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On a class of functional differential equations in Banach spaces

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Abstract

The aim of this paper is to establish the existence of solutions and some properties of set solutions for a Cauchy problem with causal operator in a separable Banach space.

Keywords: Causal operators; differential equations in Banach space.

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1 Introduction

The study of functional equations with causal operators has a rapid development in the last years and some results are assembled in a recent monograph [4]. The term of causal operators is adopted from engineering literature and the theory of these operators has the powerful quality of unifying ordinary differential equations, integrodifferential equations, differential equations with finite or infinite delay, Volterra integral equations, and neutral functional equations, to name a few (see, [1], [3], [10], [11], [17], [19], [21], [22]).

Let $\mathcal{S}$ be the class of all infinite - dimensional nonlinear M- input u, M- output y systems (ρ, f, g, Q) given by the following controlled nonlinear functional equation

$$\begin{cases} y'(t) = f(p(t), (\widehat{Q}y)(t)) + g(p(t), (\widehat{Q}y)(t), u(t)), \\ y|_{[-\sigma,0]} = y^0 \in C([-\sigma, 0], E) \end{cases} \tag{1.1}$$

where $\sigma \geq 0$ quantifies the memory of the system, p is a perturbation term, $\widehat{Q}$ is a nonlinear causal operator, and E is a real Banach space. The aim of the control objective is the development of a adaptive servomechanism which ensures practical tracking, by the system output, of an arbitrary reference signal assumed to be in the class $\mathcal{R}$ of all locally absolutely continuous and bounded with essentially bounded derivative. In fact, the control objective is to determine an $(\mathcal{R}, \mathcal{S})-$ servomechanism, that is, to determine the continuous functions $\Phi : E \to E$ and $\psi_\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ (parametrized by $\lambda > 0$) such that, for every system of class $\mathcal{S}$ and every reference signal $r \in \mathcal{R}$, the control

$$u(t) = -k(t)\Phi(y(t) - r(t)),\ k(t) = \psi_\lambda(||y(t) - r(t)||),\ k|_{[-\sigma,0]} = k^0 \tag{1.2}$$

applied to (1.1) ensures convergence of controller gain,and tracking of $r(\cdot)$ with asymptotic accuracy quantified by $\lambda > 0$, in the sense that $\max\{||y(t) - r(t)|| - \lambda, 0\} \to 0$ as $t \to \infty$. For more details see the papers [12], [20].

Using (1.2), we can write (1.1) as

$$x'(t) = F(t, (Qx)(t)), \;\; x|_{[-\sigma,0]} = x^0 \in C([-\sigma, 0], E \times \mathbb{R}) \tag{1.3}$$

where $x(t) := (y(t), k(t))$, $x^0 = (y^0, k^0)$, and Q is an operator defined on $C([-\sigma, 0], E \times \mathbb{R})$ by

$$(Qx)(t) = (Q(y, k))(t) := ((\widehat{Q}y)(t), y(t), k(t)).$$

The purpose of this article is to study the topological properties of the initial value problem (1.3) in a Banach space. For this we will use ideas from papers [8], [9]. Also, we give an existence result for this problem, assuming only the continuity of the operator Q. In the paper [12] is also obtained an existence result assuming that the operator Q is a locally Lipschitz operator.

2 Preliminaries

Let E be a real separable Banach space with norm $||\cdot||$. For $x \in E$ and $r > 0$ let $B_r(x) := \{y \in E; ||y - x|| < r\}$ be the open ball centered at x with radius r, and let $B_r[x]$ be its closure. If $\sigma > 0$, we denote by $\mathcal{C}([-\sigma, b), E)$ the Banach space of continuous bounded functions from $[-\sigma, b)$ into E and we denote by $\mathcal{C}_\sigma$ the space $\mathcal{C}([-\sigma, 0], E)$ with the norm $||\varphi||_\sigma = \sup\limits_{-\sigma \leq s \leq 0} ||\varphi(s)||$. By $L^p_{loc}([0, b), E)$, $1 \leq p \leq \infty$, we denote the space of all functions which are L^p-Bochner integrable on each compact interval of $[0, b)$.

By $\alpha(A)$, we denote the Hausdorff measure of non-compactness of nonempty bounded set $A \subset E$, defined as follows([2], [14]):

$$\alpha(A) = \inf\{\varepsilon > 0; A \text{ admits a finite cover by balls of radius} \leq \varepsilon\}.$$

This is equivalent to the measure of non-compactness introduced by Kuratowski (see [2]).

If $\dim(A) = \sup\{||x - y||; x, y \in A\}$ is the diameter of the bounded set A, then we have that $\alpha(A) \leq \dim(A)$ and $\alpha(A) \leq 2d$ if $\sup_{x\in A} ||x|| \leq d$. We recall the some properties for α (see [14]).

If A, B are bounded subsets of E and $\overline{A}$ denotes the closure of A, then

(i) $\alpha(A) = 0$ if and only if $\overline{A}$ is compact;

(ii) $\alpha(A) = \alpha(\overline{A}) = \alpha(\overline{co}(A))$;

(iii) $\alpha(\lambda A) = |\lambda|\alpha(A)$ for every $\lambda \in \mathbb{R}$;

(iv) $\alpha(A) \leq \alpha(B)$ if $A \subset B$;

(v) $\alpha(A + B) = \alpha(A) + \alpha(B)$.

We recall the following lemma due to Mönch([16], Proposition 1.6).

Lemma 2.1. *Let* $\{u_m(\cdot)\}_{m\geq 1}$ *be a bounded sequence of continuous functions from* $[0,T]$ *into* E. *Then,* $\beta(t)=\alpha(\{u_m(t); m\geq 1\})$ *is measurable and*

$$\alpha\left(\left\{\int_0^T u_m(t)dt; m\geq 1\right\}\right)\leq \int_0^T \beta(t)dt. \quad \square$$

Definition 2.1. Let $\sigma\geq 0$. An operator $Q:\mathcal{C}([-\sigma,b),E)\to L^{\infty}_{loc}([0,b),E)$ is a causal operator if, for each $\tau\in[0,b)$ and for all $u(\cdot),v(\cdot)\in\mathcal{C}([-\sigma,b),E)$, with $u(t)=v(t)$ for every $t\in[-\sigma,\tau]$, we have $(Qu)(t)=(Qv)(t)$ for a.e. $t\in[0,\tau]$.

Two significant examples of causal operators are: the Niemytzki operator

$$(Qu)(t)=f(t,u(t))$$

and the Volterra-Hammerstein integral operator

$$(Qu)(t)=g(t)+\int_0^t k(t,s)f(s,u(s))ds.$$

For $i=0,1,...,p$, we consider the functions $F_i:\mathbb{R}\times E\to E$, $(t,u)\to F(t,u)$, that are measurable in t and continuous in u. Set $\sigma:=\max_{i=\overline{1,p}}\sigma_i$, where $\sigma_i\geq 0$, and let

$$(Qu)(t)=\int_{-\sigma}^0 F_0(s,u(t+s))ds+\sum_{i=1}^p F_i(t,u(t-\sigma_i)),\ t\geq 0.$$

Then, the operator Q, so defined, is a causal operator (for details, see [12]). For other concrete examples which serve to illustrate that the class of causal operators is very large, we refer to the monograph [4].

We consider the initial-valued problem with causal operator

$$u'(t)=F(t,u(t),(Qu)(t)),\ \ u|_{[-\sigma,0]}=\varphi\in\mathcal{C}_\sigma, \tag{2.1}$$

under the following assumptions:

(h_1) Q is continuous;

(h_2) for each $r>0$ and each $\tau\in(0,b)$, there exists $M>0$ such that, for all $u(\cdot)\in\mathcal{C}([-\sigma,b),E)$ with $\sup\limits_{-\sigma\leq t\leq\tau}||u(t)||<r$, we have $||(Qu)(t)||\leq M$ for a.e. $t\in[0,\tau]$;

(h_3) $F:[-\sigma,b)\times E\times E\to E$ is a Carathéodory function, that is:

(a) for a.e. $t\in[-\sigma,b)$, $F(t,\cdot,\cdot)$ is continuous,

(b) for each fixed $(u,v)\in E\times E$, $F(\cdot,u,v)$ is measurable,

(c) for every bounded $B \subset E \times E$, there exists $\mu(\cdot) \in L^1_{loc}([0,b),\mathbb{R}_+)$ such that

$$||F(t,u,v)|| \leq \mu(t) \text{ for a.e. } t \in [-\sigma,b) \text{ and all } (u,v) \in B. \tag{2.2}$$

(h_4) for each bounded set $A \subset \mathcal{C}([-\sigma,b),E)$, there exists $k_0 > 0$ such that

$$\alpha((QA)(t)) \leq k_0 \int_0^t \alpha(A(s))ds \text{ for every } t \in [0,b), \tag{2.3}$$

where $A(t) = \{u(t); u \in A\}$ and $(QA)(t) = \{(Qu)(t); u \in A\}$.

(h_5) there exist c_1, $c_2 > 0$ such that

$$\alpha(F(t,B_1,B_2)) \leq c_1\alpha(B_1) + c_2\alpha(B_2) \tag{2.4}$$

for every $t \in [0,b)$ and for every bounded sets $B_1, B_2 \subset E$.

By solution of (2.1) we mean a continuous function $u(\cdot) : [-\sigma,b) \to E$ such that $u|_{[-\sigma,0]} = \varphi$, $u(\cdot)$ is local absolutely continuous on $[0,b)$ and $u'(t) = F(t,u(t),(Qu)(t))$ for a.e. $t \in [0,b)$.

We remark that $u(\cdot) \in \mathcal{C}([-\sigma,T],E)$, $T > 0$, is a solution for (2.1) on $[-\sigma,T]$, if and only if, $u|_{[-\sigma,0]} = \varphi$ and

$$u(t) = \begin{cases} \varphi(t), & \text{for } t \in [-\sigma,0] \\ \varphi(0) + \int_0^t F(s,u(s),(Qu)(s))ds, & \text{for } t \in [0,T]. \end{cases} \tag{2.5}$$

The existence of solutions for this kind of Cauchy problem has been studied in [12] for the case when $Q : \mathcal{C}([-\sigma,b),\mathbb{R}^n) \to L^\infty_{loc}([0,b),\mathbb{R}^n)$ is a locally Lipschitz operator. This problem has been studied in [6] for a Lipschitz causal operator $Q : \mathcal{C}([0,b),E) \to \mathcal{C}([0,b),E)$, where E is a real Banach space.

The existence of solutions for this kind of Cauchy problem has been studied by [6], in the case in that $Q : \mathcal{C}([0,b),E) \to \mathcal{C}([0,b),E)$. Also, for other results see [5], [7], [13], [15], [18].

The aim of this paper is to establish the existence of solutions and some properties of set solutions for Cauchy problem (2.1). To prove the properties of set solutions, we use the same method as in [9] and [8], accordingly adapted.

3 Existence of solutions

In the first half of this section, we present an existence result of the solutions for Cauchy problem (2.1), under conditions (h_1)-(h_5).

Theorem 3.1. *Let $Q : \mathcal{C}([-\sigma,b),E) \to L^\infty_{loc}([0,b),E)$ be a causal operator such that the conditions (h_1) - (h_5) hold. Then, for every $\varphi \in \mathcal{C}_\sigma$, there exists a solution $u(\cdot) : [-\sigma,T] \to E$ for Cauchy problem (2.1) on some interval $[-\sigma,T]$ with $T \in (0,b)$.*

Proof. Let $\delta > 0$ be any number and let $r := ||\varphi||_\sigma + \delta$. Also, let $\tau \in (0, b)$. If $u^0(\cdot) \in \mathcal{C}([-\sigma, b), E)$ denotes the function defined by

$$u^0(t) = \begin{cases} \varphi(t), & \text{for } t \in [-\sigma, 0) \\ \varphi(0), & \text{for } t \in [0, b), \end{cases}$$

then $\sup_{0 \le t \le \tau} ||u^0(t)|| < r$ and therefore, by (h_2), we have $||(Qu^0)(t)|| \le M$ for a.e. $t \in [0, \tau]$. On the other hand, since F is a Carathéodory function, there exists $\mu(\cdot) \in L^1([0, \tau], \mathbb{R}_+)$ such that

$$||F(t, u, v)|| \le \mu(t) \text{ for a.e. } t \in [0, \tau] \text{ and } (u, v) \in B_r(0) \times B_M(0).$$

We choose $T \in (0, \tau]$ such that $\int_0^T \mu(t)dt < \delta$ and we consider the set B defined as follows

$$B = \{u \in \mathcal{C}([-\sigma, T], E); u|_{[-\sigma,0]} = \varphi, \sup_{0 \le t \le T} ||u(t) - u^0(t)|| \le \delta\}.$$

Further on, we consider the integral operator $P : B \to \mathcal{C}([-\sigma, T], E)$ given by

$$(Pu)(t) = \begin{cases} \varphi(t), & \text{for } t \in [-\sigma, 0) \\ \varphi(0) + \displaystyle\int_0^t F(s, u(s), (Qu)(s))ds, & \text{for } t \in [0, T], \end{cases}$$

and we prove that this is a continuous operator from B into B.

First, we observe that $u(\cdot) \in B$, then $\sup_{0 \le t \le T} ||u(t)|| < r$, and so $||(Qu^0)(t)|| \le M$ for a.e. $t \in [0, \tau]$. Hence, for each $u(\cdot) \in B$, we have

$$\begin{aligned} \sup_{0 \le t \le T} ||(Pu)(t) - u^0(t)|| &= \sup_{0 \le t \le T} ||\int_0^t F(s, u(s), (Qu)(s))ds|| \\ &\le \int_0^T ||F(s, u(s), (Qu)(s))||ds \\ &\le \int_0^T \mu(t)dt < \delta \end{aligned}$$

and thus, $P(B) \subset B$.

Further on, let $u_m \to u$ in B. We have

$$\sup_{0\le t\le T} ||(Pu_m)(t) - (Pu)(t)|| =$$

$$= \sup_{0\le t\le T} || \int_0^t [F(s, u_m(s), (Qu_m)(s)) - F(s, u(s), (Qu)(s))]ds||$$

$$\le \sup_{0\le t\le T} \int_0^t ||F(s, u_m(s), (Qu_m)(s)) - F(s, u(s), (Qu)(s))||ds$$

$$\le \int_0^T ||F(s, u_m(s), (Qu_m)(s)) - F(s, u(s), (Qu)(s))||ds$$

$$\le T ess \sup_{0\le t\le T} ||F(s, u_m(s), (Qu_m)(s)) - F(s, u(s), (Qu)(s))||.$$

By (h_1) and (h_3) it follows that $\sup_{0\le t\le T} ||(Pu_m)(t)-(Pu)(t)|| \to 0$ as $m \to \infty$. Since $u_m|_{[-\sigma,0]} = \varphi$ for every $m \in \mathbb{N}$, we deduce that $P : B \to B$ is a continuous operator.

Moreover, it follows that $P(B)$ is uniformly bounded. Next, we show that $P(B)$ is uniformly equicontinuous on $[-\sigma, T]$. Let $\varepsilon > 0$. On the closed set $[0, T]$, the function $t \to \int_0^t \mu(s)ds$ is uniformly continuous, and so there exists $\eta' > 0$ such that

$$\left| \int_s^t \mu(\tau)d\tau \right| \le \varepsilon/2, \text{ for every } t, s \in [0, T] \text{ with } |t - s| < \eta'.$$

On the other hand, since $\varphi \in \mathcal{C}_\sigma$ is a continuous function on $[-\sigma, 0]$, then there exists $\eta'' > 0$ such that

$$||\varphi(t) - \varphi(s)|| \le \varepsilon/2, \text{ for every } t, s \in [0, T] \text{ with } |t - s| < \eta''.$$

Let $t, s \in [-\sigma, T]$ are such that $|t - s| \le \eta$, where $\eta = \min\{\eta', \eta''\}$. If $-\sigma \le s \le t \le 0$ then, for each $u(\cdot) \in B$, we have $||(Pu)(t) - (Pu)(s)|| = 0$. Next, if $-\sigma \le s \le 0 \le t \le T$ then, for each $u(\cdot) \in B$, we have

$$||(Pu)(t) - (Pu)(s)|| = ||\varphi(0) + \int_0^t F(\tau, u(\tau), (Qu)(\tau)d\tau - \varphi(s)||$$

$$\le ||\varphi(0) - \varphi(s)|| + | \int_0^t ||F(\tau, u(\tau), (Qu)(\tau)||d\tau \le ||\varphi(0) - \varphi(s)||+$$

$$\left| \int_0^t \mu(\tau)d\tau \right| \le \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} \le \varepsilon.$$

Finally, if $0 \le s \le t \le T$ then, for each $u(\cdot) \in B$, we have

$$||(Pu)(t) - (Pu)(s)|| =$$

$$= ||(\varphi(0) + \int_0^t F(\tau, u(\tau), (Qu)(\tau)d\tau) - (\varphi(0) + \int_0^s F(\tau, u(\tau), (Qu)(\tau)d\tau)||$$

$$\le || \int_0^t F(\tau, u(\tau), (Qu)(\tau)d\tau - \int_0^s F(\tau, u(\tau), (Qu)(\tau)d\tau||$$

$$\le \left| \int_s^t ||F(\tau, u(\tau), (Qu)(\tau)d\tau||d\tau \right| \le \left| \int_s^t \mu(\tau)d\tau \right| \le \varepsilon.$$

Therefore, we conclude that $P(B)$ is uniformly equicontinuous on $[-\sigma, T]$.

Further on, for each $m \ge 1$, we consider the following classical approximations

$$u_m(t) = \begin{cases} u^0(t), & \text{for } -\sigma \le t \le T/m \\ \varphi(0) + \int_0^{t-T/m} F(s, u_m(s), (Qu_m)(s))ds, & \text{for } T/m \le t \le T. \end{cases}$$

Then, for all $m \ge 1$ we have $u_m(\cdot) \in B$. Moreover, for $0 \le t \le T/m$, we have

$$||(Pu_m)(t) - u_m(t)|| \le \int_0^{T/m} ||F(s, u_m(s), (Qu_m)(s))||ds \le \int_0^{T/m} \mu(s)ds.$$

and for $T/m \le t \le T$, we have

$$||(Pu_m)(t) - u_m(t)|| = ||(Pu_m)(t) - (Pu_m)(t - T/m)|| =$$

$$|| \int_0^t F(s, u_m(s), (Qu_m)(s))ds - \int_0^{t-T/m} F(s, u_m(s), (Qu_m)(s))ds||$$

$$= || \int_{t-T/m}^t F(s, u_m(s), (Qu_m)(s))ds|| \le \int_{t-T/m}^t ||F(s, u_m(s), (Qu_m)(s))||ds$$

$$\le \int_{t-T/m}^t \mu(s)ds.$$

Therefore, it follows that

$$\sup_{0 \le t \le T} ||(Pu_m)(t) - u_m(t)|| \to 0 \text{ as } m \to \infty. \tag{3.1}$$

Let $A = \{u_m(\cdot); m \ge 1\}$. Denote by I the identity mapping on B. From 3.1 it follows that $(I - P)(A)$ is a uniformly equicontinuous subset of B. Since $A \subset (I - P)(A) + P(A)$ and the set $P(A)$ is uniformly equicontinuous, then we infer that the set A is also uniformly equicontinuous on $[-\sigma, T]$. Set $A(t) =$

$\{u_m(t); m \geq 1\}$ for $t \in [0, T]$. Then, by (2.5) and property (v) of the measure of non-compactness we have

$$\alpha(A(t)) \leq \alpha\left(\int_0^t F(s, A(s), (QA)(s))ds\right) + \alpha\left(\int_{t-T/m}^t F(s, A(s), (QA)(s))ds\right).$$

Note that, given $\varepsilon > 0$, we can find $m(\varepsilon) > 0$ such that $\int_{t-T/m}^t \mu(s)ds < \varepsilon/2$ for $t \in [0, T]$ and $m \geq m(\varepsilon)$. Hence we have that

$$\alpha\left(\int_{t-T/m}^t F(s, A(s), (QA)(s))ds\right)$$

$$= \alpha\left(\left\{\int_{t-T/m}^t F(s, u_m(s), (Qu_m)(s))ds; m \geq m(\varepsilon)\right\}\right)$$

$$\leq 2 \sup_{m \geq m(\varepsilon)} \int_{t-T/m}^t \mu(s)ds < \varepsilon.$$

Using the last inequality, we obtain that

$$\alpha(A(t)) \leq \alpha\left(\int_0^t F(s, A(s), (QA)(s))ds\right)$$

Since for every $t \in [0, T]$, $A(t)$ is bounded then, by Lemma 2.1, (h_4) and (h_5), we have that

$$\begin{aligned}
\alpha(A(t)) &\leq \int_0^t \alpha\left(F(s, A(s), (QA)(s))\right) ds \\
&\leq \int_0^t [c_1\alpha(A(s)) + c_2\alpha((QA)(s))]ds \\
&\leq \int_0^t [c_1\alpha(A(s)) + c_2 k_0 \int_0^s \alpha(A(\tau))d\tau]ds \\
&\leq \int_0^t c_1\alpha(A(s))ds + c_2 k_0 \int_0^t ds \int_0^s \alpha(A(\tau))d\tau \\
&= \int_0^t c_1\alpha(A(s))ds + c_2 k_0 \int_0^t (t-\tau)\alpha(A(\tau))d\tau,
\end{aligned}$$

for every $t \in [0, T]$. Therefore,

$$\alpha(A(t)) \leq K \int\limits_0^t \alpha(A(s))ds,$$

for every $t \in [0, T]$, where $K := c_1 + c_2 k_0 T$.

Then, by Gronwall's lemma, we must have that $\alpha(A(t)) = 0$ for every $t \in [0, T]$. Moreover, since (see [14], Theorem 1.4.2) $\alpha(A) = \sup\limits_{0 \leq t \leq T} \alpha(A(t))$ and $A|_{[-\sigma,0]} = \{\varphi\}$ we deduce that $\alpha(A) = 0$. Therefore, A is relatively compact subset of $\mathcal{C}([-\sigma, T], E)$. Then, by Arzela-Ascoli theorem (see [14], Theorem 1.1.5), and extracting a subsequence if necessary, we may assume that the sequence $\{u_m(\cdot)\}_{m \geq 1}$ converges uniformly on $[0, T]$ to a continuous function $u(\cdot) \in B$. Therefore, since

$$\sup_{0 \leq t \leq T} ||(Pu)(t) - u(t)|| \leq \sup_{0 \leq t \leq T} ||(Pu)(t) - (Pu_m)(t)||$$

$$+ \sup_{0 \leq t \leq T} ||(Pu_m)(t) - u_m(t)|| + \sup_{0 \leq t \leq T} ||u_n(t) - u(t)||$$

then, by (3.1) and by the fact that P is a continuous operator, we obtain that $\sup\limits_{0 \leq t \leq T} ||(Pu)(t) - u(t)|| = 0$. It follows that $u(t) = (Pu)(t) = u_0 + \int_0^t F(s, u(s), (Qu)(s))ds$ for every $t \in [0, T]$. Hence

$$u(t) = \begin{cases} \varphi(t), & \text{for } t \in [-\sigma, 0) \\ \varphi(0) + \int_0^t F(s, u(s), (Qu)(s))ds, & \text{for } t \in [0, b), \end{cases}$$

solve the Cauchy problem (2.1). □

Theorem 3.2. *Let $Q : \mathcal{C}([-\sigma, b), E) \to L^\infty_{loc}([0, b), E)$ be a causal operator such that the conditions $(h_1) - (h_5)$ hold. Then, the largest interval of existence for any bounded solution of Cauchy problem (2.1) is $[0, b)$.*

Proof. Let $u(\cdot) : [-\sigma, \beta) \to E$ be any solution of Cauchy problem (2.1) existing on $[-\sigma, \beta)$, $0 < \beta < b$. Also, we suppose, by contradiction, that the value of β cannot be increased. Since $u(\cdot)$ is bounded, then there exists $r > 0$ such that $\sup\limits_{-\sigma \leq t < \beta} ||u(t)|| \leq r$ and so, by (h_2), there exists $M > 0$ such that $||(Qu)(t)|| \leq M$ for $t \in [0, b)$. By (h_3), it follows that there exists a function there exists $\mu(\cdot) \in L^1_{loc}([0, b), \mathbb{R}_+)$ such that

$$||F(t, u, v)|| \leq \mu(t) \text{ for a.e. } t \in [0, \beta) \text{ and } u \in B_r(0) \times B_M(0).$$

For every t_1, t_2 such that $0 < t_1 < t_2 < \beta$, we have

$$||u(t_2) - u(t_1)|| =$$

$$|| \int_0^{t_1} F(s, u(s), (Qu)(s))ds - \int_0^{t_2} F(s, u(s), (Qu)(s))ds||$$

$$\leq \int_{t_1}^{t_2} ||F(s, u(s), (Qu)(s))||ds \leq \int_{t_1}^{t_2} \mu(s)ds$$

Since $\mu(\cdot) \in L^1([0, \beta], \mathbb{R}_+)$ then $\displaystyle\int_{t_1}^{t_2} \mu(s)ds \to 0$ as t_1, $t_2 \to \beta^-$, which implies that $\lim_{t\to\beta^-} u(t)$ exists. Hence, if we take $u(\beta) = \lim_{t\to\beta^-} u(t)$, then the function $u(\cdot)$ can be extended by continuity on $[0, \beta]$. Further on, we consider the Cauchy problem

$$\begin{cases} v'(t) = F(t, u(t + \beta), (Qv(\cdot - \beta)(t + \beta)), \ 0 \leq t < b - \beta \\ v|_{[-(\sigma+\beta),0]} = \psi \end{cases} \tag{3.2}$$

where $\psi(\cdot) \in \mathcal{C}_{\sigma+\beta}$ is defined by $\psi(s) = u(s + \beta)$, for all $s \in [-(\sigma + \beta), 0]$.

By Theorem 3.1, there exists a solution $v(\cdot) : [-(\sigma + \beta), \tau) \to E$ of Cauchy problem (3.2), where $\tau \in (0, b - \beta]$. It follows that $w(\cdot) : [-\sigma, \beta + \tau] \to E$, given by

$$w(t) = \begin{cases} u(t), & \text{for } t \in [-\sigma, \beta] \\ v(t - \beta), & \text{for } t \in [\beta, \beta + \tau], \end{cases}$$

is a solution of Cauchy problem (2.1) because, for a.e. $t \in [\beta, \beta + \tau]$, we have that

$$w'(t) = v'(t - \beta) = F(t, u(t), (Qv(\cdot - \beta)(t)) = F(t, u(t), (Qw)(t)).$$

Therefore, the solution $u(\cdot)$.can be continued beyond β, contradicting the assumption that β cannot be increased. It follows that $\beta = b$. □

4 Some properties of solution sets

In the following, for a fixed $\varphi \in \mathcal{C}_\sigma$ and a bounded set $K \subset E$, by $\mathcal{S}_T(\varphi, K)$ we denote the set of all solutions $u(\cdot)$ of Cauchy problem (2.1) on $[-\sigma, T]$ with $T \in (0, b]$ and such that $u(t) \in K$ for all $t \in [-\sigma, T]$. By $\mathcal{A}_T(\varphi, K)$ we denote the attainable set; that is, $\mathcal{A}_T(\varphi, K) = \{u(T); u(\cdot) \in \mathcal{S}_T(\varphi, K)\}$.

Theorem 4.1. *Assume that $Q : \mathcal{C}([-\sigma, b), E) \to L^\infty_{loc}([0, b), E)$ is a causal operator such that the conditions $(h_1) - (h_5)$ hold. Then, for every $\varphi \in \mathcal{C}_\sigma$, the set $\mathcal{S}_T(\varphi, K)$ is a compact set in $\mathcal{C}([-\sigma, T], E)$.*

Proof. We consider a sequence $\{u_m(\cdot)\}_{m\geq 1}$ in $\mathcal{S}_T(\varphi, K)$ and we shall show that this sequence contains a subsequence which converges, uniformly on $[-\sigma, T]$, to a solution $u(\cdot) \in \mathcal{S}_T(\varphi, K)$. Since K is a bounded set, then there exists $r > 0$ such that $K \subset B_r(0)$. By (h_2), there exists $M > 0$ such that $||(Qu)(t)|| \leq M$ for every $u(\cdot) \in ([-\sigma, T], E)$ with $\sup_{-\sigma\leq t\leq T} ||u(t)|| < r$. Since F is a Carathéodory function, there exists $\mu(\cdot) \in L^1([0, T], \mathbb{R}_+)$ such that

$$||F(t, u, v)|| \leq \mu(t) \text{ for a.e. } t \in [0, T] \text{ and } u \in B_r(0) \times B_M(0).$$

Since $u_m|_{[-\sigma,0]} = \varphi$, we have that $u_m(\cdot) \to \varphi(\cdot)$ uniformly on $[-\sigma, 0]$. On the other hand, since

$$u_m(t) = \varphi(0) + \int_0^t F(s, u_m(s), (Qu_m)(s))ds$$

for all $t \in [0, T]$, then we have that

$$||u_m(t) - u_m(s)|| \leq \left| \int_s^t ||F(\tau, u_m(\tau), (Qu_m)(\tau))||d\tau \right|$$

$$\leq \left| \int_s^t \mu(\tau)d\tau \right| \text{ for } s, t \in [0, T].$$

Therefore, $\{u_m(\cdot)\}_{m\geq 1}$ is uniformly equicontinuous on $[0, T]$. As in proof of Theorem 3.1 we can show that $A = \{u_m(\cdot); n \geq 1\}$ is relatively compact subset of $\mathcal{C}([0, T], E)$. Moreover, since $\alpha(A) = \sup\limits_{0\leq t\leq T} \alpha(A(t))$, we deduce that $\alpha(A) = 0$. Therefore, A is relatively compact subset of $\mathcal{C}([0, T], E)$. Further, by the Ascoli-Arzela theorem and extracting a subsequence if necessary, we may assume that the sequence $\{u_m(\cdot)\}_{m\geq 1}$ converges uniformly on $[0, T]$ to a continuous function $u(\cdot)$. If we extend $u(\cdot)$ to $[-\sigma, T]$ such that $u|_{[-\sigma,0]} = \varphi$ then is clearly that $u_m(\cdot) \to u(\cdot)$ uniformly on $[-\sigma, T]$. Now, by *(h_1)* we have that $\lim\limits_{n\to\infty} Qu_m = Qu$ in $L^\infty([0, T], E)$ and so

$$\lim_{n\to\infty} (Qu_m)(t) = (Qu)(t) \text{ for a.e. } t \in [0, T].$$

Since $||F(t, u_m(t), (Qu_m)(t))|| \leq \mu(t)$ for almost all $t \in [0, T]$ and all $m \geq 1$, by the Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty} \int_0^t F(s, u_m(s), (Qu_m)(s))ds = \int_0^t F(s, u(s), (Qu)(s))ds \text{ for all } t \in [0, T].$$

It follows that $u(t) = \lim\limits_{n\to\infty} u_m(t) = \varphi(0) + \displaystyle\int_0^t F(s, u(s), (Qu)(s))ds$ for all $t \in [0, T]$ and so $u(\cdot) \in \mathcal{S}_T(\varphi, K)$. $\square$

Theorem 4.2. *Assume that* $Q : \mathcal{C}([-\sigma, b), E) \to L^\infty_{loc}([0, b), E)$ *is a causal operator such that the condition* $(h_1) - (h_5)$ *hold. Then the multifunction* $S_T : \mathcal{C}_\sigma \mapsto \mathcal{C}([-\sigma, T], E)$ *is upper semicontinuous.*

Proof. Let $\mathcal{K}$ be a closed set in $\mathcal{C}([-\sigma,T],E)$ and $\mathcal{G}=\{\varphi\in\mathcal{C}_\sigma;\mathcal{S}_T(\varphi,K)\cap\mathcal{K}\neq\emptyset\}$. We must show that $\mathcal{G}$ is closed in $\mathcal{C}_\sigma$. For this, let $\{\varphi_m\}_{m\geq 1}$ be a sequence in $\mathcal{G}$ such that $\varphi_m\to\varphi$ on $[-\sigma,0]$. Further on, for each $m\geq 1$, let $u_m(\cdot)\in\mathcal{S}_T(\varphi_n,K)\cap\mathcal{K}$. Then, $u_m=\varphi_m$ on $[-\sigma,0]$ for every $m\geq 1$, and $u_m(t)=\varphi_m(0)+\displaystyle\int_0^t F(s,u_m(s),(Qu_m)(s))ds$ for every $t\in(0,T]$ and $m\geq 1$. As in proof of Theorem 3.1 we can show that $\{u_m(\cdot)\}_{n\geq 1}$ converges uniformly on $[-\sigma,T]$ to a continuous function $u(\cdot)\in\mathcal{K}$. Since $u(t)=\lim_{m\to\infty}u_m(t)=\varphi(0)+\displaystyle\int_0^t F(s,u(s),(Qu)(s))ds$ for every $t\in[0,T]$, we deduce that $u(\cdot)\in\mathcal{S}_T(\varphi,K)\cap\mathcal{K}$. This proves that $\mathcal{G}$ is closed and so $\varphi\mapsto\mathcal{S}_T(\varphi,K)$ is upper semicontinuous. □

Corollary 4.1. *Assume that $Q:\mathcal{C}([-\sigma,b),E)\to L^\infty_{loc}([0,b),E)$ is a causal operator such that the conditions $(h_1)-(h_5)$ hold. Then, for any $\varphi\in\mathcal{C}_\sigma$ and any $t\in[0,T]$ the attainable set $\mathcal{A}_t(\varphi,K)$ is compact in $\mathcal{C}([-\sigma,t],E)$ and the multifunction $(t,\varphi)\mapsto\mathcal{A}_t(\varphi,K)$ is upper semicontinuous.* □

In the following, we consider a control problem:

$$\begin{cases} u'(t)=F(t,u(t),(Qu)(t)), \text{ for a.e. } t\in[0,T]\\ u|_{[-\sigma,0]}=\varphi\\ \textit{minimize } g(u(T)), \end{cases} \tag{4.1}$$

where $g:E\to\mathbb{R}$ is a given function.

Theorem 4.3. *Let $\mathcal{K}_0$ be a compact set in $\mathcal{C}_\sigma$ and let $g:E\to\mathbb{R}$ be a lower semicontinuous function. If $Q:\mathcal{C}([-\sigma,b),E)\to L^\infty_{loc}([0,b),E)$ is a causal operator such that the conditions $(h_1)-(h_5)$ hold, then the control problem (4.1) has an optimal solution; that is, there exists $\varphi_0\in\mathcal{K}_0$ and $u_0(\cdot)\in\mathcal{S}_T(\varphi_0,K)$ such that*

$$g(u_0(T))=\inf\{g(u(T));u(\cdot)\in\mathcal{S}_T(\varphi),\varphi\in\mathcal{K}_0\}.$$

Proof. From Corollary 4.1 we deduce that the attainable set $\mathcal{A}_T(\varphi,K)$ is upper semicontinuous. Then the set $\mathcal{A}_T(\mathcal{K}_0)=\{u(T);u(\cdot)\in\mathcal{S}_T(\varphi,K),\varphi\in\mathcal{K}_0\}=\cup_{\varphi\in\mathcal{K}_0}\mathcal{A}_T(\varphi,K)$ is compact in E and so, since g is lower semicontinuous, there exists $\varphi_0\in\mathcal{K}_0$ such that $g(u_0(T))=\inf\{g(u(T));u(\cdot)\in\mathcal{S}_T(\varphi,K),\varphi\in\mathcal{K}_0\}$. □

5 Monotone iterative technique

In this section, we suppose, in addition, that E is an ordered Banach space with partial order $\leq$, whose positive cone $P=\{x\in E;x\geq 0\}$ is normal with normal constant N. Evidently, $\mathcal{C}([0,b],E)$ is also an ordered Banach space with the partial order $\leq$ defined by the positive function cone $K=\{u\in\mathcal{C}([0,b],E);u(t)\geq 0,t\in[0,b]\}$. K is also normal cone with same constant N. For $v,w\in\mathcal{C}([0,b],E)$, we use $[v,w]$ to denote the order interval $\{u\in\mathcal{C}([0,b],E);v\leq u\leq w\}$, and $[v(t),w(t)]$ to denote the order interval $\{x\in E;v(t)\leq x\leq w(t)\}$ in E.

Also, we recall the following lemma [14].

Lemma 5.1. If $V \subset \mathcal{C}([0,b],E)$ is bounded and equicontinuous, then the function $t \mapsto \alpha(V(t))$ is continuous on $[0,b]$ and

$$\alpha(\{\int_0^b v(t)dt; v(\cdot) \in V\}) \leq \int_0^b \alpha(V(t))dt. \ \square$$

In the following, consider the initial-valued problem

$$u'(t) = F(t, u(t), (Qu)(t)), \ \ u(0) = x_0, \tag{5.1}$$

under the following assumptions:

($\widetilde{h}_1$) $Q : \mathcal{C}([0,b],E) \to \mathcal{C}([0,b],E)$ is a causal continuous operator;

($\widetilde{h}_2$) $F : [0,b] \times E \times E \to E$ is a continuous function,

($\widetilde{h}_3$) for each bounded set $A \subset \mathcal{C}([0,b],E)$, there exists $k_0 > 0$ such that

$$\alpha((QA)(t)) \leq k_0 \int_0^t \alpha(A(s))ds \text{ for every } t \in [0,b], \tag{5.2}$$

where $A(t) = \{u(t); u \in A\}$ and $(QA)(t) = \{(Qu)(t); u \in A\}$.

A continuous function $u(\cdot) \in \mathcal{C}^1([0,b],E)$ is said to be a lower solution of (5.1) if

$$\begin{cases} u'(t) \leq F(t, u(t), (Qu)(t)), & t \in [0,b] \\ u(0) \leq x_0. \end{cases} \tag{5.3}$$

Also, $u(\cdot)$ is said to be an upper solution of (5.1), if the inequalities of (5.3) are reversed.

Theorem 5.1. *Assume that the conditions* ($\widetilde{h}_1$) − ($\widetilde{h}_3$) *holds, and that the initial value problem (5.3) has a lower solution* $v_0 \in \mathcal{C}^1([0,b],E)$ *and an upper solution* $w_0 \in \mathcal{C}^1([0,b],E)$ *with* $v_0 \leq w_0$*. If, in addition, the following conditions are satisfied:*

($\widetilde{h}_4$) *there exists* $M > 0$ *such that*

$$F(t, x_2, y_2) - F(t, x_1, y_1) \geq -M(x_2 - x_1)$$

for all $t \in [0,b]$*, and* $v_0(t) \leq x_1 \leq x_2 \leq w_0(t)$*,* $(Qv_0)(t) \leq y_1 \leq y_2 \leq (Qw_0)(t)$*,*

($\widetilde{h}_5$) *there exist* c_1*,* $c_2 > 0$ *such that*

$$\alpha(F(t, B_1, B_2)) \leq c_1\alpha(B_1) + c_2\alpha(B_2)$$

for every bounded sets $B_1, B_2 \subset E$*.*

Then the initial value problem (5.1) has minimal and maximal solutions between v_0 and w_0.

Proof. First, for any $h(\cdot) \in [v_0, w_0]$, consider the differential equation

$$\begin{cases} u'(t) + Mu(t) = \sigma(t), t \in [0, b] \\ u(0) = x_0, \end{cases} \tag{5.4}$$

where $\sigma(t) = F(s, u(t), (Qu)(t)) + Mu(t)$, $M > 0$ and $x_0 \in E$. It is easy to see that $u(\cdot) \in \mathcal{C}^1([0, b], E)$ is a solution of (5.4) if and only if $u(\cdot) \in \mathcal{C}([0, b], E)$ is a solution of the following integral equation

$$u(t) = x_0 e^{-Mt} + \int_0^t e^{-M(t-s)} \sigma(s) ds, t \in [0, b]. \tag{5.5}$$

We consider the operator $A : \mathcal{C}([0, b], E) \to \mathcal{C}([0, b], E)$ given by the formula

$$(Au)(t) = x_0 e^{-Mt} + \int_0^t e^{-M(t-s)} [F(s, u(s), (Qu)(s)) + Mu(s)] ds, t \in [0, b].$$

¿From (5.4) we have that u is a solution of (5.1) if and only if $Au = u$. Obviously, A is a continuous operator. By $(\widetilde{h}_4)$, the operator A is increasing in $[v_0, w_0]$, and maps any bounded set in $[v_0, w_0]$ into a bounded set. We shall show that $v_0 \le Av_0$ and $Aw_0 \le w_0$. If we put $\sigma(t) = v_0'(t) + Mv_0(t)$ for $t \in [0, b]$ then, by the definition of lower solution, we have that $\sigma \in \mathcal{C}([0, b], E)$ and $\sigma(t) \le F(t, v_0(t), (Qv_0)(t)) + Mv_0(t)$ for $t \in [0, b]$. Since v_0 is a solution of (5.4) with initial condition $v_0(0) = x_0$, then

$$\begin{aligned} v_0(t) &= x_0 e^{-Mt} + \int_0^t e^{-M(t-s)} \sigma(s) ds \\ &\le x_0 e^{-Mt} + \int_0^t e^{-M(t-s)} [F(s, v_0(s), (Qv_0)(s)) + Mv_0(s)] ds \\ &= (Av_0)(t), t \in [0, b], \end{aligned}$$

and so, $v_0 \le Av_0$. Similarly, we can show that $Aw_0 \le w_0$. Therefore, since A is an increasing operator in $[v_0, w_0]$, we obtain that A maps $[v_0, w_0]$ into itself. Further, we define the sequences $\{v_m(\cdot)\}_{m \ge 0}$ and $\{w_m(\cdot)\}_{m \ge 0}$ by iterative scheme

$$v_m = Av_{m-1},\ w_m = Aw_{m-1},\ m = 1, 2, ... \tag{5.6}$$

Then from monotonicity property of A, it follows that

$$v_0(t) \le v_2(t) \le ...v_m(t) \le ... \le w_m(t) \le ... \le w_m(t) \le ... \le w_1(t) \le w_0(t), \tag{5.7}$$

for every $t \in [0, T]$ and $m = 0, 1, 2, ...$We prove that $\{v_m(\cdot)\}_{m \ge 0}$ and $\{w_m(\cdot)\}_{m \ge 0}$ are uniformly convergent in $[0, T]$. For this, let $V = \{v_m(\cdot); m = 0, 1, 2, ...\}$ and $V(t) = \{v_m(t); m = 0, 1, 2, ...\}$. First, the normality of P implies that

$V = \{v_m(\cdot); m = 0, 1, 2, ...\}$ is bounded set in $\mathcal{C}([0, b], E)$. Since (5.2) implies that V is bounded, then we deduce that QV is bounded in $\mathcal{C}([0, b], E)$. Therefore, since $F([0, b], B_1, B_2)$ is bounded for every bounded sets $B_1, B_2 \subset E,$ there exists $c_0 > 0$ such that

$$||F(t, v_m(t), (Qv_m)(t)) + Mv_m(t)|| \leq c_0 \tag{5.8}$$

for every $t \in [0, T]$ and $m = 0, 1, 2, ...$ From the definition of $v_m(\cdot)$ and (5.5), we have

$$v_m(t) = x_0 e^{-Mt} + \int_0^t e^{-M(t-s)}[F(s, v_{m-1}(s), (Qv_{m-1})(s)) + Mv_{m-1}(s)]ds \tag{5.9}$$

for every $t \in [0, T]$ and $m = 0, 1, 2, ...$ Then, from (5.8) and (5.9), it follows that V is equicontinuous on $[0, b]$, and so, by Lemma 5.1 it follows that the function $t \mapsto \alpha(V(t))$ is continuous on $[0, b]$. Next, by Lemma 5.1, $(\widetilde{h}_3)$, $(\widetilde{h}_5)$ and (5.9), we obtain that

$$\begin{aligned}
\alpha(V(t)) &\leq \int_0^t \alpha(\{e^{-M(t-s)}[F(s, v_{m-1}(s), (Qv_{m-1})(s)) + Mv_{m-1}(s)]ds; m \geq 1\}) \\
&\leq \int_0^t [\alpha(F(s, V(s), (QV)(s)) + M\alpha(V(s))]ds \\
&\leq \int_0^t [c_1\alpha(V(s)) + c_2\alpha((QV)(s)) + M\alpha(V(s))]ds \\
&\leq \int_0^t [c_1\alpha(V(s)) + c_2 k_0 \int_0^s \alpha(V(\tau))d\tau + M\alpha(V(s))]ds \\
&= (c_1 + M)\int_0^t \alpha(V(s))ds + c_2 k_0 \int_0^t (t - s)\alpha(V(s))ds.
\end{aligned}$$

Therefore,

$$\alpha(V(t)) \leq K_0 \int_0^t \alpha(V(s))ds, \quad t \in [0, b],$$

and so, by Gronwall's lemma, we have that $\alpha(V(t)) = 0$ for every $t \in [0, T]$. Moreover, since (see [14], Theorem 1.4.2) $\alpha(V) = \sup_{0 \leq t \leq T} \alpha(V(t))$, we deduce that $\alpha(V) = 0$. Therefore, V is a relatively compact subset of $\mathcal{C}([0, b], E)$, and so, there exists a subsequence of $\{v_m(\cdot)\}_{m \geq 0}$ which converges uniformly on $[0, b]$ to some $\overline{v} \in \mathcal{C}([0, b], E)$. Since $\{v_m(\cdot)\}_{m \geq 0}$ is nondecreasing and P is normal, we easily prove that $\{v_m(\cdot)\}_{m \geq 0}$ converges uniformly on $[0, b]$ to $\overline{v}$. Next, we have

$$\lim_{m \to \infty}[F(t, v_{m-1}(t), (Qv_{m-1})(t)) + Mv_{m-1}(t)]$$

$$= [F(t, \overline{v}(t), (Q\overline{v})(t)) + M\overline{v}(t)], \text{ as } m \to \infty,$$

for all $t \in [0, b]$. Also, by (5.8), we have

$$||F(t, v_{m-1}(t), (Qv_{m-1})(t)) + Mv_{m-1}(t)$$

$$-F(t, \overline{v}(t), (Q\overline{v})(t)) + M\overline{v}(t)|| \leq 2c_0$$

for every $t \in [0, b]$ and $m = 1, 2, ...$Therefore, taking limits as $m \to \infty$ in (5.9) we obtain that

$$\overline{v}(t) = x_0 e^{-Mt} + \int_0^t e^{-M(t-s)}[F(s, \overline{v}(s), (Q\overline{v})(s)) + M\overline{v}(s)]ds,$$

for all $t \in [0, b]$. It follows that $\overline{v} \in \mathcal{C}^1([0, b], E)$ and $\overline{v}$ is a solution of (5.1). Similarly, we can show that $\{w_m(\cdot)\}_{m \geq 0}$ uniformly converges on $[0, b]$ to some $\overline{w}$ and $\overline{w}$ is a solution of (5.1) in $\mathcal{C}^1([0, b], E)$. Next, letting $m \to \infty$ in (5.6) and (5.7), we infer that $v_0 \leq \overline{v} \leq \overline{w} \leq w_0$ and $\overline{v} = A\overline{v}$, $\overline{w} = A\overline{w}$. By the monotonicity of A, it is easy to see that $\overline{v}$ and $\overline{w}$ are the minimal and maximal fixed points of A in $[v_0, w_0]$, respectively. This completes the proof of our theorem. □

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