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Triple positive solutions of nth order impulsive integro-differential equations*

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Abstract: In this paper, we prove the existence of at least three positive solutions of boundary value problems for nth order nonlinear impulsive integro-differential equations of mixed type on infinite interval with infinite number of impulsive times. Our results are obtained by applying a new fixed point theorem introduced by Avery and Peterson.
Keywords: Impulsive integro-differential equation; Cone and partial ordering; Positive solution; Fixed point.

1 Introduction

The branch of modern applied analysis known as "impulsive" differential equations furnishes a natural framework to mathematically describe some "jumping processes". Consequently, the area of impulsive differential equations has been developing at a rapid rate(see [2-5]). Most of the works in this area discussed the first- and second- order problems (see e. g. [2,3,6-12]), though the theory of nth order nonlinear impulsive integro-differential equations of mixed type has received attention and some significant results have been obtained in very recent years (see [4,5,13,14]). For instance, Guo [5] has established the existence of solutions for a class of nth order problems on infinite interval with infinite number of impulsive times in Banach spaces by means of the Schauder fixed point theorem. By using the fixed point index theory of completely continuous operators, in [4] Guo has investigated the existence of twin positive solutions of a boundary value problem (BVP) for nth-order nonlinear impulsive integro-differential equation of mixed type as follows:

$$\begin{cases} u^{(n)}(t) = f(t, u(t), u'(t), \cdots, u^{(n-1)}(t), (Tu)(t), (Su)(t)), \qquad \forall t \in J' \\ \Delta u^{(i)}|_{t=t_k} = I_{ik}(u(t_k), u'(t_k), \cdots, u^{(n-1)}(t_k)) \\ (i = 0, 1, \cdots, n-1,\ k = 1, 2, \cdots), \\ u^{(i)}(0) = \theta\ (i = 0, 1, \cdots, n-2),\ u^{(n-1)}(\infty) = \rho u^{(n-1)}(0), \end{cases} \tag{1}$$

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where $J = [0, \infty)$, $0 < t_1 < \cdots < t_k < \cdots$, $t_k \to \infty$, $J' = J/\{t_1, \cdots, t_k \cdots\}$, $f \in C[J \times P \times P \times \cdots \times P \times P \times P, P]$, $I_{ik} \in C[P \times P \times \cdots \times P, P]$ $(i = 0, 1, \cdots, n-1,\ k = 1, 2, \cdots)$, $\rho > 1, u^{(n-1)}(\infty) = \lim_{t\to\infty} u^{(n-1)}(t)$ and

$$(Tu)(t) = \int_0^t K(t,s)u(s)ds, \quad (Su)(t) = \int_0^\infty H(t,s)u(s)ds \tag{2}$$

$K \in C[D, \mathbb{R}_+]$ with $D = \{(t,s) \in J \times J : t \geq s\}, H \in C[J \times J, \mathbb{R}_+]$, $\mathbb{R}_+$ denotes the set of all nonnegative numbers. $\Delta u^{(i)}|_{t=t_k}$ denotes the jump of $u^{(i)}(t)$ at $t = t_k$, i. e.

$$\Delta u^{(i)}|_{t=t_k} = u^{(i)}(t_k^+) - u^{(i)}(t_k^-),$$

where $u^{(i)}(t_k^+)$ and $u^{(i)}(t_k^-)$ represent the right and left limits of $u^{(i)}(t)$ at $t = t_k$, respectively $(i = 0, 1, \cdots, n-1)$.

Here, $(E, |\cdot|)$ is a real Banach spaces, the nonempty convex closed set $P \subset E$ is a cone, that is, $au \in P$ for all $u \in P$ and all $a \geq 0$, and $u, -u \in P$ implies $u = 0$.

But to our best knowledge, there are no results on triple positive solutions for such impulsive equations. The purpose for us to present this paper is to obtain sufficient conditions for the existence of at least three positive solutions for (1). This is also an application of a new fixed point theorem introduced by Avery and Peterson [1] which has been used to verify the existence of three positive solutions for ordinary differential equations in [15] and for p-Laplacian dynamic equations on time scales in [16].

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from recent references (see, e.g., [1, 4, 13, 14]) which are used throughout this paper.

For the Banach space E, by a cone $P \subset E$ we introduce a partial ordering in E, that is, $x \leq y$ if and only if $y - x \in P$.

Let $PC[J, E] = \{u : u$ is a map from J into E such that $u(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$, and $u(t_k^+)$ exists, $k = 1, 2, \cdots\}$, $BPC[J, E] = \{u \in PC[J, E] : u$ is bounded on J with respect to the norm $|\cdot|$ $\}$ with norm

$$\|u\| = \sup\{e^{-t}|u(t)| : t \in J\}.$$

It is easy to see that $BPC[J, E]$ is a Banach space. Let $PC^{n-1}[J, E] = \{u \in PC[J, E] : u^{(n-1)}(t)$ exists and is continuous at $t \neq t_k$, and $u^{(n-1)}(t_k^+)$, $u^{(n-1)}(t_k^-)$ exist for $k = 1, 2, \cdots\}$. For $u \in PC^{n-1}[J, E]$, as shown in [4], $u^{(i)}(t_k^+)$ and $u^{(i)}(t_k^-)$ exist and $u^{(i)} \in PC[J, E]$, where $i = 1, 2, \cdots, n-2, k = 1, 2, \cdots$. We define $u^{(i)}(t_k) = u^{(i)}(t_k^-)$ and, in (1) and in what follows, $u^{(i)}(t_k)$ is understood as $u^{(i)}(t_k^-)$ $(i = 1, 2, \cdots, n-1)$. Let $DPC[J, E] = \{u \in PC^{n-1}[J, E] : u^{(i)} \in BPC[J, E], i = 1, 2, \cdots, n-1\}$, then $DPC[J, E]$ is a Banach space with norm

$$\|u\|_D = \max\{\|u\|, \|u'\|, \cdots, \|u^{(n-1)}\|\}.$$

Let $BPC[J, P] = \{u \in BPC[J, E] : u(t) \geq 0, t \in J\}$ and $DPC^{n-1}[J, P] = \{u \in DPC^{n-1}[J, E] : u^{(i)}(t) \geq 0, t \in J : i = 1, 2, \cdots, n-1\}$. Evidently, $BPC[J, P]$ is a cone in space $BPC[J, E]$ and $DPC^{n-1}[J, P]$ is a cone in space $DPC^{n-1}[J, E]$.

An operator is called completely continuous if it is continuous and maps bounded sets into relatively compact sets.

For a given cone P in a real Banach space E, the map $\chi : P \to [0, \infty)$ is called a nonnegative continuous concave function on P provided that χ is continuous and

$$\chi(tx + (1-t)y) \geq t\chi(x) + (1-t)\chi(y)$$

For $x, y \in P$ and $0 \leq t \leq 1$. Dual to this, we call the map $\varphi : P \to [0, \infty)$ a nonnegative continuous convex function on P provided that φ is continuous and

$$\varphi(tx + (1-t)y) \leq t\varphi(x) + (1-t)\varphi(y)$$

For $x, y \in P$ and $0 \leq t \leq 1$.

Let θ and γ be nonnegative continuous convex functions on P, α a nonnegative continuous concave function on P and ψ a nonnegative continuous function on P. Let a, b, c and d be positive real numbers. We define the following convex sets.

$$\begin{aligned}
&P(\gamma, d) = \{x \in P : \gamma(x) < d\},\\
&P(\gamma, \alpha, b, d) = \{x \in P : b \leq \alpha(x), \gamma(x) \leq d\},\\
&P(\gamma, \theta, \alpha, b, c, d) = \{x \in P : b \leq \alpha(x), \theta(x) \leq c, \gamma(x) \leq d\}
\end{aligned}$$

and a closed set

$$R(\gamma, \psi, a, d) = \{x \in P : a \leq \psi(x), \gamma(x) \leq d\}.$$

The following Lemma 1 is due to Avery and Peterson [1] which play an important role in this paper.

Lemma 1. Let P be a cone in E and $\theta, \gamma, \alpha, \psi$ be defined as above, moreover, ψ satisfy $\psi(\lambda x) \leq \lambda\psi(x)$ for $0 \leq \lambda \leq 1$ such that, for some positive numbers h and d,

$$\alpha(x) \leq \psi(x), \quad |x| \leq h\gamma(x) \tag{3}$$

for all $x \in \overline{P(\gamma, d)}$. Suppose that $A : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$ is a completely continuous operator and there exist positive real numbers a, b and c with $a < b$ such that the following conditions are satisfied:

(h1) $\{x \in P(\gamma, \theta, \alpha, b, c, d) : \alpha(x) > b\} \neq \emptyset$ and

$$\alpha(Ax) > b \quad \text{for } x \in P(\gamma, \theta, \alpha, b, c, d).$$

(h2) $\alpha(Ax) > b$ for $x \in P(\gamma, \alpha, b, d)$ with $\theta(Ax) > c$.

(h3) $0 \notin R(\gamma, \psi, a, d)$ and $\psi(Ax) < a$ for $x \in R(\gamma, \psi, a, d)$ with $\psi(x) = a$.

Then A has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, d)}$ such that

$$\begin{aligned}
&\gamma(x_i) \leq d \quad for\ i = 1, 2, 3,\\
&b < \alpha(x_1),\\
&a < \psi(x_2) \quad with\ \alpha(x_2) < b,\\
&\psi(x_3) < a.
\end{aligned}$$

Let $E = \mathbb{R}$. For the sake of convenience, we list the following hypotheses.

(H1) $\sup_{t\in J}\left(\int_0^t K(t,s)ds\right)\le 1$, $\sup_{t\in J}\left(\int_0^\infty H(t,s)ds\right)\le 1$ and

$$k^*=\sup_{t\in J}\left(e^{-t}\int_0^t K(t,s)e^s ds\right)<\infty,\quad h^*=\sup_{t\in J}\left(e^{-t}\int_0^\infty H(t,s)e^s ds\right)<\infty$$

Remark 1. Similar to [4, Lemma 1], if condition (H1) is satisfied, then the operators T and S defined by (2) are bounded linear operators from $BPC[J,\mathbb{R}_+]$ into itself. Moreover, $T(BPC[J,\mathbb{R}_+])\subset BPC[J,\mathbb{R}_+]$, $S(BPC[J,\mathbb{R}_+])\subset BPC[J,\mathbb{R}_+]$.

Assume there exist the function $\lambda\in C[J,\mathbb{R}_+]$ such that

$$0<\lambda^*=\int_0^\infty \lambda(t)dt<\infty,\quad \lambda(t)\ge\lambda_0>0$$

for some given positive number λ_0 and any $t\in[0,t_1]$. Moreover, assume that there exist positive constants η_{ik} $(i=0,1,\cdots,n-1;k=1,2,3,\cdots)$ with

$$\eta_i^*=\sum_{k=1}^{\infty}\eta_{ik}<\infty(i=0,1,\cdots,n-1).$$

Let

$$L=\frac{\rho}{\rho-1}(\lambda^*+\eta_{n-1}^*)+\sum_{i=0}^{n-2}\eta_i^*. \tag{4}$$

We assume ulteriorly there exist constants a,b,d,l,k_1,k_2 and m satisfying

$$\begin{cases} 0<l<t_1,\\ k_1=\max\left\{1,\frac{1}{l},\frac{2!}{l^2},\cdots,\frac{(n-1)!}{l^{n-1}}\right\},\\ k_2=\min\left\{1,\frac{1}{l},\frac{2!}{l^2},\cdots,\frac{(n-1)!}{l^{n-1}}\right\},\\ m>\max\{k_1,1\}, mk_2>k_1,\\ 0<a<b<\min\{\frac{Ld}{m},\frac{dl\lambda_0}{k_1}\}\end{cases} \tag{5}$$

such that

(H2) $f\in C[J\times\mathbb{R}_+\times\mathbb{R}_+\times\cdots\times\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}_+,\mathbb{R}_+]$, $I_{ik}\in C[\mathbb{R}_+\times\mathbb{R}_+\times\cdots\times\mathbb{R}_+,\mathbb{R}_+]$ $(i=0,1,\cdots,n-1,\ \ k=1,2,\cdots)$. For any $u\in DPC^{n-1}[J,\mathbb{R}_+]$ with $\|u^{(i)}\|\le Ld$ $(i=0,1,\cdots,n-1)$ and any $t\in J$, we have

$$|f(t,u(t),u'(t),\cdots,u^{(n-1)}(t),(Tu)(t),(Su)(t))|\le d\lambda(t),$$

in addition,

$$|I_{ik}(u(t_k),u'(t_k),\cdots,u^{(n-1)}(t_k))|\le d\eta_{ik}\quad(i=0,1,\cdots,n-1;k=1,2,3,\cdots).$$

(H3) For $b\le u_i\le Ld$ for $i=0,1,\cdots,n-2$, $b\le u_{n-1}\le mb$, $0\le u_n,u_{n+1}\le Ld$ and $t\in[0,l]$, we have

$$f(t,u_0,u_1,\cdots,u_{n+1})>\frac{k_1 b}{l}.$$

(H4) There exists $q_0 \in (l, \infty)$ such that, for all $t \in J$, $u \in DPC^{n-1}[J, \mathbb{R}_+]$ with $\sup_{t \in [0,q_0]} |u^{(i)}(t)| \leq a$ $(i = 0, 1, \cdots, n-1)$, f and I_{ik} satisfy, respectively,

$$|f(t, u(t), u'(t), \cdots, u^{(n-1)}(t), (Tu)(t), (Su)(t))| < a\delta c(t)$$

and

$$|I_{ik}(u(t_k), u'(t_k), \cdots, u^{(n-1)}(t_k))| \leq a\delta\mu_{ik} \quad (i = 0, 1, \cdots, n-1, k = 1, 2, \cdots),$$

where, $c \in C[J, \mathbb{R}_+]$ satisfies

$$c^* = \int_0^\infty c(t)dt < \infty,$$

the positive constants μ_{ik} $(i = 0, 1, \cdots, n-1, k = 1, 2, \cdots)$ satisfy

$$\mu_i^* = \sum_{k=1}^\infty \mu_{ik} < \infty,$$

$$\delta = e^{-q_0} \left[\frac{\rho}{\rho - 1} c^* + \frac{\rho}{\rho - 1} \mu_{n-1}^* + \sum_{i=0}^{n-2} \mu_i^* \right]^{-1}.$$

3 Main Results

Throughout this section we will work in the Banach space $DPC^{n-1}[J, \mathbb{R}]$ and our considerations are placed in the Banach space $DPC^{n-1}[J, \mathbb{R}]$ considered previously. Let us denote

$$P = DPC^{n-1}[J, \mathbb{R}_+].$$

For any $x, y \in DPC^{n-1}[J, \mathbb{R}]$, define $x \leq y$ if and only if $x(t) \leq y(t)$ for each $t \in J$, $x < y$ if and only if $x \leq y$ and there exists some $t \in J$ such that $x(t) \neq y(t)$.

Let $h = L^{-1}$. For $x \in P$ and the positive real number l given in (5), define

$$\gamma(x) = h\|x\|_D, \quad \theta(x) = \max_{t \in [0,l]} |x^{(n-1)}(t)|$$

$$\alpha(x) = \min \left\{ \min_{t \in [l,\infty)} |x^{(i)}(t)| : i = 0, 1, \cdots, n-1 \right\},$$

$$\psi(x) = \max \left\{ \sup_{t \in [0,q_0]} |x^{(i)}(t)| : i = 0, 1, \cdots, n-1 \right\}.$$

Remark 2. Distinctly, γ and θ are nonnegative continuous convex functions, α is the nonnegative continuous concave function and ψ is nonnegative continuous function on the cone P. Furthermore, from the fact that $x^{(i)} \geq 0$, we see that $x^{(i)}$ is increasing on J $(i = 0, 1, \cdots, n-1)$. This yields $\alpha(x) = \min \left\{ x^{(i)}(l) : i = 0, 1, \cdots, n-1 \right\} \leq \psi(x)$. Hence, condition (3) is satisfied. We also have that $\psi(\lambda x) = \lambda\psi(x)$ for $\lambda \in [0, 1]$ and $x \in P$.

Let us define that a function $u \in PC^{n-1}[J,\mathbb{R}] \cap C^n[J',\mathbb{R}]$ is called a nonnegative solution of BVP(1) if $u^{(i)}(t) \geq 0$ $(i = 0,1,\cdots,n-1)$ for $t \in J$ and $u(t)$ satisfies (1). A function $u \in PC^{n-1}[J,\mathbb{R}] \cap C^n[J',\mathbb{R}]$ is called a positive solution of BVP(1) if it is a nonnegative solution and $u(t) \not\equiv 0$.

Theorem 1. If the conditions (H1)-(H4) hold, then BVP(1) has at least three positive solutions x_1, x_2 and x_3 satisfying

$$\begin{aligned}
&\|x_i\|_D \leq Ld \quad \text{for } i = 1,2,3;\\
&b < \min\left\{\min_{t\in[l,\infty)} x_1^{(i)}(t) : i = 0,1,\cdots,n-1\right\};\\
&a < \max\left\{\sup_{t\in[0,q_0]} x_2^{(i)}(t) : i = 0,1,\cdots,n-1\right\}\\
&\text{with } \min\left\{\min_{t\in[l,\infty)} x_1^{(i)}(t) : i = 0,1,\cdots,n-1\right\} < b;\\
&\max\left\{\sup_{t\in[0,q_0]} x_3^{(i)}(t) : i = 0,1,\cdots,n-1\right\} < a.
\end{aligned}$$

Proof. Define a operator A as follows:

$$\begin{aligned}
(Au)(t) \;=\;& \frac{t^{n-1}}{(\rho-1)(n-1)!}\left\{\int_0^\infty f(s,u(s),u'(s),\cdots,u^{(n-1)}(s),(Tu)(s),(Su)(s))ds\right.\\
&\left.+\sum_{k=1}^{\infty} I_{n-1k}(u(t_k),u'(t_k),\cdots,u^{(n-1)}(t_k))\right\}\\
&+\frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} f(s,u(s),u'(s),\cdots,u^{(n-1)}(s),(Tu)(s),(Su)(s))ds\\
&+\sum_{0<t_k<t}\sum_{j=0}^{n-1}\frac{(t-t_k)^j}{j!} I_{jk}(u(t_k),u'(t_k),\cdots,u^{(n-1)}(t_k)), \quad \forall t \in J. \qquad (6)
\end{aligned}$$

[4, Lemma 3] has proved that $u \in DPC^{n-1}[J,E] \cap C^n[J',E]$ is a solution of BVP(1) if and only if u is a fixed point of A.

In what follows, we write $J_1 = [0,t_1]$, $J_k = (t_{k-1},t_k]$ for $k = 2,3,\cdots$.

We are now in a position to prove that the operator A has three fixed points by means of Lemma 1. To verify that all conditions of Lemma 1 hold, we shall divide this proof into three steps.

Step 1. We will prove that the operator A maps $\overline{P(\gamma,d)}$ into itself. Note that $(Au)(t) \geq 0$ for any $u \in \overline{P(\gamma,d)}$ and $t \in J$, also, differentiating (6) i times for $i = 0,1,\cdots,n-1$, we have

$$
\begin{aligned}
(A^{(i)}u)(t) \;=\; & \frac{t^{n-i-1}}{(\rho-1)(n-i-1)!}\left\{\int_0^\infty f(s,u(s),u'(s),\cdots,u^{(n-1)}(s),(Tu)(s),(Su)(s))ds\right.\\
& \left.+\sum_{k=1}^{\infty} I_{n-1k}(u(t_k),u'(t_k),\cdots,u^{(n-1)}(t_k))\right\}\\
& +\frac{1}{(n-i-1)!}\int_0^t (t-s)^{n-i-1} f(s,u(s),u'(s),\cdots,u^{(n-1)}(s),(Tu)(s),(Su)(s))ds\\
& +\sum_{0<t_k<t}\sum_{j=i}^{n-1}\frac{(t-t_k)^{j-i}}{(j-i)!} I_{jk}(u(t_k),u'(t_k),\cdots,u^{(n-1)}(t_k)),\quad \forall t\in J. \qquad (7)
\end{aligned}
$$

This shows $(Au)^{(i)}(t)\geq 0$.

For any $u\in \overline{P(\gamma,d)}$, from $\gamma(u)=h\|u\|_D\leq d$ and condition (H2) it follows that

$$|f(t,u_0(t),u_1(t),\cdots,u_{n+1}(t))|\leq d\lambda(t) \qquad (8)$$

and

$$|I_{ik}(u_0(t),u_1(t),\cdots,u_{n+1}(t))|\leq d\eta_{ik} \qquad (9)$$

for all $t\in J$. (8) and Remark 1 guarantee that the infinite integral

$$\int_0^\infty f(t,u(t),u'(t),\cdots,u^{(n-1)}(t),(Tu)(t),(Su)(t))dt$$

is convergent and

$$\int_0^\infty |f(t,u(t),u'(t),\cdots,u^{(n-1)}(t),(Tu)(t),(Su)(t))|dt\leq d\lambda^*. \qquad (10)$$

On the other hand, (9) and (H2) guarantee that the infinite series

$$\sum_{k=1}^{\infty} I_{ik}(u(t_k),u'(t_k),\cdots,u^{(n-1)}(t_k)) \quad (i=0,1,\cdots,n-1)$$

is convergent and

$$\sum_{k=1}^{\infty} |I_{ik}(u(t_k),u'(t_k),\cdots,u^{(n-1)}(t_k))|\leq d\eta_i^* \quad (i=0,1,\cdots,n-1) \qquad (11)$$

By (4), (7), (10) and (11), we have the following estimate:

$$
\begin{aligned}
& e^{-t}|(Au)^{(i)}(t)| \\
\leq\ & e^{-t}\left(\frac{\rho}{\rho-1}\right)\frac{t^{n-i-1}}{(n-i-1)!}\int_0^\infty |f(t,u(t),u'(t),\cdots,u^{(n-1)}(t),(Tu)(t),(Su)(t))|dt \\
& + e^{-t}\frac{t^{n-i-1}}{(\rho-1)(n-i-1)!}\sum_{k=1}^{\infty}|I_{n-1k}(u(t_k),u'(t_k),\cdots,u^{(n-1)}(t_k))| \\
& + e^{-t}\sum_{j=i}^{n-1}\frac{t^{j-1}}{(j-i)!}\sum_{0<t_r<t}|I_{ik}(u(t_k),u'(t_k),\cdots,u^{(n-1)}(t_k))| \\
\leq\ & \frac{\rho}{\rho-1}\int_0^\infty |f(t,u(t),u'(t),\cdots,u^{(n-1)}(t),(Tu)(t),(Su)(t))|dt \\
& + \frac{1}{(\rho-1)}\sum_{k=1}^{\infty}|I_{n-1k}(u(t_k),u'(t_k),\cdots,u^{(n-1)}(t_k))| \\
& + \sum_{j=i}^{n-1}\sum_{k=1}^{\infty}|I_{ik}(u(t_k),u'(t_k),\cdots,u^{(n-1)}(t_k))| \\
\leq\ & d\frac{\rho\lambda^*}{\rho-1}+\frac{d\eta^*_{n-1}}{\rho-1}+d\sum_{i=0}^{n-1}\eta_i^* = Ld
\end{aligned}
$$

for all $t\in J$ and $i=0,1,\cdots,n-1$. This implies A is bounded on $\overline{P(\gamma,d)}$ and

$$\|Au\|_D \leq Ld.$$

Hence we deduce that $\gamma(Au)\leq d$, i.e.,$A(\overline{P(\gamma,d)})\subset\overline{P(\gamma,d)}$.

Similar to the proof of [4, Lemma 2] , we can get that A is continuous. As a consequence of Arzela-Ascoli theorem we get that A is a completely continuous operator.

Step 2. To check condition (h1) of Lemma 1, we choose $k_0\in(k_1,mk_2)$ and $v(t)=\frac{k_0 b}{(n-1)!}t^{n-1}$, then $v^{(i)}(t)=\frac{k_0 b}{(n-i-1)!}t^{n-i-1}$. It is easy to see

$$b<v^{(i)}(l)=\frac{k_0 b}{(n-i-1)!}l^{n-i-1}<mb \quad (i=0,1,\cdots,n-1),$$

which implies that $\alpha(v)>b$ and $\theta(v)<mb$. Hence, $v\in P(\gamma,\theta,\alpha,b,mb,d)$ and $\{u\in(\gamma,\theta,\alpha,b,mb,d):\alpha(u)>b\}\neq\emptyset$. For any $u\in P(\gamma,\theta,\alpha,b,mb,d)$ and all $t\in[l,\infty)$, then $b\leq u^{(i)}(t)\leq Ld$ for $(i=0,1,\cdots,n-2)$, $b\leq u^{(n-1)}(t)\leq mb$ and $0\leq(Tu)(t),(Su)(t)\leq Ld$ (in virtue of (H1)). Since $Au\in P$, by Remark 2 we have $\alpha(Au)=\min\{(Au)^{(i)}(l):i=0,1,\cdots,n-1\}$. Assumption (H3) and (7) guarantee

$$
\begin{aligned}
(Au)^{(i)}(l) &= \frac{l^{n-i-1}}{(\rho-1)(n-i-1)!}\left\{\int_0^\infty f(s,u(s),u'(s),\cdots,u^{(n-1)}(s),(Tu)(s),(Su)(s))ds\right. \\
&\quad \left.+\sum_{k=1}^{\infty} I_{n-1k}(u(t_k),u'(t_k),\cdots,u^{(n-1)}(t_k))\right\} \\
&\quad +\frac{1}{(n-i-1)!}\int_0^l (l-s)^{n-i-1} f(s,u(s),u'(s),\cdots,u^{(n-1)}(s),(Tu)(s),(Su)(s))ds \\
&\geq \frac{1}{(n-i-1)!}\int_0^l (l-s)^{n-i-1} f(s,u(s),u'(s),\cdots,u^{(n-1)}(s),(Tu)(s),(Su)(s))ds \\
&> \frac{k_1 b}{l}\frac{1}{(n-i-1)!}\int_0^l (l-s)^{n-i-1}ds \\
&\geq \frac{k_1 b l^{n-i-1}}{(n-i-1)!} \geq b \quad (i=0,1,\cdots,n-1).
\end{aligned}
$$

This shows that condition (h1) is true.

Step 3. It remains to prove (in virtue of Lemma 1) that the conditions (h2) and (h3) hold.

We first check (h2). For any $u \in P(\gamma,\alpha,b,d)$ with $\theta(Au) > mb$. Note that $\theta(Au) = (Au)^{(n-1)}(l)$ and by (7) we have

$$
\begin{aligned}
(Au)^{(n-1)}(l) &= \frac{1}{(\rho-1)}\left\{\int_0^\infty f(s,u(s),u'(s),\cdots,u^{(n-1)}(s),(Tu)(s),(Su)(s))ds\right. \\
&\quad \left.+\sum_{k=1}^{\infty} I_{n-1k}(u(t_k),u'(t_k),\cdots,u^{(n-1)}(t_k))\right\} \\
&\quad +\int_0^l f(s,u(s),u'(s),\cdots,u^{(n-1)}(s),(Tu)(s),(Su)(s))ds \\
&> mb \geq k_1 b.
\end{aligned} \tag{12}
$$

So

$$
\begin{aligned}
(Au)^{(i)}(l) &= \frac{l^{n-i-1}}{(\rho-1)(n-i-1)!}\int_0^\infty f(s,u(s),u'(s),\cdots,u^{(n-1)}(s),(Tu)(s),(Su)(s))ds \\
&\quad +\frac{l^{n-i-1}}{(\rho-1)(n-i-1)!}\sum_{k=1}^{\infty} I_{n-1k}(u(t_k),u'(t_k),\cdots,u^{(n-1)}(t_k)) \\
&\quad +\frac{1}{(n-i-1)!}\int_0^l (l-s)^{n-i-1} f(s,u(s),u'(s),\cdots,u^{(n-1)}(s),(Tu)(s),(Su)(s))ds \\
&\geq \frac{l^{n-i-1}}{(n-i-1)!}\left\{\frac{1}{(\rho-1)}\int_0^\infty f(s,u(s),u'(s),\cdots,u^{(n-1)}(s),(Tu)(s),(Su)(s))ds\right. \\
&\quad +\frac{1}{\rho-1}\sum_{k=1}^{\infty} I_{n-1k}(u(t_k),u'(t_k),\cdots,u^{(n-1)}(t_k))
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^l f(s,u(s),u'(s),\cdots,u^{(n-1)}(s),(Tu)(s),(Su)(s))ds\Big\} \\
> &\frac{l^{n-i-1}}{(n-i-1)!}k_1 b \geq b \quad (i=0,1,\cdots,n-2).
\end{aligned}
$$

This, together with (12), implies that $\alpha(Au) > b$, (h2) is true.

Finally, we check condition (h3). Clearly, as $\psi(0) = 0 < a$, we have $0 \notin R(\gamma,\psi,a,d)$. Suppose that $x \in R(\gamma,\psi,a,d)$ with $\psi(x) = \max\{\sup_{t\in[0,q_0]} |x^{(i)}(t)| : i = 0,1,\cdots,n-1\} = \max\{x^{(i)}(q_0) : i = 0,1,\cdots,n-1\} = a$. By (7) and assumption (H4), we have

$$
\begin{aligned}
&\sup_{t\in[0,q_0]} |(Ax)^{(i)}(t)| = (Ax)^{(i)}(q_0) \\
= &\frac{q_0^{n-i-1}}{(\rho-1)(n-i-1)!}\Big\{\int_0^\infty f(s,u(s),u'(s),\cdots,u^{(n-1)}(s),(Tu)(s),(Su)(s))ds \\
&+\sum_{k=1}^{\infty} I_{n-1k}(u(t_k),u'(t_k),\cdots,u^{(n-1)}(t_k))\Big\} \\
&+\frac{1}{(n-i-1)!}\int_0^{q_0}(q_0-s)^{n-i-1}f(s,u(s),u'(s),\cdots,u^{(n-1)}(s),(Tu)(s),(Su)(s))ds \\
&+\sum_{0<t_k<q_0}\sum_{j=i}^{n-1}\frac{(q_0-t_k)^{j-i}}{(j-i)!}I_{jk}(u(t_k),u'(t_k),\cdots,u^{(n-1)}(t_k)) \\
\leq &\left(\frac{\rho}{\rho-1}\right)\frac{q_0^{n-i-1}}{(n-i-1)!}\int_0^\infty |f(t,u(t),u'(t),\cdots,u^{(n-1)}(t),(Tu)(t),(Su)(t))|dt \\
&+\frac{q_0^{n-i-1}}{(\rho-1)(n-i-1)!}\sum_{k=1}^{\infty}|I_{n-1k}(u(t_k),u'(t_k),\cdots,u^{(n-1)}(t_k))| \\
&+\sum_{j=i}^{n-1}\frac{q_0^{j-1}}{(j-i)!}\sum_{0<t_r<t}|I_{ik}(u(t_k),u'(t_k),\cdots,u^{(n-1)}(t_k))| \\
< &a\delta e^{q_0}\left[\frac{\rho}{\rho-1}\int_0^\infty c(t)dt + \frac{1}{\rho-1}\sum_{k=1}^{\infty}\mu_{n-1k} + \sum_{i=0}^{n-1}\sum_{k=1}^{\infty}\mu_{ik}\right] \\
\leq &a\delta e^{q_0}\left[\frac{\rho}{\rho-1}c^* + \frac{\rho}{\rho-1}\mu_{n-1}^* + \sum_{i=0}^{n-2}\mu_i^*\right] = a \quad (i=0,1,\cdots,n-1).
\end{aligned}
$$

which implies $\psi(Ax) < a$. So, condition (h3) holds.

Sum up the conclusions we obtain that BVP(1) has at least three solutions x_1, x_2 and x_3 satisfying Theorem 1. The proof is completed.

4 An Example

Consider the infinite system of scalar third order impulsive integro-differential equations

$$\begin{cases} u''(t) = f(t, u_0, u_1, u_2, u_3), \quad \forall t \in J, t \neq 2^k \ (k = 0, 1, 2, \cdots); \\ \Delta u|_{t=2^k} = 2^{-k}[u(2^k)]^2(15 + [u(2^k) + u'(2^k)]^2)^{-1}, \quad (k = 0, 1, 2, \cdots), \\ \Delta u'|_{t=2^k} = 4^{-k}[u'(2^k)]^{3/2}(5 + (u(2^k) + u'(2^k))^{3/2})^{-1}, \quad (k = 0, 1, 2, \cdots), \\ u(0) = 0, \ u'(\infty) = 2u'(0). \end{cases} \tag{13}$$

where

$$f(t, u_0, u_1, u_2, u_3) = \begin{cases} 18e^{-2t-8}g(t, u_0, u_1, u_2, u_3), & u_i < 8(i = 0, 1, 2, 3) \\ 18e^{-2t}e^{-2(10-u_0)(10-u_1)}g(t, u_0, u_1, u_2, u_3) & 8 \leq u_i < 10(i = 0, 1, 2, 3) \\ 18e^{-2t}g(t, u_0, u_1, u_2, u_3) & u_i \geq 10(i = 0, 1, 2, 3). \end{cases}$$

with

$$\begin{aligned} g(t, u_0, u_1, u_2, u_3) \ &= \ \left(3u(t) + 4u'(t) + 5\int_0^t e^{-(t+1)s}u(s)ds + 6\int_0^\infty e^{-2s}\sin^2(t-s)u(s)ds\right)^2 \\ &\cdot \left(1 + u(t) + u'(t) + \int_0^t e^{-(t+1)s}u(s)ds + \int_0^\infty e^{-2s}\sin^2(t-s)u(s)ds\right)^{-2}. \end{aligned}$$

Conclusion. BVP(13) has at least three positive solutions $x_1(t), x_2(t), x_3(t)$ such that

$$\|x_i\|_D \leq 2160 \quad \text{for } i = 1, 2, 3;$$

$$10 < \min\left\{\min_{t\in[l,\infty)} x_1^{(i)}(t) : i = 0, 1, \cdots, n-1\right\};$$

$$8 < \max\left\{\sup_{t\in[0,q_0]} x_2^{(i)}(t) : i = 0, 1, \cdots, n-1\right\} \text{ with } \min\left\{\min_{t\in[l,\infty)} x_1^{(i)}(t) : i = 0, 1, \cdots, n-1\right\} < 10;$$

$$\max\left\{\sup_{t\in[0,q_0]} x_3^{(i)}(t) : i = 0, 1, \cdots, n-1\right\} < 8.$$

Proof. Let $E = \mathbb{R}$, $P = \mathbb{R}_+$. Thus, (13) can be regarded as BVP of the form (1) in E. In this case, $K(t, s) = e^{-(t+s)s}$, $H(t, s) = e^{-2s}\sin^2(t-s)$, $t_{k+1} = 2^k$ $(k = 0, 1, 2, \cdots)$, $\rho = 2$, in which

$$\begin{aligned} &g(t, u_0, u_1, u_2, u_3) = \left(\frac{3u_0 + 4u_1 + 5u_2 + 6u_3}{1 + u_0 + u_1 + u_2 + u_3}\right)^2 \\ &\quad \forall t \in J = [0, \infty), u_i \geq 0(i = 0, 1, 2, 3), \\ &I_{0k}(u_0, u_1) = 2^{-k}u_0^2(15 + (u_0 + u_1)^2)^{-1}, \\ &I_{1k}(u_0, u_1) = 4^{-k}u_1^{3/2}(5 + (u_0 + u_1)^{3/2})^{-1}, \quad \forall u_0 \geq 0, u_1 \geq 0, (k = 0, 1, 2, \cdots). \end{aligned}$$

Obviously, $f \in C[J \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+]$, $I_{0k}, I_{1k} \in C[J, \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+]$. Moreover,

$$\int_0^t e^{-(t+1)s}ds = -\frac{e^{-(t+1)t}}{t+1} + \frac{1}{t+1} < 1, \quad \int_0^\infty e^{-2s}\sin^2(t-s)ds \leq \frac{1}{2},$$

$$k^* = \sup_{t\in J}\left(e^{-t}\int_0^t e^{-(t+1)s}e^s ds\right) \leq \sup_{t\in J}(te^{-t}) = e^{-1},$$

$$h^* = \left(e^{-t}\int_0^\infty e^{-2s}\sin^2(t-s)e^s ds\right) \leq \sup_{t\in J}(e^{-t}) = 1.$$

Hence, condition (H1) is satisfied. From the definitions of f, I_{0k} and I_{1k} we have

$$0 \leq f(t, u_0, u_1, u_2, u_3) \leq 648e^{-2t} \left(\frac{u_0 + u_1 + u_2 + u_3}{1 + u_0 + u_1 + u_2 + u_3} \right)^2 < 648e^{-2t}$$

for any $t \in J$, $u_i \geq 0$ $(i = 0, 1, 2, 3)$.

$$\begin{aligned} 0 \leq I_{0k}(u_0, u_1) \leq 2^{-k} \frac{(u_0 + u_1)^2}{15 + (u_0 + u_1)^2} \leq 2^{-k}, \\ 0 \leq I_{1k}(u_0, u_1) \leq 4^{-k} \frac{(u_0 + u_1)^{3/2}}{5 + (u_0 + u_1)^{3/2}} \leq 4^{-k} \end{aligned}$$

for any $u_0 \geq 0, u_1 \geq 0$ $(k = 0, 1, 2, \cdots)$.

We now take $\rho = 2, \lambda(t) = c(t) = e^{-2t}, \eta_{0k} == \mu_{0k} = 2^{-k}, \eta_{1k} = \mu_{1k} = 4^{-k}$, then $\lambda^* = c^* = \frac{1}{2}, \eta_0^* = \mu_0^* = 1, \eta_1^* = \mu_1^* = \frac{1}{3}, L = \frac{10}{3}$. Take $a = 8, b = 10, d = 648$, then the condition (H2) holds.

Take $l = \frac{1}{2}$, then $k_1 = 1, k_2 = \frac{1}{2}$. Take $m = 3$. Since $t_1 = 1$, $\lambda_0 = e^{-2}$. For $0 \leq t \leq \frac{1}{2}$ and $u_0 \geq 10, u_1 \geq 10, u_2 \geq 0, u_3 \geq 0$, we have

$$\begin{aligned} f(t, u_0, u_1, u_2, u_3) &\geq 18e^{-2t} \times 9 \left(\frac{u_0 + u_1 + u_2 + u_3}{1 + u_0 + u_2 + u_2 + u_3} \right)^2 \\ &\geq 72e^{-1} \left(\frac{20}{21} \right)^2 > 20 = \frac{k_1 b}{l}. \end{aligned}$$

This implies that the condition (H3) is true.

Take $q_0 = 1$, then $\delta = \frac{3}{10e}$. If $0 \leq u_0 \leq 8, 0 \leq u_1 \leq 8$, then $0 \leq u_2 \leq 8, 0 \leq u_3 \leq 4$. Thus, we get

$$\begin{aligned} f(t, u_0, u_1, u_2, u_3) &= 18e^{-2t-8} \left(\frac{3u_0 + 4u_1 + 5u_2 + 6u_3}{1 + u_0 + u_1 + u_2 + u_3} \right)^2 \\ &\leq 18e^{-2t-8} \left(\frac{120}{29} \right)^2 < \frac{24}{10e} e^{-2t} = a\delta c(t). \\ I_{0k}(u_0, u_1) &= 2^{-k} \frac{u_0^2}{15 + (u_0 + u_1)^2} \leq \frac{64}{79} \times 2^{-k} < a\delta\mu_{0k}, \\ I_{1k}(u_0, u_1) &= 4^{-k} \frac{u_1^{3/2}}{5 + (u_0 + u_1)^{3/2}} \leq \frac{8^{3/2}}{5 + 8^{3/2}} \times 4^{-k} < a\delta\mu_{1k}. \end{aligned}$$

So, the condition (H4) is satisfied. Consequently, our conclusion follows from Theorem 1.

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