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SOME EXISTENCE RESULTS FOR BOUNDARY VALUE PROBLEMS OF FRACTIONAL DIFFERENTIAL INCLUSIONS WITH NON-SEPARATED BOUNDARY CONDITIONS

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Abstract

In this paper, we study the existence of solutions for a boundary value problem of differential inclusions of order $q \in (1, 2]$ with non-separated boundary conditions involving convex and non-convex multivalued maps. Our results are based on the nonlinear alternative of Leray Schauder type and some suitable theorems of fixed point theory.

Key words and phrases: Fractional differential inclusions; non-separated boundary conditions; existence; nonlinear alternative of Leray Schauder type; fixed point theorems.
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1 Introduction

Fractional calculus (differentiation and integration of arbitrary order) has proved to be an important tool in the modelling of dynamical systems associated with phenomena such as fractals and chaos. In fact, this branch of calculus has found its applications in various disciplines of science and engineering such as mechanics, electricity, chemistry, biology, economics, control theory, signal and image processing, polymer rheology, regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, electro-dynamics of complex medium, viscoelasticity and damping, control theory, wave propagation, percolation, identification, fitting of experimental data, etc. [1-4].
Recently, differential equations of fractional order have been addressed by several researchers with the sphere of study ranging from the theoretical aspects of existence and uniqueness of solutions to the analytic and numerical methods for finding solutions. For some recent work on fractional differential equations, see [5-11] and the references therein.

Differential inclusions arise in the mathematical modelling of certain problems in economics, optimal control, etc. and are widely studied by many authors; see [12-15] and the references therein. For some recent development on differential inclusions of fractional order, we refer the reader to the references [16-21].

In this paper, we consider the following fractional differential inclusion with non-separated boundary conditions

$$\begin{cases} {}^cD^q x(t) \in F(t, x(t)), \quad t \in [0, T], \ T > 0, \quad 1 < q \leq 2, \\ x(0) = \lambda_1 x(T) + \mu_1, \quad x'(0) = \lambda_2 x'(T) + \mu_2, \quad \lambda_1 \neq 1, \ \lambda_2 \neq 1, \end{cases} \tag{1.1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, $F : [0, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$, and $\lambda_1, \ \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$.

2 Preliminaries

Let $C([0, T])$ denote a Banach space of continuous functions from $[0, T]$ into $\mathbb{R}$ with the norm $\|x\| = \sup_{t \in [0,T]} |x(t)|$. Let $L^1([0, T], \mathbb{R})$ be the Banach space of measurable functions $x : [0, T] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^T |x(t)|dt$.

Now we recall some basic definitions on multi-valued maps [22, 23].

For a normed space $(X, \|.\|)$, let $P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$. A multi-valued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. The map G is bounded on bounded sets if $G(\mathbb{B}) = \cup_{x \in \mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in P_b(X)$ (i.e. $\sup_{x \in \mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in P_b(X)$. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*, \ y_n \to y_*, \ y_n \in G(x_n)$ imply $y_* \in G(x_*)$. G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$. A multivalued map $G : [0; 1] \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.

Definition 2.1.A multivalued map $F : [0, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be L^1-Carathéodory if

(**i**) $t \longmapsto F(t, x)$ is measurable for each $x \in \mathbb{R}$;

(ii) $x \longmapsto F(t,x)$ is upper semicontinuous for almost all $t \in [0,T]$;

(iii) for each $\alpha > 0$, there exists $\varphi_\alpha \in L^1([0,T],\mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leq \varphi_\alpha(t) \text{ for all } \|x\|_\infty \leq \alpha \text{ and for a. e. } t \in [0,T].$$

For each $y \in C([0,T],\mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1([0,T],\mathbb{R}) : v(t) \in F(t,y(t)) \text{ for a.e. } t \in [0,T]\}.$$

Let X be a nonempty closed subset of a Banach space E and $G : X \to \mathcal{P}(E)$ be a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{y \in X : G(y) \cap B \neq \emptyset\}$ is open for any open set B in E. Let A be a subset of $[0,T] \times \mathbb{R}$. A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the $\sigma-$algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in $[0,T]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$. A subset $\mathcal{A}$ of $L^1([0,T],\mathbb{R})$ is decomposable if for all $u, v \in \mathcal{A}$ and measurable $\mathcal{J} \subset [0,T] = J$, the function $\chi_{\mathcal{J}} u + \chi_{J-\mathcal{J}} v \in \mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

Definition 2.2. Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,T],\mathbb{R}))$ be a multivalued operator. We say N has a property (BC) if N is lower semi-continuous (l.s.c.) and has nonempty closed and decomposable values.

Let $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Define a multivalued operator $\mathcal{F} : C([0,T] \times \mathbb{R}) \to \mathcal{P}(L^1([0,T],\mathbb{R}))$ associated with F as

$$\mathcal{F}(x) = \{w \in L^1([0,1],\mathbb{R}) : w(t) \in F(t,x(t)) \text{ for a.e. } t \in [0,T]\},$$

which is called the Nymetzki operator associated with F.

Definition 2.3. Let $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Nymetzki operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Let (X,d) be a metric space induced from the normed space $(X; \|.\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A,B) = \max\{\sup_{a \in A} d(a,B), \sup_{b \in B} d(A,b)\},$$

where $d(A,b) = \inf_{a \in A} d(a;b)$ and $d(a,B) = \inf_{b \in B} d(a;b)$. Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space (see [24]).

Definition 2.4. A multivalued operator $N : X \to P_{cl}(X)$ is called (**a**) $\gamma-$Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \leq \gamma d(x,y) \text{ for each } x, y \in X;$$

(**b**) a contraction if and only if it is $\gamma-$Lipschitz with $\gamma < 1$.

The following lemmas will be used in the sequel.

Lemma 2.1. ([25]) Let X be a Banach space. Let $F : [0,T] \times \mathbb{R} \to P_{cp,c}(X)$ be an L^1- Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([0,T],X)$ to $C([0,T],X)$. Then the operator

$$\Theta \circ S_F : C([0,T],X) \to P_{cp,c}(C([0,T],X)), \quad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C([0,T],X) \times C([0,T],X)$.

Lemma 2.2. ([26]) Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,T],\mathbb{R}))$ be a multivalued operator satisfying the property (BC). Then N has a continuous selection, that is, there exists a continuous function (single-valued) $g : Y \to L^1([0,T],\mathbb{R})$ such that $g(x) \in N(x)$ for every $x \in Y$.

Lemma 2.3. ([27]) Let (X,d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.

Let us recall some definitions on fractional calculus [1-3].

Definition 2.5. For a function $g : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s) ds, \quad n-1 < q < n, \ n = [q]+1, \ q > 0,$$

where $[q]$ denotes the integer part of the real number q and Γ denotes the gamma function.

Definition 2.6. The Riemann-Liouville fractional integral of order q for a function g is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \quad q > 0,$$

provided the right hand side is pointwise defined on $(0,\infty)$.

Definition 2.7. The Riemann-Liouville fractional derivative of order q for a function g is defined by

$$D^q g(t) = \frac{1}{\Gamma(n-q)} \Big(\frac{d}{dt}\Big)^n \int_0^t \frac{g(s)}{(t-s)^{q-n+1}} ds, \quad n = [q]+1, \ q > 0,$$

provided the right hand side is pointwise defined on $(0,\infty)$.

In order to define a solution of (1.1), we consider the following lemma.

Lemma 2.4. For a given $\rho \in C[0,T]$, the unique solution of the boundary value problem

$$\begin{cases} {}^cD^q x(t) = \rho(t), \quad 0 < t < T, \quad 1 < q \le 2, \\ x(0) = \lambda_1 x(T) + \mu_1, \qquad x'(0) = \lambda_2 x'(T) + \mu_2, \end{cases} \tag{2.1}$$

is given by

$$x(t) = \int_0^T G(t,s)\rho(s)ds + \frac{\mu_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2-1)(\lambda_1-1)} - \frac{\mu_1}{(\lambda_1-1)},$$

where $G(t,s)$ is the Green's function given by

$$G(t,s) = \begin{cases} \frac{(t-s)^{q-1}}{\Gamma(q)} - \frac{\lambda_1(T-s)^{q-1}}{(\lambda_1-1)\Gamma(q)} + \frac{\lambda_2[\lambda_1 T+(1-\lambda_1)t](T-s)^{q-2}}{(\lambda_2-1)(\lambda_1-1)\Gamma(q-1)}, \quad 0 \le s \le t \le T, \\ \\ -\frac{\lambda_1(T-s)^{q-1}}{(\lambda_1-1)\Gamma(q)} + \frac{\lambda_2[\lambda_1 T+(1-\lambda_1)t](T-s)^{q-2}}{(\lambda_2-1)(\lambda_1-1)\Gamma(q-1)}, \quad 0 \le t \le s \le T. \end{cases} \tag{2.2}$$

Proof. As argued in [8], for some constants $c_0, c_1 \in \mathbb{R}$, we have

$$x(t) = I^q\rho(t) - c_0 - c_1 t = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\rho(s)ds - c_0 - c_1 t. \tag{2.3}$$

In view of the relations ${}^cD^q\ I^q x(t) = x(t)$ and $I^q\ I^p x(t) = I^{q+p}x(t)$ for $q,p > 0, x \in L(0,T)$, we obtain

$$x'(t) = \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)}\rho(s)ds - c_1.$$

Applying the boundary conditions for (2.1), we find that

$$c_0 = \frac{\lambda_1}{(\lambda_1-1)}\Big[\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}\rho(s)ds - \frac{T\lambda_2}{(\lambda_2-1)}\Big(\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}\rho(s)ds + \frac{\mu_2}{\lambda_2}\Big) + \frac{\mu_1}{\lambda_1}\Big],$$

$$c_1 = \frac{\lambda_2}{(\lambda_2-1)}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}\rho(s)ds + \frac{\mu_2}{(\lambda_2-1)}.$$

Substituting the values of c_0 and c_1 in (2.3), we obtain the unique solution of (2.1) given by

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\rho(s)ds - \frac{\lambda_1}{(\lambda_1-1)}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}\rho(s)ds \\ &\quad + \frac{\lambda_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2-1)(\lambda_1-1)}\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}\rho(s)ds \\ &\quad + \frac{\mu_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2-1)(\lambda_1-1)} - \frac{\mu_1}{(\lambda_1-1)} \\ &= \int_0^T G(t,s)\rho(s)ds + \frac{\mu_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2-1)(\lambda_1-1)} - \frac{\mu_1}{(\lambda_1-1)}, \end{aligned}$$

where $G(t,s)$ is given by (2.2). This completes the proof.

Definition 2.8. A function $x \in C^2([0,T])$ is a solution of the problem (1.1) if there exists a function $f \in L^1([0,T],\mathbb{R})$ such that $f(t) \in F(t,x(t))$ a.e. on $[0,T]$ and

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds - \frac{\lambda_1}{(\lambda_1-1)} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)ds \\ &+ \frac{\lambda_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2-1)(\lambda_1-1)} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)ds \\ &+ \frac{\mu_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2-1)(\lambda_1-1)} - \frac{\mu_1}{(\lambda_1-1)}. \end{aligned}$$

3 Main results

Theorem 3.1. Assume that

($\mathbf{H_1}$) $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is L^1-Carathéodory and has convex values;

($\mathbf{H_2}$) there exists a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in L^1([0,T],\mathbb{R}^+)$ such that

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\psi(\|x\|_\infty) \ \text{ for each}(t,x) \in [0,T] \times \mathbb{R};$$

($\mathbf{H_3}$) there exists a number $M > 0$ such that

$$\frac{M}{\nu_1 \psi(M)\|p\|_{L^1} + \nu_2} > 1,$$

where

$$\nu_1 = \frac{T^{q-1}}{\Gamma(q)}\Big(1+\frac{|\lambda_1|}{|\lambda_1-1|}+\frac{|\lambda_2(1+\lambda_1)(q-1)|}{|(\lambda_2-1)(\lambda_1-1)|}\Big), \ \ \nu_2 = \frac{|\mu_2(1+\lambda_1)|T}{|(\lambda_2-1)(\lambda_1-1)|}+\frac{|\mu_1|}{|\lambda_1-1|}. \tag{3.1}$$

Then the boundary value problem (1.1) has at least one solution on $[0,T]$.

Proof. Define an operator

$$\begin{aligned} \Omega(x) = \Big\{ & h \in C([0,T],\mathbb{R}) : h(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds \\ & - \frac{\lambda_1}{(\lambda_1-1)} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)ds + \frac{\lambda_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2-1)(\lambda_1-1)} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)ds \\ & + \frac{\mu_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2-1)(\lambda_1-1)} - \frac{\mu_1}{(\lambda_1-1)}, \ \ f \in S_{F,x} \Big\}. \end{aligned}$$

We will show that Ω satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof consists of several steps. As a first step, we show that $\Omega(x)$ is convex for each $x \in C([0,T],\mathbb{R})$. For that, let $h_1, h_2 \in \Omega(x)$. Then there exist $f_1, f_2 \in S_{F,x}$ such that for each $t \in [0,T]$, we have

$$\begin{aligned} h_i(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_i(s)ds - \frac{\lambda_1}{(\lambda_1-1)} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f_i(s)ds \\ &\quad + \frac{\lambda_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2-1)(\lambda_1-1)} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f_i(s)ds \\ &\quad + \frac{\mu_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2-1)(\lambda_1-1)} - \frac{\mu_1}{(\lambda_1-1)}, \quad i = 1,2. \end{aligned}$$

Let $0 \le \omega \le 1$. Then, for each $t \in [0,T]$, we have

$$\begin{aligned} [\omega h_1 + (1-\omega)h_2](t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} [\omega f_1(s) + (1-\omega) f_2(s)](s)ds \\ &\quad - \frac{\lambda_1}{(\lambda_1-1)} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} [\omega f_1(s) + (1-\omega) f_2(s)]ds \\ &\quad + \frac{\lambda_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2-1)(\lambda_1-1)} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} [\omega f_1(s) + (1-\omega) f_2(s)]ds \\ &\quad + \frac{\mu_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2-1)(\lambda_1-1)} - \frac{\mu_1}{(\lambda_1-1)}. \end{aligned}$$

Since $S_{F,x}$ is convex (F has convex values), therefore it follows that $\omega h_1 + (1-\omega)h_2 \in \Omega(x)$.
Next, we show that $\Omega(x)$ maps bounded sets into bounded sets in $C([0,T],\mathbb{R})$. For a positive number r, let $B_r = \{x \in C([0,T],\mathbb{R}) : \|x\|_\infty \le r\}$ be a bounded set in $C([0,T],\mathbb{R})$. Then, for each $h \in \Omega(x), x \in B_r$, there exists $f \in S_{F,x}$ such that

$$\begin{aligned} h(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds - \frac{\lambda_1}{(\lambda_1-1)} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)ds \\ &\quad + \frac{\lambda_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2-1)(\lambda_1-1)} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)ds + \frac{\mu_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2-1)(\lambda_1-1)} - \frac{\mu_1}{(\lambda_1-1)} \end{aligned}$$

and

$$\begin{aligned} |h(t)| &\le \int_0^t \frac{|t-s|^{q-1}}{\Gamma(q)} |f(s)|ds + \left|\frac{\lambda_1}{(\lambda_1-1)}\right| \int_0^T \frac{|T-s|^{q-1}}{\Gamma(q)} |f(s)|ds \\ &\quad + \left|\frac{\lambda_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2-1)(\lambda_1-1)}\right| \int_0^T \frac{|T-s|^{q-2}}{\Gamma(q)} |f(s)|ds \\ &\quad + \left|\frac{\mu_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2-1)(\lambda_1-1)}\right| + \left|\frac{\mu_1}{(\lambda_1-1)}\right| \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{T^{q-1}}{\Gamma(q)}\Big(1+\frac{|\lambda_1|}{|\lambda_1-1|}+\frac{|\lambda_2(1+\lambda_1)(q-1)|}{|(\lambda_2-1)(\lambda_1-1)|}\Big)\int_0^T \varphi_r(s)ds\\
&\quad +\frac{|\mu_2(1+\lambda_1)|T}{|(\lambda_2-1)(\lambda_1-1)|}+\frac{|\mu_1|}{|\lambda_1-1|}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|h\|_\infty &\leq \frac{T^{q-1}}{\Gamma(q)}\Big(1+\frac{|\lambda_1|}{|\lambda_1-1|}+\frac{|\lambda_2(1+\lambda_1)(q-1)|}{|(\lambda_2-1)(\lambda_1-1)|}\Big)\int_0^T \varphi_r(s)ds\\
&\quad +\frac{|\mu_2(1+\lambda_1)|T}{|(\lambda_2-1)(\lambda_1-1)|}+\frac{|\mu_1|}{|\lambda_1-1|}.
\end{aligned}
$$

Now we show that Ω maps bounded sets into equicontinuous sets of $C([0,T],\mathbb{R})$. Let $t',t''\in[0,T]$ with $t'<t''$ and $x\in B_r$, where B_r is a bounded set of $C([0,T],\mathbb{R})$. For each $h\in\Omega(x)$, we obtain

$$
\begin{aligned}
&|h(t'')-h(t')|\\
= &\Big|\int_0^{t''}\frac{(t''-s)^{q-1}}{\Gamma(q)}f(s)ds+\frac{\lambda_2[\lambda_1T+(1-\lambda_1)t'']}{(\lambda_2-1)(\lambda_1-1)}\int_0^T\frac{(T-s)^{q-2}}{\Gamma(q-1)}f(s)ds\\
&+\frac{\mu_2[\lambda_1T+(1-\lambda_1)t'']}{(\lambda_2-1)(\lambda_1-1)}-\int_0^{t'}\frac{(t'-s)^{q-1}}{\Gamma(q)}f(s)ds\\
&-\frac{\lambda_2[\lambda_1T+(1-\lambda_1)t']}{(\lambda_2-1)(\lambda_1-1)}\int_0^T\frac{(T-s)^{q-2}}{\Gamma(q-1)}f(s)ds-\frac{\mu_2[\lambda_1T+(1-\lambda_1)t']}{(\lambda_2-1)(\lambda_1-1)}\Big|\\
\leq &\Big|\int_0^{t'}\frac{[(t''-s)^{q-1}-(t'-s)^{q-1}]}{\Gamma(q)}f(s)ds\Big|+\Big|\int_{t'}^{t''}\frac{(t''-s)^{q-1}}{\Gamma(q)}f(s)ds\Big|\\
&+\Big|\frac{(1-\lambda_1)(t''-t')}{(\lambda_2-1)(\lambda_1-1)}\Big(\lambda_2\int_0^T\frac{(T-s)^{q-2}}{\Gamma(q-1)}f(s)ds+\mu_2\Big)\Big|.
\end{aligned}
$$

Obviously the right hand side of the above inequality tends to zero independently of $x\in B_{r'}$ as $t''-t'\to 0$. As Ω satisfies the above three assumptions, therefore it follows by the Ascoli-Arzelá theorem that $\Omega: C([0,T],\mathbb{R})\to\mathcal{P}(C([0,T],\mathbb{R}))$ is completely continuous.

In our next step, we show that Ω has a closed graph. Let $x_n\to x_*$, $h_n\in\Omega(x_n)$ and $h_n\to h_*$. Then we need to show that $h_*\in\Omega(x_*)$. Associated with $h_n\in\Omega(x_n)$, there exists $f_n\in S_{F,x_n}$ such that for each $t\in[0,T]$,

$$
\begin{aligned}
h_n(t) &= \int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f_n(s)ds-\frac{\lambda_1}{(\lambda_1-1)}\int_0^T\frac{(T-s)^{q-1}}{\Gamma(q)}f_n(s)ds\\
&\quad+\frac{\lambda_2[\lambda_1T+(1-\lambda_1)t]}{(\lambda_2-1)(\lambda_1-1)}\int_0^T\frac{(T-s)^{q-2}}{\Gamma(q-1)}f_n(s)ds\\
&\quad+\frac{\mu_2[\lambda_1T+(1-\lambda_1)t]}{(\lambda_2-1)(\lambda_1-1)}-\frac{\mu_1}{(\lambda_1-1)}.
\end{aligned}
$$

Thus we have to show that there exists $f_* \in S_{F,x_*}$ such that for each $t \in [0, T]$,

$$\begin{aligned} h_*(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_*(s)ds - \frac{\lambda_1}{(\lambda_1 - 1)} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f_*(s)ds \\ &+ \frac{\lambda_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2 - 1)(\lambda_1 - 1)} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f_*(s)ds \\ &+ \frac{\mu_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2 - 1)(\lambda_1 - 1)} - \frac{\mu_1}{(\lambda_1 - 1)}. \end{aligned}$$

Let us consider the continuous linear operator $\Theta : L^1([0, T], \mathbb{R}) \to C([0, T], \mathbb{R})$ given by

$$\begin{aligned} f \mapsto \Theta(f)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds - \frac{\lambda_1}{(\lambda_1 - 1)} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)ds \\ &+ \frac{\lambda_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2 - 1)(\lambda_1 - 1)} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)ds \\ &+ \frac{\mu_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2 - 1)(\lambda_1 - 1)} - \frac{\mu_1}{(\lambda_1 - 1)}. \end{aligned}$$

Observe that

$$\begin{aligned} & \|h_n(t) - h_*(t)\| \\ = & \Big\| \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} (f_n(s) - f_*(s))ds \\ & - \frac{\lambda_1}{(\lambda_1 - 1)} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} (f_n(s) - f_*(s))ds \\ & + \frac{\lambda_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2 - 1)(\lambda_1 - 1)} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} (f_n(s) - f_*(s))ds \Big\| \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Thus, it follows by Lemma 2.1 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$\begin{aligned} h_*(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_*(s)ds - \frac{\lambda_1}{(\lambda_1 - 1)} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f_*(s)ds \\ &+ \frac{\lambda_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2 - 1)(\lambda_1 - 1)} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f_*(s)ds \\ &+ \frac{\mu_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2 - 1)(\lambda_1 - 1)} - \frac{\mu_1}{(\lambda_1 - 1)}. \end{aligned}$$

for some $f_* \in S_{F,x_*}$.

Finally, we discuss a priori bounds on solutions. Let x be a solution of (1.1). Then there exists $f \in L^1([0,T],\mathbb{R})$ with $f \in S_{F,x}$ such that, for $t \in [0,T]$, we have

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds - \frac{\lambda_1}{(\lambda_1-1)} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)ds \\ &+ \frac{\lambda_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2-1)(\lambda_1-1)} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)ds \\ &+ \frac{\mu_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2-1)(\lambda_1-1)} - \frac{\mu_1}{(\lambda_1-1)}. \end{aligned}$$

In view of (H_2), for each $t \in [0,T]$, we obtain

$$\begin{aligned} |x(t)| &\leq \frac{T^{q-1}}{\Gamma(q)}\Big(1 + \frac{|\lambda_1|}{|\lambda_1-1|} + \frac{|\lambda_2(1+\lambda_1)(q-1)|}{|(\lambda_2-1)(\lambda_1-1)|}\Big)\psi(\|x\|_\infty)\int_0^T p(s)ds \\ &+ \frac{|\mu_2(1+\lambda_1)|T}{|(\lambda_2-1)(\lambda_1-1)|} + \frac{|\mu_1|}{|\lambda_1-1|}. \end{aligned}$$

Consequently, by virtue of (3.1), we have

$$\frac{\|x\|_\infty}{\nu_1\psi(\|x\|_\infty)\|p\|_{L^1} + \nu_2} \leq 1.$$

In view of (H_3), there exists M such that $\|x\|_\infty \neq M$. Let us set

$$U = \{x \in C([0,T],\mathbb{R}) : \|x\|_\infty < M+1\}.$$

Note that the operator $\Omega : \overline{U} \to \mathcal{P}(C([0,T],\mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x \in \mu\Omega(x)$ for some $\mu \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type [28], we deduce that Ω has a fixed point $x \in \overline{U}$ which is a solution of the problem (1.1). This completes the proof. □

As a next result, we study the case when F is not necessarily convex valued. Our strategy to deal with this problems is based on the nonlinear alternative of Leray Schauder type together with the selection theorem of Bressan and Colombo [26] for lower semi-continuous maps with decomposable values.

Theorem 3.2. Assume that $(H_2), (H_3)$ and the following conditions hold:

($\mathbf{H_4}$) $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a nonempty compact-valued multivalued map such that

- (**a**) $(t,x) \longmapsto F(t,x)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable,
- (**b**) $x \longmapsto F(t,x)$ is lower semicontinuous for each $t \in [0,T]$;

($\mathbf{H_5}$) for each $\sigma > 0$, there exists $\varphi_\sigma \in L^1([0,T],\mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|y| : y \in F(t,x)\} \leq \varphi_\sigma(t) \text{ for all } \|x\|_\infty \leq \sigma \text{ and for a.e.} t \in [0,T].$$

Then the boundary value problem (1.1) has at least one solution on $[0,T]$.

Proof. It follows from (H_4) and (H_5) that F is of l.s.c. type. Then from Lemma 2.1, there exists a continuous function $f : C([0,T],\mathbb{R}) \to L^1([0,T],\mathbb{R})$ such that $f(x) \in \mathcal{F}(x)$ for all $x \in C([0,T],\mathbb{R})$.

Consider the problem

$$\begin{cases} {}^cD^q x(t) = f(x(t)), \quad t \in [0,T], \ T > 0, \quad 1 < q \leq 2, \\ x(0) = \lambda_1 x(T) + \mu_1, \quad x'(0) = \lambda_2 x'(T) + \mu_2, \quad \lambda_1 \neq 1, \ \lambda_2 \neq 1, \end{cases} \tag{3.2}$$

Observe that if $x \in C^2([0,T])$ is a solution of (3.2), then x is a solution to the problem (1.1). In order to transform the problem (3.2) into a fixed point problem, we define the operator $\overline{\Omega}$ as

$$\begin{aligned} \overline{\Omega}x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(x(s))ds - \frac{\lambda_1}{(\lambda_1-1)} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(x(s))ds \\ &\quad + \frac{\lambda_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2-1)(\lambda_1-1)} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(x(s))ds \\ &\quad + \frac{\mu_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2-1)(\lambda_1-1)} - \frac{\mu_1}{(\lambda_1-1)}. \end{aligned}$$

It can easily be shown that $\overline{\Omega}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 3.1. So we omit it. This completes the proof. □

Now we prove the existence of solutions for the problem (1.1) with a nonconvex valued right hand side by applying a fixed point theorem for multivalued map due to Covitz and Nadler [27].

Theorem 3.3. Assume that the following conditions hold:

($\mathbf{H_6}$) $F : [0,T] \times \mathbb{R} \to P_{cp}(\mathbb{R})$ is such that $F(.,x) : [0,T] \to P_{cp}(\mathbb{R})$ is measurable for each $x \in \mathbb{R}$;

($\mathbf{H_7}$) $H_d(F(t,x), F(t,\bar{x})) \leq m(t)|x-\bar{x}|$ for almost all $t \in [0,T]$ and $x, \bar{x} \in \mathbb{R}$ with $m \in L^1([0,T],\mathbb{R}^+)$ and $d(0, F(t,0)) \leq m(t)$ for almost all $t \in [0,T]$.

Then the boundary value problem (1.1) has at least one solution on $[0,T]$ if

$$\frac{T^{q-1}\|m\|_{L^1}}{\Gamma(q)} \Big(1 + \frac{|\lambda_1|}{|\lambda_1 - 1|} + \frac{|\lambda_2(1+\lambda_1)(q-1)|}{|(\lambda_2-1)(\lambda_1-1)|} \Big) < 1.$$

Proof. Observe that the set $S_{F,x}$ is nonempty for each $x \in C([0,T],\mathbb{R})$ by the assumption (H_6), so F has a measurable selection (see Theorem III.6 [28]). Now we show that the operator Ω satisfies the assumptions of Lemma 2.2. To show that $\Omega(x) \in P_{cl}((C[0,T],\mathbb{R}))$ for each $x \in C([0,T],\mathbb{R})$, let $\{u_n\}_{n\geq 0} \in \Omega(x)$ be such that $u_n \to u$ $(n \to \infty)$ in $C([0,T],\mathbb{R})$. Then $u \in C([0,T],\mathbb{R})$ and there exists $v_n \in S_{F,x}$ such that, for each $t \in [0,T]$,

$$\begin{aligned} u_n(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_n(s)ds - \frac{\lambda_1}{(\lambda_1-1)} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} v_n(s)ds \\ &+ \frac{\lambda_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2-1)(\lambda_1-1)} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} v_n(s)ds \\ &+ \frac{\mu_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2-1)(\lambda_1-1)} - \frac{\mu_1}{(\lambda_1-1)}. \end{aligned}$$

As F has compact values, we pass onto a subsequence to obtain that v_n converges to v in $L^1([0,T],\mathbb{R})$. Thus, $v \in S_{F,x}$ and for each $t \in [0,T]$,

$$\begin{aligned} u_n(t) \to u(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v(s)ds - \frac{\lambda_1}{(\lambda_1-1)} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} v(s)ds \\ &+ \frac{\lambda_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2-1)(\lambda_1-1)} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} v(s)ds \\ &+ \frac{\mu_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2-1)(\lambda_1-1)} - \frac{\mu_1}{(\lambda_1-1)}. \end{aligned}$$

Hence, $u \in \Omega(x)$.

Next we show that there exists $\gamma < 1$ such that

$$H_d(\Omega(x), \Omega(\bar{x})) \leq \gamma \|x - \bar{x}\|_\infty \text{ for each } x, \bar{x} \in C([0,T],\mathbb{R}).$$

Let $x, \bar{x} \in C([0,T],\mathbb{R})$ and $h_1 \in \Omega(x)$. Then there exists $v_1(t) \in F(t, x(t))$ such that, for each $t \in [0,T]$,

$$\begin{aligned} h_1(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_1(s)ds - \frac{\lambda_1}{(\lambda_1-1)} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} v_1(s)ds \\ &+ \frac{\lambda_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2-1)(\lambda_1-1)} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} v_1(s)ds \\ &+ \frac{\mu_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2-1)(\lambda_1-1)} - \frac{\mu_1}{(\lambda_1-1)}. \end{aligned}$$

By (H_7), we have

$$H_d(F(t,x), F(t,\bar{x})) \leq m(t)|x(t) - \bar{x}(t)|.$$

So, there exists $w \in F(t, \bar{x}(t))$ such that

$$|v_1(t) - w| \leq m(t)|x(t) - \bar{x}(t)|, \quad t \in [0, T].$$

Define $U : [0, T] \to \mathcal{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq m(t)|x(t) - \bar{x}(t)|\}.$$

Since the multivalued operator $V(t) \cap F(t, \bar{x}(t))$ is measurable (Proposition III.4 [29]), there exists a function $v_2(t)$ which is a measurable selection for V. So $v_2(t) \in F(t, \bar{x}(t))$ and for each $t \in [0, T]$, we have $|v_1(t) - v_2(t)| \leq m(t)|x(t) - \bar{x}(t)|$.

For each $t \in [0, T]$, let us define

$$\begin{aligned} h_2(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_2(s)ds - \frac{\lambda_1}{(\lambda_1 - 1)} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} v_2(s)ds \\ &\quad + \frac{\lambda_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2 - 1)(\lambda_1 - 1)} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} v_2(s)ds \\ &\quad + \frac{\mu_2[\lambda_1 T + (1-\lambda_1)t]}{(\lambda_2 - 1)(\lambda_1 - 1)} - \frac{\mu_1}{(\lambda_1 - 1)}. \end{aligned}$$

Thus,

$$\begin{aligned} & |h_1(t) - h_2(t)| \\ \leq & \int_0^t \frac{|t-s|^{q-1}}{\Gamma(q)} |v_1(s) - v_2(s)|ds \\ & + \left|\frac{\lambda_1}{(\lambda_1 - 1)}\right| \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} |v_1(s) - v_2(s)|ds \\ & + \left|\frac{\lambda_2}{(\lambda_2 - 1)(\lambda_1 - 1)}\right| \int_0^T \frac{|(T-s)^{q-2}[\lambda_1 T + (1-\lambda_1)t]|}{\Gamma(q-1)} |v_1(s) - v_2(s)|ds \\ \leq & \ \nu_1 \int_0^T m(s)\|x - \overline{x}\|ds, \end{aligned}$$

where ν_1 is given by (3.1). Hence,

$$\|h_1(t) - h_2(t)\|_\infty \leq \frac{T^{q-1}\|m\|_{L^1}}{\Gamma(q)} \Big(1 + \frac{|\lambda_1|}{|\lambda_1 - 1|} + \frac{|\lambda_2(1+\lambda_1)(q-1)|}{|(\lambda_2 - 1)(\lambda_1 - 1)|}\Big) \|x - \overline{x}\|_\infty.$$

Analogously, interchanging the roles of x and $\overline{x}$, we obtain

$$\begin{aligned} H_d(\Omega(x), \Omega(\bar{x})) &\leq \gamma \|x - \bar{x}\|_\infty \\ &\leq \frac{T^{q-1}\|m\|_{L^1}}{\Gamma(q)} \Big(1 + \frac{|\lambda_1|}{|\lambda_1 - 1|} + \frac{|\lambda_2(1+\lambda_1)(q-1)|}{|(\lambda_2 - 1)(\lambda_1 - 1)|}\Big) \|x - \overline{x}\|_\infty. \end{aligned}$$

Since Ω is a contraction, it follows by Lemma 2.2 that Ω has a fixed point x which is a solution of (1.1). This completes the proof. □

4 Discussion

In this paper, we have presented some existence results for fractional differential inclusions of order $q \in (1,2]$ involving convex and non-convex multivalued maps with non-separated boundary conditions. Our results give rise to various interesting situations. Some of them are listed below:

(i) The results for an anti-periodic boundary value problem of fractional differential inclusions of order $q \in (1,2]$ follow as a special case by taking $\lambda_1 = -1 = \lambda_2,\ \mu_1 = 0 = \mu_2$ in the results of this paper. In this case, the operator $\Omega(x)$ takes the form

$$\Omega(x) = \Big\{ h \in C([0,T],\mathbb{R}) : h(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds \\ - \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)ds + \frac{1}{4}(T-2t)\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s)ds, \quad f \in S_{F,x} \Big\},$$

and the condition (H_3) becomes

$$\frac{4M\Gamma(q)}{(5+q)T^{q-1}\psi(M)\|p\|_{L^1}} > 1,$$

while the condition ensuring the existence of at least one solution of the problem (1.1) in Theorem (3.3) reduces to

$$\frac{T^{q-1}(5+q)\|m\|_{L^1}}{4\Gamma(q)} < 1.$$

(ii) For $q = 2$, we obtain new results for second order differential inclusions with non-separated boundary conditions. In this case, the Green's function $G(t,s)$ is

$$G(t,s) = \begin{cases} \frac{-\lambda_1(\lambda_2-1)(T-s)+\lambda_2[\lambda_1 T+(1-\lambda_1)t]}{(\lambda_1-1)(\lambda_2-1)}, & 0 \le t < s \le T, \\ \\ \frac{(\lambda_1-1)(\lambda_2-1)(t-s)-\lambda_1(\lambda_2-1)(T-s)+\lambda_2[\lambda_1 T+(1-\lambda_1)t]}{(\lambda_1-1)(\lambda_2-1)}, & 0 \le s \le t \le T, \end{cases}$$

which takes the following form for the second order anti-periodic boundary value problem ($\lambda_1 = -1 = \lambda_2$):

$$G(t,s) = \begin{cases} \frac{1}{4}(-T-2t+2s), & 0 \le t < s \le T, \\ \\ \frac{1}{4}(-T+2t-2s), & 0 \le s \le t \le T. \end{cases}$$

(iii) The results for an initial value problem of differential inclusions of fractional order $q \in (1,2]$ can be obtained by taking $\lambda_1 = 0 = \lambda_2$ in the present results with the operator $\Omega(x)$ taking the form

$$\Omega(x) = \Big\{ h \in C([0,T],\mathbb{R}) : h(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds + \mu_2 t + \mu_1, \quad f \in S_{F,x} \Big\}.$$

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References

[1] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[2] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[3] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[4] J. Sabatier, O.P. Agrawal, J.A.T. Machado (Eds.), *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, Springer, Dordrecht, 2007.

[5] V. Daftardar-Gejji, S. Bhalekar, Boundary value problems for multi-term fractional differential equations, *J. Math. Anal. Appl.* **345** (2008), 754-765.

[6] B. Ahmad, J.R. Graef, Coupled systems of nonlinear fractional differential equations with nonlocal boundary conditions, *Pan American Math Journal* **19** (2009), 29-39.

[7] B. Ahmad, J.J. Nieto, Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions, *Comput. Math. Appl.* **58** (2009), 1838-1843.

[8] V. Lakshmikantham, S. Leela, J. Vasundhara Devi, *Theory of Fractional Dynamic Systems*, Cambridge Academic Publishers, Cambridge, 2009.

[9] V. Gafiychuk, B. Datsko, V. Meleshko, Mathematical modeling of different types of instabilities in time fractional reaction-diffusion systems, *Comput. Math. Appl.* **59** (2010), 1101-1107.

[10] R.P. Agarwal, V. Lakshmikantham, J.J. Nieto, On the concept of solution for fractional differential equations with uncertainty, *Nonlinear Anal.* **72** (2010), 2859-2862.

[11] B. Ahmad, Existence of solutions for irregular boundary value problems of nonlinear fractional differential equations, *Appl. Math. Lett.* **23** (2010), 390-394.

[12] G.V. Smirnov, *Introduction to the theory of differential inclusions*, American Mathematical Society, Providence, RI, 2002.

[13] Y.-K. Chang, W.T. Li, J.J. Nieto, Controllability of evolution differential inclusions in Banach spaces, *Nonlinear Anal.* **67** (2007), 623-632.

[14] W.S. Li, Y.K. Chang, J.J. Nieto, Solvability of impulsive neutral evolution differential inclusions with state-dependent delay, *Math. Comput. Modelling* **49** (2009), 1920-1927.

[15] S.K. Ntouyas, Neumann boundary value problems for impulsive differential inclusions,*Electron. J. Qual. Theory Differ. Equ. 2009, Special Edition I, No. 22, 13 pp.*

[16] J. Henderson, A. Ouahab, Fractional functional differential inclusions with finite delay, *Nonlinear Anal.* **70** (2009), 2091-2105.

[17] A. Ouahab, Some results for fractional boundary value problem of differential inclusions, *Nonlinear Anal.* **69** (2008), 3877-3896.

[18] B. Ahmad, V. Otero-Espinar, Existence of solutions for fractional differential inclusions with anti-periodic boundary conditions, *Bound. Value Probl. 2009, Art. ID 625347, 11 pages.*

[19] R.P. Agarwal, M. Belmekki, M. Benchohra, A survey on semilinear differential equations and inclusions involving Riemann-Liouville fractional derivative, *Adv. Difference Equ. 2009, Art. ID 981728, 47 pp.*

[20] S. Hamani, M. Benchohra, J.R. Graef, Existence results for boundary-value problems with nonlinear fractional differential inclusions and integral conditions, *Electron. J. Differential Equations 2010, No. 20, 16 pp.*

[21] M.A. Darwish, S.K. Ntouyas, On initial and boundary value problems for fractional order mixed type functional differential inclusions, *Comput. Math. Appl.* **59** (2010), no. 3, 1253-1265.

[22] K. Deimling, *Multivalued Differential Equations*, Walter De Gruyter, Berlin-New York, 1992.

[23] Sh. Hu, N. Papageorgiou, *Handbook of Multivalued Analysis, Theory I*, Kluwer, Dordrecht, 1997.

[24] M. Kisielewicz, *Differential Inclusions and Optimal Control*, Kluwer, Dordrecht, The Netherlands, 1991.

[25] A. Lasota, Z. Opial, An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.* **13** (1965), 781-786.

[26] A. Bressan, G. Colombo, Extensions and selections of maps with decomposable values, *Studia Math.* **90** (1988), 69-86.

[27] H. Covitz, S. B. Nadler Jr., Multivalued contraction mappings in generalized metric spaces, *Israel J. Math.* **8** (1970), 5-11.

[28] A. Granas, J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2005.

[29] C. Castaing, M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Mathematics 580, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

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