatory.

**Corollary 5.** *Assume that (2.1) and (2.2) hold. If there exists a positive constant $\epsilon$ such that*

$$\int_{t_0}^{\infty}\tau^{\alpha\epsilon}\left(s\right)p\left(s\right)ds=\infty,\ 0<\epsilon<1, \tag{2.18}$$

*then Eq. (1.1) with $r\left(t\right)\equiv 1$ is oscillatory.*

**Corollary 6.** *Assume that (2.1) and (2.2) hold. If*

$$\liminf_{t\to\infty}\frac{\tau^{\alpha+1}\left(t\right)p\left(t\right)}{\tau'\left(t\right)}>\mu, \tag{2.19}$$

*then Eq. (1.1) with $r\left(t\right)\equiv 1$ is oscillatory.*

From Theorem 2, we get the following oscillation criterion.

**Corollary 7.** *Assume that (2.1) and (2.2) hold. If*

$$\liminf_{t\to\infty}R^{\alpha}\left(\tau\left(t\right)\right)\int_{t}^{\infty}p\left(s\right)ds>\mu\frac{1}{\alpha}, \tag{2.20}$$

*then Eq. (1.1) is oscillatory.*

*Proof.* It is sufficient to prove that (2.20) implies (2.14). Suppose that (2.14) fails, that is for all $\epsilon > 0$ there exists a $t_1$ such that for all $t \geq t_1$

$$\int_t^\infty R^\alpha(\tau(s)) \left( p(s) - \mu \frac{\tau'(s)}{R^{\alpha+1}(\tau(s)) r^{1/\alpha}(\tau(s))} \right) ds < \epsilon.$$

Since $R^\alpha(\tau(t))$ is nondecreasing, this inequality implies

$$R^\alpha(\tau(t)) \int_t^\infty \left( p(s) - \mu \frac{\tau'(s)}{R^{\alpha+1}(\tau(s)) r^{1/\alpha}(\tau(s))} \right) ds < \epsilon.$$

Therefore

$$R^\alpha(\tau(t)) \int_t^\infty p(s)\, ds < \epsilon + \frac{\mu}{\alpha}$$

for all $\epsilon > 0$ which contradicts to (2.20). ∎

Similarly, we have the following result from Theorem 3.

**Corollary 8.** *Assume that (2.1) and (2.2) hold. If*

$$\liminf_{t\to\infty} \tau^\alpha(t) \int_t^\infty p(s)\, ds > \frac{\mu}{\alpha}, \tag{2.21}$$

*then Eq. (1.1) with $r(t) \equiv 1$ is oscillatory.*

**Remark 1.** *When $\alpha = 1$ and $r(t) \equiv 1$, Theorem 1 and Corollary 2 with $\rho(t) = t$, Theorem 3, Corollary 6 reduce to Theorem 2.1, Corollary 2.1, Theorem 2.2, and Corollary 2.5 in* [7], *respectively.*

## 3. Philos, Kamenev- type oscillation criteria for Equation (1.1)

In this section, by using the generalized Riccatti technique (2.4) and the integral averaging technique, similar to that Grace [12], Kamenev [14], Philos [25], Rogovchenko [27], Tiryaki [28], and Wong [31], we give new oscillation criteria for Eq. (1.1) thereby improving our main results.

For this purpose, we first define the sets

$$D_0 = \{(t,s) : t > s \geq t_0\} \text{ and } D = \{(t,s) : t \geq s \geq t_0\}.$$

We introduce a general class of parameter functions $H : D \to R$ which have continuous partial derivative on $D$ with respect to the second variable and satisfy

$H_1$ : $H(t, t) = 0$ for $t \geq t_0$ and $H(t,s) > 0$ for all $(t,s) \in D_0$,

$H_2$ : $-\dfrac{\partial H(t,s)}{\partial s} \geq 0$ for all $(t,s) \in D$.

Suppose that $h : D_0 \to R$ is a continuous function such that

$$\alpha \frac{\rho'(s)}{\rho(s)} H(t,s) - \frac{\partial H(t,s)}{\partial s} = h(t,s) (H(t,s))^{\alpha/\alpha+1} \text{ for all } (t,s) \in D_0.$$

Note that, by choosing specific functions $H$, it is possible to derive several oscillation criteria for a wide range of differential equations, see [5, 6, 21, 22, 27]. More general types of such functions have been constructed in [28, 31].

**Theorem 4.** *Assume that (2.1) and (2.2) hold. If there exist a positive differentiable function $\rho \in C^1([t_0,\ \infty),\ R^+)$ and $H$ such that $(H_1)$ and $(H_2)$ hold, and*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\left\{p(s)\rho^{\alpha}(s)-\mu\frac{r(\tau(s))\rho'^{\alpha+1}(s)}{\tau'^{\alpha}(s)\rho(s)}\right\}ds=\infty, \tag{3.1}$$

*then Eq. (1.1) is oscillatory.*

*Proof.* Using the function $w(t)$ defined in (2.4) and proceeding similarly as in the proof of Theorem 1, we have inequality (2.9).

$$w'(t)+\left\{p(t)\rho^{\alpha}(t)-\mu\frac{r(\tau(t))\rho'^{\alpha+1}(t)}{\rho(t)\tau'^{\alpha}(t)}\right\}\le 0,\ t\ge t_1.$$

Multiplying this inequality by $H(t,s)$, $t>s$, and next integrating from $t_1$ to $t$ after simple computation we have

$$\begin{aligned}\int_{t_1}^{t}H(t,s)\left\{p(s)\rho^{\alpha}(s)-\mu\frac{r(\tau(s))\rho'^{\alpha+1}(s)}{\rho(s)\tau'^{\alpha}(s)}\right\}ds &\le H(t,t_1)w(t_1)\\ &\le H(t,t_0)w(t_1).\end{aligned}$$

Therefore

$$\begin{aligned}&\int_{t_0}^{t}H(t,s)\left\{p(s)\rho^{\alpha}(s)-\mu\frac{r(\tau(s))\rho'^{\alpha+1}(s)}{\rho(s)\tau'^{\alpha}(s)}\right\}ds\\ =&\int_{t_0}^{t_1}H(t,s)\left\{p(s)\rho^{\alpha}(s)-\mu\frac{r(\tau(s))\rho'^{\alpha+1}(s)}{\rho(s)\tau'^{\alpha}(s)}\right\}ds\\ &+\int_{t_1}^{t}H(t,s)\left\{p(s)\rho^{\alpha}(s)-\mu\frac{r(\tau(s))\rho'^{\alpha+1}(s)}{\rho(s)\tau'^{\alpha}(s)}\right\}ds\\ \le& H(t,t_0)\int_{t_0}^{t_1}p(s)\rho^{\alpha}(s)\,ds+H(t,t_0)w(t_1)\ \text{ for all } t\ge t_0.\end{aligned}\tag{3.2}$$

This gives

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\left\{p(s)\rho^{\alpha}(s)-\mu\frac{r(\tau(s))\rho'^{\alpha+1}(s)}{\rho(s)\tau'^{\alpha}(s)}\right\}ds\\ \le&\int_{t_0}^{t_1}p(s)\rho^{\alpha}(s)\,ds+w(t_1)\end{aligned}$$

which contradicts (3.1). This completes the proof of the theorem. ∎

**Theorem 5.** *Assume that (2.1) and (2.2) hold. If there exist a positive differentiable function $\rho \in C^1([t_0,\ \infty),\ R^+)$ and $H$ such that $(H_1)$, $(H_2)$ and $(H_3)$ hold, and*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left\{H(t,s)p(s)\rho^{\alpha}(s)-\mu_1\frac{r(\tau(s))\rho^{\alpha}(s)}{\tau'^{\alpha}(s)}(h(t,s))^{\alpha+1}\right\}ds=\infty, \tag{3.3}$$

*where* $\mu_1 := \dfrac{1}{(\alpha+1)^{\alpha+1}}\left(\dfrac{\alpha}{k}\right)^{\alpha}$, *then Eq. (1.1) is oscillatory.*

*Proof.* Using the function $w(t)$ defined in (2.4) and proceeding similarly as in the proof of Theorem 1, we obtain inequality (2.7).

$$w'(t) \le \alpha\frac{\rho'(t)}{\rho(t)}w(t) - \rho^{\alpha}(t)\,p(t) - k\frac{(w(t))^{1+1/\alpha}\tau'(t)}{r(\tau(t))^{1/\alpha}\rho(t)},\ t \ge t_1. \tag{2.7}$$

Multiplying this inequality by $H$ and next integrating from $t_1$ to $t$ we have

$$\int_{t_1}^{t} H(t,s)\,\rho^{\alpha}(s)\,p(s)\,ds \le H(t,t_1)\,w(t_1)$$

$$-\int_{t_1}^{t}\left[\left(\frac{\partial H(t,s)}{\partial s} - \alpha\frac{\rho'(s)}{\rho(s)}H(t,s)\right)w(s)\right.$$

$$\left.+H(t,s)\,\frac{\tau'(s)\,k}{r(\tau(s))^{1/\alpha}\rho(s)}(w(s))^{1+1/\alpha}\right]ds.$$

By using the inequality (2.8), we get

$$\int_{t_1}^{t} H(t,s)\,\rho^{\alpha}(s)\,p(s)\,ds \le H(t,t_1)\,w(t_1) - \mu_1\int_{t_1}^{t}\rho^{\alpha}(s)\,\frac{r(\tau(s))}{(\tau'(s))^{\alpha}}(h(t,s))^{\alpha+1}ds$$

Next proceeding similarly as in the inequality (3.2) we obtain

$$\int_{t_0}^{t}\left\{H(t,s)\,\rho^{\alpha}(s)\,p(s) - \mu_1\rho^{\alpha}(s)\,\frac{r(\tau(s))}{(\tau'(s))^{\alpha}}(h(t,s))^{\alpha+1}\right\}ds$$

$$\le\ H(t,t_0)\left(\int_{t_0}^{t_1}\rho^{\alpha}(s)\,p(s)\,ds + w(t_1)\right)$$

which contradicts (3.3). Thus, the proof is complete. ∎

**Corollary 9.** *The conclusion of Theorem 5 remains valid , if assumption (3.3) is replaced by*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)\,p(s)\,\rho^{\alpha}(s)\,ds = \infty \tag{3.4}$$

*and*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{r(\tau(s))\,\rho^{\alpha}(s)}{(\tau'(s))^{\alpha}}(h(t,s))^{\alpha+1}ds < \infty. \tag{3.5}$$

Note that, by choosing specific functions $\rho$ and $H$, it is possible to derive several oscillation criteria for Eq. (1.1) and its special cases Half-linear Eq. (1.2) and generalized Emden-Fowler Eq. (1.5) with $\beta \ge \alpha$.

In particular, by choosing $\alpha = 1$, $r(t) \equiv 1$, $H(t,s) = (t-s)^n$, $\rho(t) = t$ in Theorem 4 and by choosing $\alpha = 1$, $r(t) \equiv 1$, $\rho(t) = t$ and also $\rho(t) = \tau(t)$ in Theorem 5 we obtain the corresponding results given in [7], respectively.

**Remark 2.** *If we take* $f(x) = |x|^{\alpha-1} x$, *then* $\dfrac{f'(x)}{|f(x)|^{1-1/\alpha}} = k = \alpha$ *is satisfied for all* $x \neq 0 \in R$. *Therefore the condition (2.2) is not required. That is, all above oscillation criteria are valid for the half-linear Eq. (1.2) without using the condition (2.2). Hence, Theorem 2, Corollary 3, Theorem 3, and Corollary 5, improve Theorem 1, Corollary 1, Theorem 3, and Corollary 2 in* [10], *respectively. Also Theorem 2 improves Theorem 1 in* [8] *and Theorem 2.3 in* [4]. *Finally, Theorem 1 gives Corollary 1 in* [3] *by changing* $\rho(t)$ *with* $\rho^{1/\alpha}(t)$ *and Corollary 8 reduces Theorem 3 in* [17].

**Remark 3.** *Let* $\beta \geq \alpha$. *If we take* $f(x) = |x|^{\beta-1} x$, *then* $\dfrac{f'(x)}{|f(x)|^{1-1/\alpha}} = \beta |x|^{\beta/\alpha - 1}$ *and we can find always a positive constant k large enough such that* $f'(x) / |f(x)|^{1-1/\alpha} \geq k$ *for all* $x \in R_D$. *Hence, all above oscillation criteria are valid for the generalized Emden-Fowler Eq. (1.5). We note that these conclusions, for Eq. (1.5) with* $\beta \geq \alpha$, *complement Theorem 3.1 in* [4] *and they do not appear to follow from the known oscillation criteria in the literature.*

**Example 1.** *Consider the generalized Emden-Fowler delay differential equation*

$$\left(|x'(t)|^{\alpha-1} x'(t)\right)' + p(t) \left|x\left(\frac{t}{2}\right)\right|^{\beta-1} x\left(\frac{t}{2}\right) = 0, \tag{3.6}$$

*where* $\alpha$, $\beta$ *are positive constants such that* $\beta \geq \alpha$ *and* $p \in C([1, \infty), R^{+})$. $f(u) = |u|^{\beta-1} u$ *and there exists a* $k > 0$ *such that* $f'(u) \geq k$ *for all* $u \in R_D$, *k large enough. Hence (2.1) holds.*

*Here we choose* $\rho(t) = t^{\lambda/\alpha}$ *and* $H(t,s) = (t-s)^{\lambda}$, *for* $t \geq s \geq 1$ *such that* $\alpha + 1 < \lambda < \alpha^2$ *and* $\rho(t) p(t) \geq \dfrac{c}{t}$ *where c is a positive constant.*

$$\begin{aligned} \lim_{t\to\infty} \int_1^t \left(\int_s^\infty p(z)\,dz\right)^{1/\alpha} ds &\geq \lim_{t\to\infty} \int_1^t \left(\int_s^\infty c z^{-1-\lambda/\alpha} dz\right)^{1/\alpha} ds \\ &= \lim_{t\to\infty} \frac{\alpha^2}{\alpha^2 - \lambda} \left(\frac{c\alpha}{\lambda}\right)^{1/\alpha} \left(t^{1-\frac{\lambda}{\alpha^2}} - 1\right) = \infty. \end{aligned}$$

*Using the inequality*

$$(t-s)^{\lambda} \geq t^{\lambda} - \lambda s t^{\lambda-1}, \text{ for } t \geq s \geq 1$$

*given in* [21], *we have*

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda}} \int_1^t (t-s)^{\lambda} \rho(s) p(s)\,ds \geq \limsup_{t\to\infty} \frac{c}{t^{\lambda}} \int_1^t \frac{t^{\lambda} - \lambda s t^{\lambda-1}}{s} ds = \infty.$$

*On the other hand, observing that* $H(t,s) = \lambda (t-s)^{\frac{\lambda-\alpha-1}{\alpha+1}} t s^{-1}$, *we obtain*

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{t^{\lambda}} \int_1^t \frac{s^{\lambda}}{2^{-\alpha}} \left[\lambda (t-s)^{\frac{\lambda-\alpha-1}{\alpha+1}} t s^{-1}\right]^{\alpha+1} ds \\ \leq\ &\limsup_{t\to\infty} \frac{2^{\alpha} t^{\lambda+1}}{\lambda - \alpha} \left(1 - \frac{t_0}{t}\right)^{\lambda-\alpha-1} \left(t^{\lambda-\alpha} - t_0^{\lambda-\alpha}\right) < \infty. \end{aligned}$$

*Consequently, all condition of Corollary 9 are satisfied and hence Eq. (3.6) is oscillatory.*

Note that criteria reported in the references do not apply to Eq. (3.6).

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