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ON THE SOLVABILITY OF THE PERIODIC PROBLEM FOR SYSTEMS OF LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH REGULAR OPERATORS

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Abstract. Systems of two linear functional differential equations of the first order with regular operators are considered. General necessary and sufficient conditions for the unique solvability of the periodic problem are obtained. For one system with monotone operators we get effective necessary and sufficient conditions for the unique solvability of the periodic problem.

1. Introduction

We consider some classes of two-dimensional systems of first order linear functional differential equations with regular operators. General necessary and sufficient conditions for the solvability of the periodic problem for such classes are obtained. These conditions mean that some function on a set in a finite-dimensional space is positive (this functions is quadratic with respect to all variables). Moreover, in terms of norms of the operators appearing in the functional differential system, we get the necessary and sufficient conditions for the unique solvability of the periodic problem for one case of two-dimensional system with monotonic operators.

It is found there exist two domains of parameters corresponding to the unique solvability. These result do not have analogues for systems. Non-improvable results for periodic problem are known only for cyclic first order functional differential systems [32].

Necessary and sufficient conditions for the unique solvability of two-dimensional functional differential systems with *monotonic operators* were achieved only for the Cauchy problems in [37, 38, 39]. Here the similar problem is solved for periodic boundary conditions.

Some criteria for the solvability of the periodic problem for ordinary differential equations can be found, for example, in [2, 10, 14, 15, 16, 26]. The works [11, 12, 13, 17, 18, 20, 25, 36] are devoted to the investigation of the solvability conditions of the periodic problem for systems of ordinary differential equations. Conditions for the solvability of periodic problem for scalar functional differential equations are obtained in [8, 9, 19, 24, 27, 28, 29, 30, 31, 35]. Conditions for the solvability of

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the periodic problem for systems of functional differential equations are obtained in [7, 21, 22, 23, 32, 33, 34] (see also lists of literature in these articles).

All known conditions for the unique solvability were obtained with the help of some a priori estimates of solutions and fixed point theorems. In this paper it is proving that the unique solvability of periodic problem for all functional differential systems with regular operators from some class (where norms of positive and negative parts of operators are given) are equivalent to the existence only the trivial solutions for all systems from a corresponding class of systems with operators of simple structure. Every such an operator has the form

$$(Tx)(t) = p_1(t)x(\tau_1) + p_2(t)x(\tau_2),$$

where τ_1 and τ_2 are points from $[a, b]$, functions p_1 and p_2 are integrable. We can often get all solutions of functional differential systems with such operators in the explicit form. So, we have necessary and sufficient conditions for the solvability of the whole class of the original problems. In [3, 4, 5, 6] this approach is applied to other boundary value problems for functional differential equations and systems of such equations.

The main results are necessary and sufficient conditions for the unique solvability of the periodic problem (Theorem 7) for systems of two functional differential equations with regular operators and effective necessary and sufficient conditions of the unique solvability of the periodic problem for a system of functional differential equations with monotonic operators with given norms (Theorem 9, Corollaries 13, 15, 17).

Throughout the paper we use the following notation:

$\mathbb{R} \equiv (-\infty, \infty)$; $\mathbf{L}$ is the Banach space of integrable functions $z : [0, \omega] \to \mathbb{R}$ equipped with the norm $\| z\|_{\mathbf{L}} = \int_0^\omega |z(t)|\, dt$, any equalities and inequalities with functions from $\mathbf{L}$ are understood as equalities or inequalities almost everywhere on $[0, \omega]$; $\mathbf{C}$ is the Banach space of integrable functions $x : [0, \omega] \to \mathbb{R}$ equipped with the norm $\| x\|_{\mathbf{C}} = \max_{t\in[0,\omega]} |x(t)|$; $\mathbf{AC}$ is the Banach space of absolutely continuous functions $x : [0, \omega] \to \mathbb{R}$ with the norm $\| x\|_{\mathbf{AC}} = |x(0)| + \int_0^\omega |\dot{x}(t)|\, dt$; a linear operator $T : \mathbf{C} \to \mathbf{L}$ is called non-negative if it maps every non-negative continuous function into an almost everywhere non-negative function, the norm of such an operator T is defined by the equality

$$\| T\| = \int_0^\omega (T\mathbf{1})(t)\, dt,$$

where $\mathbf{1}$ is the unit function; an operator T is called monotonic if T or $-T$ is a non-negative operator; if an operator can be represented by the difference of non-negative operators, it is called regular; using the notation with a double index, for example $T^{+/-}$, means two propositions: one for T^+, another for T^-.

2. The periodic problem for systems of functional differential equations

Consider the periodic problem for a two-dimensional system of functional differential equations:

$$
\left\{\begin{array}{ll}
\dot{x}(t)=(T_{11}x)(t)+(T_{12}y)(t)+f_1(t), & t\in[0,\omega],\\
\dot{y}(t)=(T_{21}x)(t)+(T_{22}y)(t)+f_2(t), & t\in[0,\omega],\\
x(0)=x(\omega), \quad y(0)=y(\omega), &
\end{array}\right. \tag{1}
$$

where $T_{ij}=T_{ij}^{+}-T_{ij}^{-}$, $T_{ij}^{+/-}:\mathbf{C}\to\mathbf{L}$, $i,j=1,2$, are linear non-negative operators; the components x and y of the solution belong to the space of absolutely continuous functions $\mathbf{AC}$.

Boundary value problem (1) is called uniquely solvable if it has a unique solution for all f_1, $f_2\in\mathbf{L}$. It is well known that problem (1) has the Fredholm property (see, for example, [1, 40]). Therefore (1) is uniquely solvable if and only if the homogeneous problem

$$
\left\{\begin{array}{ll}
\dot{x}(t)=(T_{11}x)(t)+(T_{12}y)(t), & t\in[0,\omega],\\
\dot{y}(t)=(T_{21}x)(t)+(T_{22}y)(t), & t\in[0,\omega],\\
x(0)=x(\omega), \quad y(0)=y(\omega), &
\end{array}\right. \tag{2}
$$

has only the trivial solution.

The following assertion is a basic for finding of solvability conditions.

Lemma 1. *If problem* (2) *has a non-trivial solution, then the system*

$$
\left\{\begin{array}{l}
\dot{x}(t)=p_{11*}(t)x(\tau_1)+p_{11}^{*}(t)x(\tau_2)+p_{12*}(t)y(\theta_1)+p_{12}^{*}y(\theta_2),\ t\in[0,\omega],\\
\dot{y}(t)=p_{21*}(t)x(\tau_1)+p_{21}^{*}(t)x(\tau_2)+p_{22*}(t)y(\theta_1)+p_{22}^{*}(t)y(\theta_2),\ t\in[0,\omega],\\
x(0)=x(\omega), \quad y(0)=y(\omega),
\end{array}\right. \tag{3}
$$

has also a non-trivial solution for some points τ_1, τ_2, θ_1, $\theta_2\in[0,\omega]$ and for some functions p_{ij*}, $p_{ij}^{*}\in\mathbf{L}$ satisfying the conditions

$$
-T_{ij}^{-}\mathbf{1}\leqslant p_{ij*}\leqslant T_{ij}^{+}\mathbf{1}, \quad p_{ij*}+p_{ij}^{*}=T_{ij}^{+}\mathbf{1}-T_{ij}^{-}\mathbf{1}, \quad i,j=1,2. \tag{4}
$$

Proof. Suppose the homogeneous problem (2) has a non-trivial solution (x,y). Let

$$
\min_{t\in[0,\omega]}x(t)=x(\tau_1), \quad \max_{t\in[0,\omega]}x(t)=x(\tau_2), \quad \min_{t\in[0,\omega]}y(t)=y(\theta_1), \quad \max_{t\in[0,\omega]}y(t)=y(\theta_2).
$$

Using the inequalities

$$
x(\tau_1)\,\mathbf{1}\,(t)\leqslant x(t)\leqslant x(\tau_2)\,\mathbf{1}\,(t) \quad y(\theta_1)\,\mathbf{1}\,(t)\leqslant y(t)\leqslant y(\theta_2)\,\mathbf{1}\,(t), \quad t\in[0,\omega],
$$

and the non-negativeness of the operators T_{ij}^{+}, T_{ij}^{-}, from (2) we get the inequalities

$$
\begin{gathered}
T_{11}^{+}\mathbf{1}\,x(\tau_1)-T_{11}^{-}\mathbf{1}\,x(\tau_2)+T_{12}^{+}\mathbf{1}\,y(\theta_1)-T_{12}^{-}\mathbf{1}\,y(\theta_2)\leqslant\\
\leqslant\dot{x}\leqslant T_{11}^{+}\mathbf{1}\,x(\tau_2)-T_{11}^{-}\mathbf{1}\,x(\tau_1)+T_{12}^{+}\mathbf{1}\,y(\theta_2)-T_{12}^{-}\mathbf{1}\,y(\theta_1)
\end{gathered}
$$

and

$$
\begin{gathered}
T_{21}^{+}\mathbf{1}\,x(\tau_1)-T_{21}^{-}\mathbf{1}\,x(\tau_2)+T_{22}^{+}\mathbf{1}\,y(\theta_1)-T_{22}^{-}\mathbf{1}\,y(\theta_2)\leqslant\\
\leqslant\dot{y}\leqslant T_{21}^{+}\mathbf{1}\,x(\tau_2)-T_{21}^{-}\mathbf{1}\,x(\tau_1)+T_{22}^{+}\mathbf{1}\,y(\theta_2)-T_{22}^{-}\mathbf{1}\,y(\theta_1).
\end{gathered}
$$

Then for some function $\zeta : [0, \omega] \to [0, 1]$ we have

$$\dot{x} = (1-\zeta)\,\big(T_{11}^{+}\mathbf{1}\,x(\tau_1) - T_{11}^{-}\mathbf{1}\,x(\tau_2) + T_{12}^{+}\mathbf{1}\,y(\theta_1) - T_{12}^{-}\mathbf{1}\,y(\theta_2)\big) +$$
$$\zeta\,\big(T_{11}^{+}\mathbf{1}\,x(\tau_2) - T_{11}^{-}\mathbf{1}\,x(\tau_1) + T_{12}^{+}\mathbf{1}\,y(\theta_2) - T_{12}^{-}\mathbf{1}\,y(\theta_1)\big) =$$
$$p_{11*}x(\tau_1) + p_{11}^{*}x(\tau_2) + p_{12*}y(\theta_1) + p_{12}^{*}y(\theta_2),$$

where integrable functions p_{11*}, p_{11}^{*}, p_{12*}, p_{12}^{*} are defined by the equalities

$$p_{1j*} = (1-\zeta)T_{1j}^{+}\mathbf{1} - \zeta T_{1j}^{-}\mathbf{1}\,, \quad p_{1j}^{*} = \zeta T_{1j}^{+}\mathbf{1} - (1-\zeta)T_{1j}^{-}\mathbf{1}\,,\ j = 1, 2,$$

and for some function $\xi : [0, \omega] \to [0, 1]$ the following equalities hold:

$$\dot{y} = (1-\xi)\,\big(T_{21}^{+}\mathbf{1}\,x(\tau_1) - T_{21}^{-}\mathbf{1}\,x(\tau_2) + T_{22}^{+}\mathbf{1}\,y(\theta_1) - T_{22}^{-}\mathbf{1}\,y(\theta_2)\big) +$$
$$\xi\,\big(T_{21}^{+}\mathbf{1}\,x(\tau_2) - T_{21}^{-}\mathbf{1}\,x(\tau_1) + T_{22}^{+}\mathbf{1}\,y(\theta_2) - T_{22}^{-}\mathbf{1}\,y(\theta_1)\big) =$$
$$p_{21*}x(\tau_1) + p_{21}^{*}x(\tau_2) + p_{22*}y(\theta_1) + p_{22}^{*}y(\theta_2),$$

where

$$p_{2j*} = (1-\xi)T_{2j}^{+}\mathbf{1} - \xi T_{2j}^{-}\mathbf{1}\,, \quad p_{2j}^{*} = \xi T_{2j}^{+}\mathbf{1} - (1-\xi)T_{2j}^{-}\mathbf{1}\,, \quad j = 1, 2.$$

It is clear that the functions ζ and ξ are measurable and conditions (4) hold. □

The conditions for the solvability of problem (3) can be written in the explicit form. If every problem (3) under conditions (4) has only the trivial solution, then problem (2) has only the trivial solution, therefore, problem (1) is uniquely solvable.

Using Lemma 1, we get sufficient conditions for the unique solvability of (1). The inverse statement yields necessary conditions for the unique solvability of all systems with given $T_{ij}^{+/-}\mathbf{1}$ or $\| T_{ij}^{+/-} \|$, $i, j = 1, 2$.

Lemma 2. *Let non-negative functions* p_{ij}^{+}, $p_{ij}^{-} \in \mathbf{L}$, $i, j = 1, 2$, *be given. If problem* (3) *has a non-trivial solution for some* τ_1, τ_2, θ_1, $\theta_2 \in [0, \omega]$ *and for some functions* p_{ij*}, $p_{ij}^{*} \in \mathbf{L}$, $i, j = 1, 2$ *such that*

$$-p_{ij}^{-} \leqslant p_{ij*} \leqslant p_{ij}^{+}, \quad p_{ij*} + p_{ij}^{*} = p_{ij}^{+} - p_{ij}^{-}, \quad i, j = 1, 2,$$

then problem (1) *is not uniquely solvable for some operators* $T_{ij} = T_{ij}^{+} - T_{ij}^{-}$, *where* $T_{ij}^{+/-} : \mathbf{C} \to \mathbf{L}$ *are linear non-negative operators such that*

$$T_{ij}^{+/-}\mathbf{1} = p_{ij}^{+/-}, \quad i, j = 1, 2.$$

Proof. Define the linear operators $T_{ij}^{+/-} : \mathbf{C} \to \mathbf{L}$, $i, j = 1, 2$:

$$(T_{ij}^{+/-}x)(t) \equiv p_{ij*}^{+/-}(t)x(s_1) + (p_{ij}^{+/-}(t) - p_{ij*}^{+/-}(t))x(s_2), \quad t \in [0, \omega],$$

where

$$p_{ij*}^{+} = (|p_{ij*}| + p_{ij*})/2 \text{ and } p_{ij*}^{-} = (|p_{ij*}| - p_{ij*})/2$$

are the positive and negative parts of the function p_{ij*}, $s_k = \tau_k$ for $j = 1$, $s_k = \theta_k$ for $j = 2$, $k = 1, 2$. These operators are non-negative and $T_{ij}^{+/-}\mathbf{1} = p_{ij}^{+/-}$, $i, j = 1, 2$. A non-trivial solution of problem (3) is a solution of the homogeneous problem (2). Since problem (1) has the Fredholm property, we see that (1) is not uniquely solvable. □

From Lemmas 1 and 2, we get necessary and sufficient condition for the unique solvability of all functional differential systems from a given class.

Lemma 3. *Let non-negative functions* $p_{ij}^{+/-} \in \mathbf{L}$, $i,j = 1,2$, *be given. Then boundary value problem* (1) *is uniquely solvable for all linear non-negative operators* $T_{ij}^{+/-} : \mathbf{C} \to \mathbf{L}$ *such that* $T_{ij}^{+/-}\mathbf{1} = p_{ij}^{+/-}$, $i,j = 1,2$, *if and only if problem* (3) *has only the trivial solution for all* τ_1, τ_2, θ_1, $\theta_2 \in [0,\omega]$ *and all functions* p_{ij*}, $p_{ij}^{*} \in \mathbf{L}$, $i,j = 1,2$, *satisfying the following conditions*

$$-p_{ij}^{-} \leqslant p_{ij*} \leqslant p_{ij}^{+}, \quad p_{ij*} + p_{ij}^{*} = p_{ij}^{+} - p_{ij}^{-}, \quad i,j = 1,2. \tag{5}$$

Let non-negative numbers $\mathcal{T}_{ij}^{+/-}$, $i,j = 1,2$, *be given. Then problem* (1) *is uniquely solvable for all linear non-negative operators* $T_{ij}^{+/-} : \mathbf{C} \to \mathbf{L}$ *such that* $\| T_{ij}^{+/-} \| = \mathcal{T}_{ij}^{+/-}$, $i,j = 1,2$, *if and only if problem* (3) *has only the trivial solution for all* τ_1, τ_2, θ_1, $\theta_2 \in [a,b]$ *and for all functions* p_{ij*}, $p_{ij}^{*} \in \mathbf{L}$, $i,j = 1,2$ *satisfying* (5) *for all non-negative functions* p_{ij}^{+}, $p_{ij}^{-} \in \mathbf{L}$, $i,j = 1,2$, *with given norms* $\| p_{ij}^{+/-} \| = \mathcal{T}_{ij}^{+/-}$, $i,j = 1,2$.

Remark 4. In Lemma 3, it is sufficient to consider only the cases $\tau_1 < \tau_2$ and $\theta_1 < \theta_2$.

Remark 5. Obviously, Lemma 3 is valid not only for the periodic problem but for any boundary value problem.

In the following lemma we get a condition for the existence of a unique solution to the Fredholm problem (3). This condition gives a possibility to obtain criteria of the unique solvability of problem (1).

Lemma 6. *Problem* (3) *has a non-trivial solution if and only if*

$$\triangle \equiv \begin{vmatrix} \int_0^\omega p_{11*}\,ds & \int_0^\omega p_{11}^{*}\,ds & \int_0^\omega p_{12*}\,ds & \int_0^\omega p_{12}^{*}\,ds \\ -1 - \int_{\tau_1}^{\tau_2} p_{11*}\,ds & 1 - \int_{\tau_1}^{\tau_2} p_{11}^{*}\,ds & -\int_{\tau_1}^{\tau_2} p_{12*}\,ds & -\int_{\tau_1}^{\tau_2} p_{12}^{*}\,ds \\ \int_0^\omega p_{21*}\,ds & \int_0^\omega p_{21}^{*}\,ds & \int_0^\omega p_{22*}\,ds & \int_0^\omega p_{22}^{*}\,ds \\ -\int_{\theta_1}^{\theta_2} p_{21*}\,ds & -\int_{\theta_1}^{\theta_2} p_{21}^{*}\,ds & -1 - \int_{\theta_1}^{\theta_2} p_{22*}\,ds & 1 - \int_{\theta_1}^{\theta_2} p_{22}^{*}\,ds \end{vmatrix} = 0. \tag{6}$$

Proof. The periodic problem for the simplest system

$$\begin{cases} \dot{x} = f_1, \quad \dot{y} = f_2, \\ x(0) = x(\omega), \quad y(0) = y(\omega), \end{cases}$$

has a solution if and only if

$$\int_0^\omega f_1(s)\,ds = \int_0^\omega f_2(s)\,ds = 0.$$

In this case the solution is defined by the equalities

$$x(t) = x_0 + \int_0^t f_1(s)\,ds, \quad y(t) = y_0 + \int_0^t f_2(s)\,ds, \quad t \in [0,\omega],$$

for arbitrary constants x_0, y_0. Therefore, problem (3) has a solution (x, y) if and only if

$$\int_0^\omega p_{11*}\,ds\,x(\tau_1) + \int_0^\omega p_{11}^*\,ds\,x(\tau_2) + \int_0^\omega p_{12*}\,ds\,y(\theta_1) + \int_0^\omega p_{12}^*\,ds\,y(\theta_2) = 0, \tag{7}$$

$$\int_0^\omega p_{21*}\,ds\,x(\tau_1) + \int_0^\omega p_{21}^*\,ds\,x(\tau_2) + \int_0^\omega p_{22*}\,ds\,y(\theta_1) + \int_0^\omega p_{22}^*\,ds\,y(\theta_2) = 0. \tag{8}$$

Then x and y are defined by the equalities

$$x(t) = x(0) + \int_0^t p_{11*}\,ds\,x(\tau_1) + \int_0^t p_{11}^*\,ds\,x(\tau_2) + \int_0^t p_{12*}\,ds\,y(\theta_1) + \int_0^t p_{12}^*\,ds\,y(\theta_2),$$

$$y(t) = y(0) + \int_0^t p_{21*}\,ds\,x(\tau_1) + \int_0^t p_{21}^*\,ds\,x(\tau_2) + \int_0^t p_{22*}\,ds\,y(\theta_1) + \int_0^t p_{22}^*\,ds\,y(\theta_2), \quad t \in [0, \omega].$$

Hence,

$$x(\tau_1) = x(0) + \int_0^{\tau_1} p_{11*}\,ds\,x(\tau_1) + \int_0^{\tau_1} p_{11}^*\,ds\,x(\tau_2) + \int_0^{\tau_1} p_{12*}\,ds\,y(\theta_1) + \int_0^{\tau_1} p_{12}^*\,ds\,y(\theta_2), \tag{9}$$

$$x(\tau_2) = x(0) + \int_0^{\tau_2} p_{11*}\,ds\,x(\tau_1) + \int_0^{\tau_2} p_{11}^*\,ds\,x(\tau_2) + \int_0^{\tau_2} p_{12*}\,ds\,y(\theta_1) + \int_0^{\tau_2} p_{12}^*\,ds\,y(\theta_2), \tag{10}$$

$$y(\theta_1) = y(0) + \int_0^{\theta_1} p_{21*}\,ds\,x(\tau_1) + \int_0^{\theta_1} p_{21}^*\,ds\,x(\tau_2) + \int_0^{\theta_1} p_{22*}\,ds\,y(\theta_1) + \int_0^{\theta_1} p_{22}^*\,ds\,y(\theta_2), \tag{11}$$

$$y(\theta_2) = y(0) + \int_0^{\theta_2} p_{21*}\,ds\,x(\tau_1) + \int_0^{\theta_2} p_{21}^*\,ds\,x(\tau_2) + \int_0^{\theta_2} p_{22*}\,ds\,y(\theta_1) + \int_0^{\theta_2} p_{22}^*\,ds\,y(\theta_2). \tag{12}$$

Subtracting equality (9) from equality (10) and equality (11) from equality (12), we get

$$\left(-1-\int_{\tau_1}^{\tau_2} p_{11*}\,ds\right)x(\tau_1)+\left(1-\int_{\tau_1}^{\tau_2} p_{11}^{*}\,ds\right)x(\tau_2)-\int_{\tau_1}^{\tau_2} p_{12*}\,ds\,y(\theta_1)-\int_{\tau_1}^{\tau_2} p_{12}^{*}\,ds\,y(\theta_2)=0,$$

$$-\int_{\theta_1}^{\theta_2} p_{21*}\,ds\,x(\tau_1)-\int_{\theta_1}^{\theta_2} p_{21}^{*}\,ds\,x(\tau_2)+\left(-1-\int_{\theta_1}^{\theta_2} p_{22*}\,ds\right)y(\theta_1)+\left(1-\int_{\theta_1}^{\theta_2} p_{22}^{*}\,ds\right)y(\theta_2)=0.$$

Problem (3) has a non-trivial solution if and only if equalities (7), (8) hold and these two equations have a non-trivial solution with respect to the variables $x(\tau_1)$, $x(\tau_2)$, $y(\theta_1)$, $y(\theta_2)$, that is, if and only if equality (6) holds. $\square$

Now we can get a necessary and sufficient condition for the solvability of the periodic problem for all systems with the operators of given norms $T_{ij}^{+/-}$, $i,j=1,2$.

Theorem 7. *Let non-negative numbers $A^{+/-}$, $B^{+/-}$, $C^{+/-}$, $D^{+/-}$ be given. Periodic problem* (1) *is uniquely solvable for all non-negative operators $T_{ij}^{+/-}:\mathbf{C}\to\mathbf{L}$ such that*

$$\| T_{11}^{+/-}\| = A^{+/-},\ \| T_{12}^{+/-}\| = C^{+/-}, \| T_{21}^{+/-}\| = D^{+/-},\ \| T_{22}^{+/-}\| = B^{+/-},$$

if and only if

$$\triangle \equiv \begin{vmatrix} A^+-A^- & y_A+x_A & C^+-C^- & y_C+x_C \\ -(A_1^+-A_1^-) & 1-y_A & -(C_1^+-C_1^-) & -y_C \\ D^+-D^- & y_D+x_D & B^+-B^- & y_B+x_B \\ -(D_1^+-D_1^-) & -y_D & -(B_1^+-B_1^-) & 1-y_B \end{vmatrix} \neq 0, \tag{13}$$

for all variables $A_1^{+/-}$, $B_1^{+/-}$, $C_1^{+/-}$, $D_1^{+/-}$, x_A, x_B, x_C, x_D, y_A, y_B, y_C, y_D from the following sets:

$$A_1^{+/-}\in[0,A^{+/-}],\ B_1^{+/-}\in[0,B^{+/-}],\ C_1^{+/-}\in[0,C^{+/-}],\ D_1^{+/-}\in[0,D^{+/-}], \tag{14}$$

$$x_A\in[-A^-+A_1^-,A^+-A_1^+],\ x_B\in[-B^-+B_1^-,B^+-B_1^+], \tag{15}$$

$$x_C\in[-C^-+C_1^-,C^+-C_1^+],\ x_D\in[-D^-+D_1^-,D^+-D_1^+], \tag{16}$$

$$y_A\in[-A_1^-,A_1^+],\ y_B\in[-B_1^-,B_1^+],\ y_C\in[-C_1^-,C_1^+],\ y_D\in[-D_1^-,D_1^+]. \tag{17}$$

Remark 8. The problem on the necessary and sufficient conditions for the solvability of a class of functional differential equations is reduced to the problem on zeros of some algebraic function given on a finite dimensional set. This function is linear or quadratic with respect to every variable. Using the linearity of $\triangle$ with respect to x_A, y_A, x_B, y_B, x_C, y_C, x_D, y_D, we get that to check the conditions of Theorem 7 it is sufficient to prove that the determinants (13) conserve their sign for all $A_1^{+/-}$, $B_1^{+/-}$, $C_1^{+/-}$, $D_1^{+/-}$ satisfying (14) and for all other variables at the ends of segments in (15)–(17).

Proof. Add the second column of the determinant in (6) to the first column, and the forth column to the third. Using conditions (4), we get

$$\int_0^{\omega} (p_{ij*}(s) + p^*_{ij}(s))\, ds = V^+ - V^-,$$

$$\int_{s_1}^{s_2} (p_{ij*}(s) + p^*_{ij}(s))\, ds \equiv V_1^+ - V_1^-, \; V_1^{+/-} \in [0, V^{+/-}],$$

$$\int_{s_1}^{s_2} p^*_{ij}(s)\, ds \equiv y_V \in [-V_1^-, V_1^+],$$

$$\int_0^{\omega} p^*_{ij}(s)\, ds \equiv y_V + x_V, \; x_V \in [-(V^- - V_1^-), V^+ - V_1^+],$$

where $V = A$ if $(i,j) = (1,1)$, $V = C$ if $(i,j) = (1,2)$, $V = D$ if $(i,j) = (2,1)$, $V = B$ if $(i,j) = (2,2)$; $s_1 = \tau_1$, $s_2 = \tau_2$ if $i = 1$; $s_1 = \theta_1$, $s_2 = \theta_2$ if $i = 2$.

If $\tau_1 < \tau_2$ and $\theta_1 < \theta_2$ (it follows from Remark 4 that it is sufficient to consider only this case), the function $\triangle$, defined by equality (6), coincides with the function defined by equality (13). Using Lemmas 3 and 6 completes the proof. □

3. Systems with monotonic operators

Let all operators T_{ij}, $i,j = 1,2$, in problem (1) be monotonic. By various substitutes of dependent and independent variables, we can reduce problem (1) to one of two cases:

$$\begin{cases} \dot{x} = T_{11}x + T_{12}y + f_1, \\ \dot{y} = T_{21}x + T_{22}y + f_2, \\ x(0) = x(\omega), \quad y(0) = y(\omega), \end{cases} \tag{18}$$

$$\begin{cases} \dot{x} = T_{11}x + T_{12}y + f_1, \\ \dot{y} = T_{21}x - T_{22}y + f_2, \\ x(0) = x(\omega), \quad y(0) = y(\omega), \end{cases}$$

where every linear operator T_{ij}, $i,j = 1,2$, is non-negative.

Consider here problem (18) only. The following statement will be proved in § 4 with the help of Theorem 7. To prove Theorem 9 we will find extrema of $\triangle$ with respect to all variables. Two domains of the unique solvability have appeared. One of them corresponds to negative values of $\triangle$, the other to positive ones.

Theorem 9. *Let non-negative numbers A, B, C, D be given. The periodic problem* (18) *is uniquely solvable* for all *linear non-negative operators $T_{ij} : \mathbf{C} \to \mathbf{L}$ such that*

$$\| T_{11} \| = A, \quad \| T_{12} \| = C, \quad \| T_{21} \| = D, \quad \| T_{22} \| = B$$

if and only if either

$$\begin{gathered} 0 < A < 4, \quad 0 < B < 4, \\ C\,D < A\,B\,\min\left(\frac{1}{1+A}, 1 - \frac{A}{4}\right)\,\min\left(\frac{1}{1+B}, 1 - \frac{B}{4}\right) \end{gathered} \tag{19}$$

or

$$0 \leqslant A < 1,\; 0 \leqslant B < 1, \tag{20}$$

$$\frac{A\,B}{(1-A)\,(1-B)} < C\,D, \tag{21}$$

$$(C\,D)^2\,t^2(1-t)^2 + C\,D\,(A\,t^2 + B\,(1-t)^2 - 1) + A\,B < 0 \textit{ for all } t \in [0,1]. \tag{22}$$

Remark 10. Let inequalities (20) and (21) be fulfilled. Then the following condition is equivalent to inequality (22) from Theorem 9:

$$C\,D < \min_{t\in(0,1)} \frac{S(t) + \sqrt{S^2(t) - 4\,t^2(1-t)^2 A\,B}}{2t^2(1-t)^2},$$

where $S(t) = 1 - At^2 - B(1-t)^2$.

Remark 11. Let inequalities (20) and (21) be fulfilled.

Then inequality (22) holds if either

$$C\,D < 8\left(1 - \frac{1}{2}\max(A,B) + \sqrt{(1 - \frac{1}{2}\max(A,B))^2 - \frac{1}{4}A\,B}\right) \tag{23}$$

or

$$C\,D < 4\,(1 + \sqrt{1 - \max(A,B)})^2. \tag{24}$$

Proof. Inequality (22) holds if the inequality

$$(C\,D)^2\,t^2(1-t)^2 + C\,D\,(\max(A,B)(t^2 + (1-t)^2) - 1) + A\,B < 0 \tag{25}$$

holds for all $t \in [0,1]$. The left side of the latter inequality takes its maximum at $t = 0$ or $t = 1$ or $t = 1/2$. For $t = 0$ and $t = 1$ this inequality is equivalent the inequality

$$A\,B < C\,D\,(1 - \max(A,B)),$$

which is fulfilled if inequality (21) holds. For $t = 1/2$ inequality (25) holds if and only if inequality (23) and the inequality

$$C\,D > 8\left(1 - \frac{1}{2}\max(A,B) - \sqrt{(1 - \frac{1}{2}\max(A,B))^2 - \frac{1}{4}A\,B}\right) \tag{26}$$

hold. Inequality (26) is fulfilled if (21) holds.

It is easy to prove that inequality (24) implies (23). Therefore, inequality (24) implies inequality (22). □

From Theorem 9 and Remark 11, we obtain a simple sufficient condition for the solvability.

Corollary 12. *Let non-negative numbers A, B, C, D be given. Periodic problem* (18) *is uniquely solvable for all linear non-negative operators $T_{ij} : \mathbf{C} \to \mathbf{L}$ such that*

$$\| T_{11} \| = A, \quad \| T_{12} \| = C, \quad \| T_{21} \| = D, \quad \| T_{22} \| = B$$

if the following inequalities are fulfilled: (19) *or* (20), (21), (23), *or* (20), (21), (24).

Necessary and sufficient conditions for the unique solvability has the simplest form when $\| T_{11} \| = \| T_{22} \|$.

Corollary 13. *Let non-negative numbers A, C, D be given. Periodic problem* (18) *is uniquely solvable for all linear non-negative operators $T_{ij} : \mathbf{C} \to \mathbf{L}$ satisfying the conditions*

$$\| T_{11} \| = A, \quad \| T_{12} \| = C, \quad \| T_{21} \| = D, \quad \| T_{22} \| = A$$

if and only if the following inequalities hold:

$$0 < A < 4, \quad \sqrt{C\,D} < A \min\left(\frac{1}{1+A}, 1 - \frac{A}{4}\right)$$

or

$$0 \leqslant A < 1, \quad \frac{A}{1-A} < \sqrt{C\,D} < 2\,(1 + \sqrt{1-A}). \tag{27}$$

Remark 14. If the condition (27) holds, then $A < 3/4$.

Proof. Apply Theorem 9 for $B = A$. The left side of inequality (22) takes its maximum at $t = 0$ or $t = 1$ or $t = 1/2$. For $t = 0$ or $t = 1$ inequality (22) is equivalent to inequality

$$A^2 < C\,D\,(1-A),$$

which is valid if inequality (21) holds for $A = B$.

For $t = 1/2$ inequality (22) is equivalent to the inequality

$$\frac{1}{16}(C\,D)^2 + C\,D\,(\frac{1}{2}A - 1) + A^2 < 0,$$

which is valid if

$$\sqrt{C\,D} < 2\,(1 + \sqrt{1-A}) \text{ and } \sqrt{C\,D} > 2\,(1 - \sqrt{1-A}).$$

The latter inequality holds because

$$\frac{A}{1-A} > 2\,(1 - \sqrt{1-A})$$

for all $A \in [0, 1)$ and inequality (21) is fulfilled for $B = A$. So, the corollary is proved. □

Now with the help of Theorem 9 we write out the conditions for the unique solvability of (18) for the zero operator T_{22}.

Corollary 15. *Let non-negative numbers A, C, D are given. The periodic problem* (18) *is uniquely solvable for all linear non-negative operators $T_{ij} : \mathbf{C} \to \mathbf{L}$ such that*

$$\| T_{11}\| = A, \quad \| T_{12}\| = C, \quad \| T_{21}\| = D, \quad T_{22} = 0$$

if and only if

$$0 \leqslant A < 1, \quad 0 < C\,D < \frac{1 - A\,t^2}{t^2\,(1-t)^2} \text{ for all } t \in (0,1).$$

Remark 16. Under the conditions of Corollary 15, it is sufficient to check the last inequality only at t satisfying the equation $A\,t^3 - 2\,t + 1 = 0$ on the segment $t \in [0,1]$.

Corollary 12 yields simple sufficient conditions for the solvability of (18) with the zero operator T_{22}.

Corollary 17. *Let non-negative numbers A, C, D be given. Periodic problem* (18) *is uniquely solvable for all linear non-negative operators $T_{ij} : \mathbf{C} \to \mathbf{L}$ such that*

$$\| T_{11}\| = A, \quad \| T_{12}\| = C, \quad \| T_{21}\| = D, \quad T_{22} = 0$$

if

$$0 \leqslant A < 1, \quad 0 < C\,D < 16 - 8\,A.$$

4. The proof of Theorem 9

It follows from Theorem 7 that the periodic problem for all systems of equations from the given class is uniquely solvable if and only if

$$\triangle = \begin{vmatrix} \triangle_A & \triangle_C \\ \triangle_D & \triangle_B \end{vmatrix} \neq 0, \tag{28}$$

where

$$\triangle_Z = \begin{pmatrix} Z & y_Z + x_Z \\ -Z_1 & 1 - y_Z \end{pmatrix} \text{ if } Z = A \text{ or } Z = B,$$

$$\triangle_Z = \begin{pmatrix} Z & y_Z + x_Z \\ -Z_1 & -y_Z \end{pmatrix} \text{ if } Z = C \text{ or } Z = D,$$

for all Z_1, y_Z, x_Z from the following intervals:

$$Z_1 \in [0, Z], \quad Z \in \{A, B, C, D\}, \tag{29}$$

$$y_Z \in [0, Z_1], \quad Z \in \{A, B, C, D\}, \tag{30}$$

$$x_Z \in [0, Z - Z_1], \quad Z \in \{A, B, C, D\}. \tag{31}$$

The determinant $\triangle$ depends on all variables y_Z, x_Z linearly, therefore, it is sufficient to check that all determinants keep their signs for all values y_Z, x_Z at the ends of the intervals (30), (31).

If $y_Z = 0$, $x_Z = 0$ for all $Z \in \{A, B, C, D\}$,

$$\triangle = \begin{vmatrix} A & 0 & C & 0 \\ 0 & 1 & 0 & 0 \\ D & 0 & B & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = AB - CD.$$

Consider two cases: I) $AB - CD > 0$, II) $AB - CD < 0$. The determinant $\triangle$ is a function of the variables A_1, B_1, C_1, D_1 for all y_Z, x_Z at the ends of the segments (30), (31). Moreover, the dependence of all variables is linear or quadratic.

Case I: $AB - CD > 0$. It is necessary to check whether the minimum of $\triangle$ is positive for all y_Z, x_Z at the ends of the segments (30), (31). Clearly, if the coefficient of Z_1^2 in $\triangle$ is non-positive, the minimum with respect to Z_1 is taken either at $Z_1 = 0$ or at $Z_1 = Z$ (here $Z \in \{A, B, C, D\}$). Every matrix $\triangle_Z$ takes four values at the ends of the segments (30), (31), (29). Moreover, for every $Z \in \{A, B, C, D\}$, there are only two values $\triangle_Z$ for which the function $\triangle_Z$ of Z_1 is quadratic. We put

$$\triangle_Z^{(1)} = \begin{pmatrix} Z & Z \\ 0 & K \end{pmatrix}, \ \triangle_Z^{(2)} = \begin{pmatrix} Z & 0 \\ -Z & K \end{pmatrix},$$

$$\triangle_Z^{(3)} = \begin{pmatrix} Z & 0 \\ 0 & K \end{pmatrix}, \ \triangle_Z^{(4)} = \begin{pmatrix} Z & Z \\ -Z & K-Z \end{pmatrix},$$

$$\triangle_Z^{(5)} = \begin{pmatrix} Z & Z-Z_1 \\ -Z_1 & K \end{pmatrix}, \ \triangle_Z^{(6)} = \begin{pmatrix} Z & Z_1 \\ -Z_1 & K-Z_1 \end{pmatrix},$$

where $K = 1$ if $Z = A$ or $Z = B$ and $K = 0$ if $Z = C$ or $Z = D$.

Denote

$$r_{ijkm} = \begin{vmatrix} \triangle_A^{(i)} & \triangle_C^{(j)} \\ \triangle_D^{(k)} & \triangle_B^{(m)} \end{vmatrix}.$$

To prove that all determinants are positive it is sufficient to check that $r_{ijkm} > 0$ in the following cases only:

1) there is no dependence on Z_1 for all $Z \in \{A, B, C, D\}$, then $i, j, k, m \in \{1, 2, 3, 4\}$;

2) r_{ijkm} is quadratic with respect to A_1 only, then $i = 6$, $j, k, m \in \{1, 2, 3, 4\}$, or with respect to B_1 only, then $m = 6$, $i, j, k \in \{1, 2, 3, 4\}$;

3) r_{ijkm} is quadratic with respect to A_1 and B_1 only, then $i = 6$, $m = 6$, $j, k \in \{1, 2, 3, 4\}$;

4) r_{ijkm} is quadratic with respect to C_1 and D_1 only, then $(j, k) = (5, 6)$ or $(j, k) = (6, 5)$, $i, m \in \{1, 2, 3, 4\}$;

5) r_{ijkm} is quadratic with respect to A_1, C_1, and D_1 only, then $i = 6$, $m \in \{1, 2, 3, 4\}$, $(j, k) = (5, 6)$ or $(j, k) = (6, 5)$;

6) r_{ijkm} is quadratic with respect to B_1, C_1, and D_1 only, then $m = 6$, $i \in \{1, 2, 3, 4\}$, $(j, k) = (5, 6)$ or $(j, k) = (6, 5)$;

7) r_{ijkm} is quadratic with respect to all variables A_1, B_1, C_1, and D_1, then $(i, j, k, m) = (6, 5, 6, 6)$ or $(i, j, k, m) = (6, 6, 5, 6)$.

To obtain conditions for positiveness of every function r_{ijkm} is an elementary problem. The consideration of various symmetries can reduce the numbers of variants. We give only the main results of the computations.

In case 1) all determinants are positive if and only if

$$CD < \frac{AB}{(1+A)(1+B)}. \tag{32}$$

In case 4), using various changing of the variables C_1 and D_1, we see that the function r_{1561} is minimal. It is positive if inequality (32) holds.

In case 2), when r_{ijkm} is quadratic with respect to A_1 only, two functions $r_{6134} = -BA_1(A-A_1) - CD(1+B) + AB$ and $r_{6114} = -BA_1(A-A_1) - CD(B-1)(A_1 - 1) + AB$ can be minimal. The minimum of r_{6134} is taken at $A_1 = A/2$, therefore, $r_{6134} > 0$ if and only if

$$A < 4, \quad CD < \frac{B}{1+B} A(1 - A/4). \tag{33}$$

If r_{ijkm} is quadratic with respect to B_1 only, then all determinants are positive if

$$B < 4, \quad CD < \frac{A}{1+A} B(1 - B/4). \tag{34}$$

It is easy to check that if the conditions (33) and (34) are fulfilled, then $r_{6114} > 0$ for all A_1.

In case 3), when r_{ijkm} is quadratic with respect to A_1 and B_1, the function r_{6226} is positive if and only if

$$A < 4, \quad B < 4, \quad CD < A(1 - A/4)B(1 - B/4). \tag{35}$$

All rest determinants are positive if the conditions (33), (34), (35) are fulfilled.

In cases 5), 6), and 7), it can be shown by elementary methods that all determinants are positive if the conditions (33), (34), (35) are fulfilled. The most difficulties arise with proving the positiveness of r_{6566} (it is shown that $r_{6566} \geqslant r_{6226}$ if (35) holds) and r_{6561}, r_{6562} (it is proved that the minimum is taken at $C_1 = D_1$ and all functions are positive if (32) holds).

Obviously, the joint fulfilment of (32), (33), (34), and (35) are equivalent to condition (19) of the Theorem.

Consider case II: $A\,B - C\,D < 0$. It is necessary to check if the maximum of the determinants $\triangle$ are negative for all y_Z, x_Z at the ends of the segments (30), (31). Obviously, if the coefficient of Z_1^2 in $\triangle$ is non-negative, then the maximum with respect to Z_1 is taken either at $Z_1 = 0$ or at $Z_1 = Z$ (here $Z \in \{A, B, C, D\}$).

Therefore, it is necessary to prove the inequality $r_{ijkm} < 0$ in the following cases:

1) there is no dependence on Z_1 for all $Z \in \{A, B, C, D\}$, then $i, j, k, m \in \{1, 2, 3, 4\}$;

2) r_{ijkm} is quadratic with respect to A_1 only, then $i = 5$, $j, k, m \in \{1, 2, 3, 4\}$, or with respect to B_1, then $m = 5$, $i, j, k \in \{1, 2, 3, 4\}$;

3) r_{ijkm} is quadratic with respect to A_1 and B_1 only, then $i = 5$, $m = 5$, $j, k \in \{1, 2, 3, 4\}$;

4) r_{ijkm} is quadratic with respect to C_1 and D_1 only, then $(j, k) = (5, 5)$ or $(k, j) = (6, 6)$, $i, m \in \{1, 2, 3, 4\}$;

5) r_{ijkm} is quadratic with respect to A_1, C_1, and D_1, then $i = 5$, $m \in \{1, 2, 3, 4\}$, $(j, k) = (5, 5)$ or $(k, j) = (6, 6)$;

6) r_{ijkm} is quadratic with respect to B_1, C_1, and D_1, then $m = 5$, $i \in \{1, 2, 3, 4\}$, $(j, k) = (5, 5)$ or $(k, j) = (6, 6)$;

7) r_{ijkm} is quadratic with respect to all variables A_1, B_1, C_1, D_1, in this case $(i, j, k, m) = (5, 5, 5, 5)$ or $(i, j, k, m) = (5, 6, 6, 5)$.

In case 1) all determinants are negative if and only if

$$A < 1, \quad B < 1, \quad CD > \frac{AB}{(1-A)(1-B)}. \tag{36}$$

In case 2), when r_{ijkm} is quadratic with respect to A_1 only or with respect to B_1 only, the maximal functions r_{ijkm} are negative if inequality (36) is fulfilled.

In case 3), when r_{ijkm} is quadratic with respect to A_1 and B_1, the maximal function r_{5115} is also negative if inequality (36) is fulfilled.

In case 4), using various changing of the variables C_1 and D_1, we see that the maximal functions are r_{3553} and r_{3663}.

Denote $C_1 = Ck_C$, $D_1 = Dk_D$, where k_C, $k_D \in [0,1]$. Then

$$r_{3553} = (CD)^2 k_C(1-k_C)k_D(1-k_D) + \\ CD(-1 + Ak_C + Bk_D - k_D k_C(A+B)) + AB.$$

Changing the variable k_C to $1 - k_C$, we have

$$r_{3553} = (CD)^2 k_C(1-k_C)k_D(1-k_D) + \\ CD(-1 + A(1-(k_C+k_D)) + k_D k_C(A+B)) + AB.$$

So, for a given product $k_C k_D$, r_{3553} has the maximum when the value $k_C + k_D$ is minimal, that is at $k_C = k_D = k$, $k \in [0,1]$. Then

$$r_{3553} = (CD)^2(k(1-k))^2 + CD(-1 + A(1-k)^2 + Bk^2)) + AB.$$

Obviously, $r_{3553} < 0$ for all $k \in [0,1]$ if and only if for all $k \in (0,1)$ the inequality

$$v^-(k) < CD < v^+(k)$$

is fulfilled, where

$$v^{+/-}(k) \equiv \frac{S \pm \sqrt{S^2 - 4((1-k)k)^2 AB}}{2((1-k)k)^2}, \ S \equiv 1 - A(1-k)^2 - Bk^2.$$

It is easy to prove that for $k \in (0,1)$ and $A, B \in [0,1)$ the inequalities

$$S > 0, \quad S^2 \geqslant 4((1-k)k)^2 AB$$

hold. Let us show that for all $k \in (0,1)$ the inequality

$$v^-(k) \leqslant \frac{AB}{(1-A)(1-B)}. \tag{37}$$

is fulfilled.

Since

$$v^-(k) = \frac{2AB}{S + \sqrt{S^2 - 4(k(1-k))^2 AB}},$$

we see that inequality (37) is equivalent to the inequality

$$S + \sqrt{S^2 - 4(k(1-k))^2 AB} \geqslant 2(1-A)(1-B).$$

It is easy to show that the inequalities

$$S \geqslant (1-A)(1-B), \ \sqrt{S^2 - 4(k(1-k))^2 AB} \geqslant (1-A)(1-B)$$

are fulfilled for all $k \in [0,1]$. Inequality (37) is proved. Therefore, r_{3553} is positive provided (36) if and only if

$$CD < v^+(k) \tag{38}$$

for all $k \in (0,1)$.

Consider the function r_{3663}. We have

$$r_{3663} = (CD)^2 k_C(1-k_C)k_D(1-k_D) + CD(-1 + k_D k_C(A+B-AB)) + AB.$$

Hence for a given product $k_C k_D$ function r_{3663} takes its maximum at $k_C = k_D = k$. Then

$$r_{3663} = (CD)^2(k(1-k))^2 + CD(-1 + k^2(A+B-AB)) + AB.$$

Let us show that the maximum of r_{3663} with respect to $k \in [0,1]$ is not greater than the maximum of r_{3553}. It is sufficient to prove that at least one of the inequalities

$$k^2A + (1-k)^2B \geqslant k^2(A + zb - BA), \quad (1-k)^2A + k^2B \geqslant k^2(A + zb - BA)$$

is fulfilled for every $k \in [0,1]$. If neither of these inequalities hold, we have

$$(1 - 2k + 2k^2)(A+B) = (1-k)^2 + k^2(A+B) < k^2(A+B-BA).$$

Since it is impossible, we see that if $r_{3553} < 0$ for all parameters, then r_{3663} is negative.

In cases 5), 6), the maximal functions are r_{5553} and r_{5554}. In case 7), it is sufficient to prove the inequality $r_{5555} < 0$. In all these cases, it is easily shown that all determinants are negative if inequalities (36) and (38) hold.

This completes the proof of Theorem 9.

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