Calc. Var.
DOI 10.1007/s00526-015-0823-4

Calculus of Variations

Singular Poisson equations on Finsler–Hadamard manifolds

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Received: 15 September 2014 / Accepted: 29 December 2014

Abstract In the first part of the paper we study the reflexivity of Sobolev spaces on non-compact and not necessarily reversible Finsler manifolds. Then, by using direct methods in the calculus of variations, we establish uniqueness, location and rigidity results for singular Poisson equations involving the Finsler–Laplace operator on Finsler–Hadamard manifolds having finite reversibility constant.

Mathematics Subject Classification 58J05 · 53C60 · 58J60

1 Introduction

Elliptic problems on Riemannian manifolds have been intensively studied in the last decades. On one hand, deep achievements have been done in connection with the famous Yamabe

Dedicated to Professor Gheorghe Moroşanu on the occasion of his 65th birthday.

Communicated by L. Ambrosio.

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Published online: 23 January 2015

problem on Riemannian manifolds which can be transformed into an elliptic PDE involving the Laplace–Beltrami operator, see Aubin [3] and Hebey [18]. On the other hand, various anisotropic elliptic problems are discussed on Minkowski spaces $(\mathbb{R}^n, F)$ where $F \in C^2(\mathbb{R}^n, [0, \infty))$ is convex and the leading term is given by the non-linear Finsler–Laplace operator associated with the Minkowski norm F, see Alvino et al. [1], Bellettini and Paolini [7], Belloni et al. [8,15], and references therein. In both classes of problems variational arguments are applied, the key roles being played by fine properties of Sobolev spaces as well as the lower semicontinuity of the energy functional associated to the studied problems.

In order to have a global approach, the theory of Sobolev spaces has been deeply investigated on metric measure spaces, see Ambrosio et al. [2], Cheeger [9], and Hajlasz and Koskela [17]. In [2], the authors proved that if (X, d) is doubling and separable, and m is finite on bounded sets, the Sobolev space $W^{1,2}(X, \mathsf{d}, \mathsf{m})$ is *reflexive*; here, $W^{1,2}(X, \mathsf{d}, \mathsf{m})$ contains functions $u \in L^2(X, \mathsf{m})$ with finite 2-relaxed slope endowed by the norm $u \mapsto \left(\int_X |\nabla u|_{*,2}^2 \mathrm{d}\mathsf{m} + \int_X u^2 \mathrm{d}\mathsf{m}\right)^{1/2}$, where $|\nabla u|_{*,2}(x)$ denotes the 2-relaxed slope of u at $x \in X$. This result clearly applies for differential structures. Indeed, if (M, F) is a reversible Finsler manifold (in particular, a Riemannian manifold), then for every $x \in M$ and $u \in C_0^\infty(M)$,

$$|\nabla u|_{*,2}(x) = \limsup_{z \to x} \frac{|u(z) - u(x)|}{d_F(x, z)} = F^*(x, Du(x)),$$

where d_F is the metric function associated with F, and F^* is the polar transform of F, see Ohta and Sturm [24]. Consequently, within the class of reversible Finsler manifolds, the synthetic notion of Sobolev spaces on metric measure spaces (see [2,9]) and the analytic notion of Sobolev spaces on Finsler manifolds (see Ge and Shen [16], and Ohta and Sturm [24]) coincide.

Although in the aforementioned works the involved metrics are symmetric, *asymmetry* is abundant in real life. In order to describe such phenomena, we put ourselves into the context of not necessarily reversible Finsler manifolds which model various Randers-type spaces, including the Matsumoto mountain slope metric, Finsler–Poincaré ball model, etc.; see Bao et al. [5]. If M is a connected n-dimensional C^∞ manifold and $TM = \bigcup_{x \in M} T_x M$ is its tangent bundle, the pair (M, F) is a *Finsler manifold* if the continuous function $F : TM \to [0, \infty)$ satisfies the conditions:

(a) $F \in C^\infty(TM \setminus \{0\})$;
(b) $F(x, ty) = tF(x, y)$ for all $t \geq 0$ and $(x, y) \in TM$;
(c) $g_{ij}(x, y) := \left[\frac{1}{2} F^2(x, y)\right]_{y^i y^j}$ is positive definite for all $(x, y) \in TM \setminus \{0\}$.

If $F(x, ty) = |t| F(x, y)$ for all $t \in \mathbb{R}$ and $(x, y) \in TM$, we say that the Finsler manifold (M, F) is reversible.

Let (M, F) be a Finsler manifold. Although it is possible to use an arbitrary measure on (M, F) to define Sobolev spaces (see [24]), here and in the sequel, we shall use the canonical Hausdorff measure on (M, F),

$$\mathrm{d}\mathsf{m} = \mathrm{d}V_F,$$

see Sect. 2. Having this measure in our mind, we consider the Sobolev spaces associated with (M, F), see [16,24]. To be more precise, let

$$W^{1,2}(M, F, \mathsf{m}) = \left\{ u \in W^{1,2}_{\mathrm{loc}}(M) : \int_M F^{*2}(x, Du(x)) \mathrm{d}\mathsf{m}(x) < +\infty \right\},$$

and $W_0^{1,2}(M, F, \mathsf{m})$ be the closure of $C_0^\infty(M)$ with respect to the (asymmetric) norm

$$\|u\|_F = \left(\int_M F^{*2}(x, Du(x))\mathsf{dm}(x) + \int_M u^2(x)\mathsf{dm}(x)\right)^{1/2}. \tag{1.1}$$

Let

$$r_F = \sup_{x\in M} \sup_{y\in T_xM\setminus\{0\}} \frac{F(x,y)}{F(x,-y)}$$

be the reversibility constant on (M, F). Clearly, $r_F \geq 1$ and $r_F = 1$ if and only if (M, F) is reversible. Let

$$F_s(x, y) = \left(\frac{F^2(x,y) + F^2(x,-y)}{2}\right)^{1/2}, \quad (x, y) \in TM.$$

It is clear that (M, F_s) is a reversible Finsler manifold, F_s being the *symmetrized* Finsler metric associated with F. We notice that the symmetrized Finsler metric associated with F^* may be different from F_s^*, i.e., in general $2F_s^{*2}(x,\alpha) \neq F^{*2}(x,\alpha) + F^{*2}(x,-\alpha)$; such a concrete case is shown for Randers metrics, see (2.11).

Our first result reads as follows:

Theorem 1.1 *Let (M, F) be a complete, n-dimensional Finsler manifold such that $r_F < +\infty$. Then $(W_0^{1,2}(M, F, \mathsf{m}), \|\cdot\|_{F_s})$ is a reflexive Banach space, while the norm $\|\cdot\|_{F_s}$ and the asymmetric norm $\|\cdot\|_F$ are equivalent. In particular,*

$$\left(\frac{1+r_F^2}{2}\right)^{-1/2} \|u\|_F \leq \|u\|_{F_s} \leq \left(\frac{1+r_F^{-2}}{2}\right)^{-1/2} \|u\|_F, \quad \forall u \in W_0^{1,2}(M, F, \mathsf{m}). \tag{1.2}$$

For sake of clarity, we notice that the norm $\|\cdot\|_{F_s}$ is considered also with respect to the Hausdorff measure $\mathsf{dm} = \mathrm{d}V_F$ (and not with $\mathrm{d}V_{F_s}$), i.e.,

$$\|u\|_{F_s} = \left(\int_M F_s^{*2}(x, Du(x))\mathsf{dm}(x) + \int_M u^2(x)\mathsf{dm}(x)\right)^{1/2}. \tag{1.3}$$

Some remarks are in order concerning Theorem 1.1.

Remark 1.1 (i) We emphasize that Theorem 1.1 is sharp. Indeed, let us consider the two-dimensional Finsler–Poincaré model $(B^2(0, 2), F)$ which is a forward (but not backward) complete Finsler manifold of Randers-type having the reversibility constant $r_F = +\infty$, see Sect. 3. In this framework, we shall construct a function $u \in W_0^{1,2}(B^2(0, 2), F, \mathsf{m})$ such that $-u \notin W_0^{1,2}(B^2(0, 2), F, \mathsf{m})$; in other words, $W_0^{1,2}(B^2(0, 2), F, \mathsf{m})$ *does not have* a vector space structure, and the norm $\|\cdot\|_{F_s}$ and the asymmetric norm $\|\cdot\|_F$ are not equivalent. A similar pathological situation has been already pointed out by Kristály and Rudas [19] for a Funk-type metric on the open unit ball of $\mathbb{R}^n$.

(ii) It is clear that $r_F < +\infty$ whenever (M, F) is a *compact* Finsler manifold. Thus, the reflexivity of $W_0^{1,2}(M, F, \mathsf{m})$ in [16,24] immediately follows from Theorem 1.1.

(iii) We believe that $(W_0^{1,2}(M, F, \mathsf{m}), \|\cdot\|_F)$ is a reflexive, complete asymmetric vector space; a possible proof requires a long series of arguments from functional analysis for asymmetric normed spaces, see Cobzaş [11]. However, the statement of Theorem 1.1 is enough for our purposes.

In the second part we consider that (M, F) is an n-dimensional Finsler–Hadamard manifold (i.e., simply connected, complete with non-positive flag curvature), $n \geq 3$, having its uniformity constant $l_F > 0$ (which implies in particular that $r_F < +\infty$), see Sect. 2. We shall study the model singular Poisson equation

$$\begin{cases} \boldsymbol{\Delta}(-u) - \mu \frac{u}{d_F^2(x_0,x)} = 1 & \text{in} \quad \Omega; \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \qquad (\mathcal{P}_\Omega^\mu)$$

where $\boldsymbol{\Delta}$ denotes the Finsler–Laplace operator on (M, F), $x_0 \in \Omega$ is fixed, $\mu \geq 0$ is a parameter, and $\Omega \subset M$ is an open and bounded domain with sufficiently smooth boundary. We prove that the singular energy functional associated with problem $(\mathcal{P}_\Omega^\mu)$ is *strictly convex* on $W_0^{1,2}(\Omega, F, \mathsf{m})$ whenever $\mu \in [0, l_F r_F^{-2} \overline{\mu})$, see Theorem 4.1; here, $\overline{\mu} = \frac{(n-2)^2}{4}$ is the optimal Hardy constant. By exploiting Theorem 1.1, a comparison principle for the Finsler–Laplace operator and well known arguments from calculus of variations, we prove (see also Theorem 5.1):

Theorem 1.2 *Problem* $(\mathcal{P}_\Omega^\mu)$ *has a unique, non-negative weak solution whenever* $\mu \in [0, l_F r_F^{-2} \overline{\mu})$.

Having the uniqueness theorem in our mind, we focus our attention to geometric rigidities related to the Poisson equation $(\mathcal{P}_\Omega^\mu)$. To do this, let $c \leq 0$ and the function $\mathbf{ct}_c : (0, \infty) \to \mathbb{R}$ defined by

$$\mathbf{ct}_c(r) = \begin{cases} \frac{1}{r} & \text{if} \quad c = 0, \\ \sqrt{-c}\coth(\sqrt{-c}r) & \text{if} \quad c < 0. \end{cases}$$

For every $\mu \in [0, \overline{\mu})$, $\rho > 0$ and $c \leq 0$, we consider the ordinary differential equation

$$\begin{cases} f''(r) + (n-1)f'(r)\mathbf{ct}_c(r) + \mu \frac{f(r)}{r^2} + 1 = 0, & r \in (0, \rho], \\ f(\rho) = 0, \quad \int_0^\rho f'(r)^2 r^{n-1} \mathrm{d}r < \infty. \end{cases} \qquad (\mathcal{Q}_{c,\rho}^\mu)$$

We shall show that $(\mathcal{Q}_{c,\rho}^\mu)$ has a unique, non-negative non-increasing solution $\sigma_{\mu,\rho,c} \in C^\infty(0, \rho)$, see Proposition 5.2. Although we are not able to solve explicitly $(\mathcal{Q}_{c,\rho}^\mu)$, in some particular cases we have its solution; namely,

$$\sigma_{\mu,\rho,c}(r) = \begin{cases} \frac{1}{\mu+2n}\left(\rho^2\left(\frac{r}{\rho}\right)^{-\sqrt{\overline{\mu}}+\sqrt{\overline{\mu}-\mu}} - r^2\right) & \text{if} \quad c=0; \\ \int_r^\rho \sinh(\sqrt{-c}s)^{-n+1}\int_0^s \sinh(\sqrt{-c}t)^{n-1}\mathrm{d}t\mathrm{d}s & \text{if} \quad c < 0 \text{ and } \mu = 0; \\ H(\sqrt{\overline{\mu}-\mu}, \rho)\frac{\sqrt{r}\sinh(\rho) I_{\sqrt{\overline{\mu}-\mu}}(r)}{\sqrt{\rho}\sinh(r) I_{\sqrt{\overline{\mu}-\mu}}(\rho)} - H(\sqrt{\overline{\mu}-\mu}, r) & \text{if} \quad c=-1,\ n=3 \text{ and } \mu \in [0, \frac{1}{4}), \end{cases}$$

where $H : (0, \frac{1}{2}] \times (0, \infty) \to \mathbb{R}$ is given by

$$H(\nu, r) = \frac{2\nu}{(25 - 4\nu^2)\sin(\nu\pi)\Gamma(\nu)\sinh(r)}$$
$$\times \Bigg\{ (5 - 2\nu)\,{}_3F_4\left(\left[\frac{3}{4} + \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}, \frac{5}{4} + \frac{\nu}{2}\right]; \left[\frac{3}{2}, 1 + \nu, \frac{3}{2} + \nu, \frac{9}{4} + \frac{\nu}{2}\right], r^2\right)$$
$$\times \left(2^{\nu-2}\sin(\nu\pi)K_\nu(r) + 2^{-\nu-1}\pi I_\nu(r)\right) r^{3+\nu}$$
$$-\nu(5+2\nu)2^{\nu-1}{}_3F_4\left(\left[\frac{3}{4} - \frac{\nu}{2}, \frac{5}{4} - \frac{\nu}{2}, \frac{5}{4} - \frac{\nu}{2}\right]; \left[\frac{3}{2}, 1 - \nu, \frac{3}{2} - \nu, \frac{9}{4} - \frac{\nu}{2}\right], r^2\right)$$
$$\times \Gamma(\nu)^2 \sin(\nu\pi) I_\nu(r) r^{3-\nu} \Bigg\};$$

here, I_ν and K_ν are the modified Bessel functions of the first and second kinds of order ν, while ${}_3F_4$ denotes the generalized hypergeometric function.

Before to state our rigidity results, we need some notations: $\mathbf{K} \leq c$ (resp. $c \leq \mathbf{K}$, resp. $\mathbf{K} = c$) means that the flag curvature $\mathbf{K}(\mathcal{S}; v)$ on (M, F) is bounded from above by $c \in \mathbb{R}$ (resp. bounded from below by c, resp. equal to c) for any choice of parameters $\mathcal{S}$ and v; $\mathbf{S} = 0$ means that (M, F) has vanishing mean covariation; $B^+(x_0, \rho)$ denotes the open forward metric ball with center $x_0 \in M$ and radius $\rho > 0$; for details, see Sect. 2.

Theorem 1.3 (Local estimate via curvature) *Let (M, F) be an n-dimensional $(n \geq 3)$ Finsler–Hadamard manifold with $\mathbf{S} = 0$ and $l_F > 0$, and $\Omega \subset M$ be an open bounded domain. Let $\mu \in [0, l_F r_F^{-2}\overline{\mu})$ and $x_0 \in \Omega$ be fixed. If $c_1 \leq \mathbf{K} \leq c_2 \leq 0$, then the unique weak solution u of problem $(\mathcal{P}^\mu_\Omega)$ verifies the inequalities*

$$\sigma_{\mu,\rho_1,c_1}(d_F(x_0, x)) \leq u(x) \leq \sigma_{\mu,\rho_2,c_2}(d_F(x_0, x)) \quad \text{for a.e.} \quad x \in B^+(x_0, \rho_1),$$

where $\rho_1 = \sup\{\rho > 0 : B^+(x_0, \rho) \subset \Omega\}$ and $\rho_2 = \inf\{\rho > 0 : \Omega \subset B^+(x_0, \rho)\}$.

In particular, if $\mathbf{K} = c \leq 0$ and $\Omega = B^+(x_0, \rho)$ for some $\rho > 0$, then $\sigma_{\mu,\rho,c}(d_F(x_0, \cdot))$ is the unique weak solution of problem $(\mathcal{P}^\mu_{B^+(x_0,\rho)})$, being also a pointwise solution in $B^+(x_0, \rho) \setminus \{x_0\}$.

A kind of converse statement of Theorem 1.3 can read as follows.

Theorem 1.4 (Radial curvature rigidity) *Let (M, F) be an n-dimensional $(n \geq 3)$ Finsler–Hadamard manifold with $\mathbf{S} = 0$, $l_F > 0$ and $\mathbf{K} \leq c \leq 0$. Let $\mu \in [0, l_F r_F^{-2}\overline{\mu})$ and $x_0 \in M$ be fixed. If the function $\sigma_{\mu,\rho,c}(d_F(x_0, \cdot))$ is the unique pointwise solution of $(\mathcal{P}^\mu_{B^+(x_0,\rho)})$ in $B^+(x_0, \rho) \setminus \{x_0\}$ for some $\rho > 0$, then $\mathbf{K}(\cdot; \dot{\gamma}_{x_0,y}(t)) = c$ for every $t \in [0, \rho)$ and $y \in T_{x_0}M \setminus \{0\}$, where $\gamma_{x_0,y}$ is the constant speed geodesic with $\gamma_{x_0,y}(0) = x_0$ and $\dot{\gamma}_{x_0,y}(0) = y$.*

In the generic Finsler setting, the conclusion of Theorem 1.4 does not imply necessarily that the flag curvature $\mathbf{K}$ is constant. Indeed, we just stated that the flag curvature is *radially* constant with respect to $x_0 \in M$, i.e., along geodesics emanating from the point x_0 where the flag-poles are the velocities of the geodesics. However, when $(M, F) = (M, g)$ is a *Riemannian* manifold of Hadamard type (thus the flag curvature and sectional curvature coincide and the notion of the flag loses its meaning), Theorems 1.3 and 1.4 and the classification of Riemannian space forms (see do Carmo [12, Theorem 4.1]) provide a *characterization* of the Euclidean and hyperbolic spaces up to isometries via the shape of solutions to the Poisson equation $(\mathcal{P}^\mu_\Omega)$:

Corollary 1.1 (Space forms vs. Poisson equation) *Let (M, g) be a Riemannian–Hadamard manifold with sectional curvature bounded above by $c \leq 0$. Then the following statements are equivalent:*

(a) *For some $\mu \in [0, \overline{\mu})$ and $x_0 \in M$, the function $\sigma_{\mu,\rho,c}(d_F(x_0, \cdot))$ is the unique pointwise solution of the Poisson equation $(\mathcal{P}^{\mu}_{B(x_0,\rho)})$ in $B(x_0, \rho) \setminus \{x_0\}$ for every $\rho > 0$;*
(b) *(M, g) is isometric to the n-dimensional space form with curvature c.*

A full classification of Finslerian space forms (i.e., the flag curvature is constant) is not available; however, the following characterization can be provided on *Berwald spaces*:

Theorem 1.5 (Full curvature rigidity) *Let (M, F) be an n-dimensional $(n \geq 3)$ Finsler–Hadamard manifold of Berwald type with $l_F > 0$. Then the following statements are equivalent:*

(a) *For every $\mu \in [0, l_F r_F^{-2} \overline{\mu})$ and $x_0 \in M$, the function $\sigma_{\mu,\rho,0}(d_F(x_0, \cdot))$ is the unique pointwise solution of the Poisson equation $(\mathcal{P}^{\mu}_{B^+(x_0,\rho)})$ in $B^+(x_0, \rho) \setminus \{x_0\}$ for every $\rho > 0$;*
(b) *For some $\mu \in [0, l_F r_F^{-2} \overline{\mu})$ and $x_0 \in M$, the function $\sigma_{\mu,\rho,0}(d_F(x_0, \cdot))$ is the unique pointwise solution of the Poisson equation $(\mathcal{P}^{\mu}_{B^+(x_0,\rho)})$ in $B^+(x_0, \rho) \setminus \{x_0\}$ for every $\rho > 0$;*
(c) *(M, F) is isometric to an n-dimensional Minkowski space.*

2 Preliminaries: elements from Finsler geometry

2.1 Finsler manifolds, geodesics, flag curvature, mean covariation, volume element

Let (M, F) be a Finsler manifold (i.e., (a)–(c) hold from the Introduction). If $g_{ij}(x) = g_{ij}(x, y)$ is independent of y then $(M, F) = (M, g)$ is called a *Riemannian manifold*. A *Minkowski space* consists of a finite dimensional vector space V (usually, identified with $\mathbb{R}^n$) and a Minkowski norm which induces a Finsler metric on V by translation, i.e., $F(x, y)$ is independent on the base point x; in such cases we often write $F(y)$ instead of $F(x, y)$.

A specific non-reversible Finsler structure is provided by *Randers metrics* which will serve to us as a model case. To be more precise, on a manifold M we introduce the Finsler structure $F : TM \to [0, \infty)$ defined by

$$F(x, y) = \sqrt{h_x(y, y)} + \beta_x(y), \quad (x, y) \in TM, \tag{2.1}$$

where h is a Riemannian metric on M, β is an 1-form on M, and we assume that

$$\|\beta\|_h(x) = \sqrt{h^*_x(\beta_x, \beta_x)} < 1, \quad \forall x \in M.$$

Here, the co-metric h^*_x can be identified by h_x^{-1}, the inverse of the symmetric, positive definite matrix h_x. Clearly, the Randers space (M, F) in (2.1) is symmetric if and only if $\beta = 0$. Note that Randers metrics appear in the study of the electromagnetic field of the physical space-time in general relativity, see Randers [21]. Moreover, a deep result of Bao et al. [6] shows that a Finsler metric is of Randers type if and only if it is a solution of the Zermelo navigation problem on a Riemannian manifold.

Let $\pi^* TM$ be the pull-back bundle of the tangent bundle TM generated by the natural projection $\pi : TM \setminus \{0\} \to M$, see Bao et al. [5, p. 28]. The vectors of the pull-back bundle $\pi^* TM$ are denoted by $(v; w)$ with $(x, y) = v \in TM \setminus \{0\}$ and $w \in T_x M$. For simplicity, let

$\partial_i|_v = (v; \partial/\partial x^i|_x)$ be the natural local basis for π^*TM, where $v \in T_xM$. One can introduce on π^*TM the *fundamental tensor* g by

$$g_v := g(\partial_i|_v, \partial_j|_v) = g_{ij}(x, y) \tag{2.2}$$

where $v = y^i(\partial/\partial x^i)|_x$. Unlike the Levi-Civita connection in the Riemannian case, there is no unique natural connection in the Finsler geometry. Among these connections on the pull-back bundle π^*TM, we choose a torsion free and almost metric-compatible linear connection on π^*TM, the so-called *Chern connection*, see Bao et al. [5, Theorem 2.4.1]. The coefficients of the Chern connection are denoted by Γ^i_{jk}, which are instead of the well known Christoffel symbols from Riemannian geometry. A Finsler manifold is of *Berwald type* if the coefficients $\Gamma^k_{ij}(x, y)$ in natural coordinates are independent of y. It is clear that Riemannian manifolds and (locally) Minkowski spaces are Berwald spaces. The Chern connection induces on π^*TM the *curvature tensor* R, see Bao et al. [5, Chapter 3]. By means of the connection, we also have the *covariant derivative* $D_v u$ of a vector field u in the direction $v \in T_xM$ with reference vector v. A vector field $u = u(t)$ along a curve σ is *parallel* if $D_{\dot\sigma} u = 0$. A C^∞ curve $\sigma : [0, a] \to M$ is a *geodesic* if $D_{\dot\sigma}\dot\sigma = 0$. Geodesics are considered to be parametrized proportionally to arc-length. The Finsler manifold is *forward* (resp. *backward) complete* if every geodesic segment $\sigma : [0, a] \to M$ can be extended to $[0, \infty)$ (resp. to $(-\infty, a]$). (M, F) is *complete* if it is both forward and backward complete.

Let $u, v \in T_xM$ be two non-collinear vectors and $\mathcal{S} = \text{span}\{u, v\} \subset T_xM$. By means of the curvature tensor R, the *flag curvature* of the flag $\{\mathcal{S}, v\}$ is defined by

$$\mathbf{K}(\mathcal{S}; v) = \frac{g_v(R(U, V)V, U)}{g_v(V, V)g_v(U, U) - g_v(U, V)^2}, \tag{2.3}$$

where $U = (v; u)$, $V = (v; v) \in \pi^*TM$. If (M, F) is Riemannian, the flag curvature reduces to the well known sectional curvature. If $\mathbf{K}(\mathcal{S}; v) \le 0$ for every choice of U and V, we say that (M, F) has *non-positive flag curvature*, and we denote by $\mathbf{K} \le 0$. (M, F) is a *Finsler–Hadamard manifold* if it is simply connected, forward complete with $\mathbf{K} \le 0$.

Let $\sigma : [0, r] \to M$ be a piecewise C^∞ curve. The value $L_F(\sigma) = \int_0^r F(\sigma(t), \dot\sigma(t))\,\mathrm{d}t$ denotes the *integral length* of σ. For $x_1, x_2 \in M$, denote by $\Lambda(x_1, x_2)$ the set of all piecewise C^∞ curves $\sigma : [0, r] \to M$ such that $\sigma(0) = x_1$ and $\sigma(r) = x_2$. Define the *distance function* $d_F : M \times M \to [0, \infty)$ by

$$d_F(x_1, x_2) = \inf_{\sigma \in \Lambda(x_1, x_2)} L_F(\sigma). \tag{2.4}$$

One clearly has that $d_F(x_1, x_2) = 0$ if and only if $x_1 = x_2$, and d_F verifies the triangle inequality. The open *forward* (resp. *backward*) *metric ball* with center $x_0 \in M$ and radius $\rho > 0$ is defined by $B^+(x_0, \rho) = \{x \in M : d_F(x_0, x) < \rho\}$ (resp. $B^-(x_0, \rho) = \{x \in M : d_F(x, x_0) < \rho\}$). In particular, when $(M, F) = (\mathbb{R}^n, F)$ is a Minkowski space, one has $d_F(x_1, x_2) = F(x_2 - x_1)$.

Let $\{\partial/\partial x^i\}_{i=1,\ldots,n}$ be a local basis for the tangent bundle TM, and $\{\mathrm{d}x^i\}_{i=1,\ldots,n}$ be its dual basis for T^*M. Let $B_x(1) = \{y = (y^i) : F(x, y^i\partial/\partial x^i) < 1\} \subset \mathbb{R}^n$. The *Hausdorff volume form* $\mathsf{dm} = \mathrm{d}V_F$ on (M, F) is defined by

$$\mathsf{dm}(x) = \mathrm{d}V_F(x) = \sigma_F(x)\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n, \tag{2.5}$$

where $\sigma_F(x) = \frac{\omega_n}{\text{Vol}(B_x(1))}$. Hereafter, $\text{Vol}(S)$ and ω_n denote the Euclidean volume of the set $S \subset \mathbb{R}^n$ and the n-dimensional unit ball, respectively. The *Finslerian-volume* of an open set $S \subset M$ is $\text{Vol}_F(S) = \int_S \mathsf{dm}(x)$.

Let $\{e_i\}_{i=1,\ldots,n}$ be a basis for T_xM and $g_{ij}^v = g_v(e_i, e_j)$. The *mean distortion* $\mu : TM \setminus \{0\} \to (0, \infty)$ is defined by $\mu(v) = \frac{\sqrt{\det(g_{ij}^v)}}{\sigma_F}$. The *mean covariation* $\mathbf{S} : TM \setminus \{0\} \to \mathbb{R}$ is defined by

$$\mathbf{S}(x, v) = \frac{\mathrm{d}}{\mathrm{d}t}(\ln \mu(\dot{\sigma}_v(t)))\Big|_{t=0},$$

where σ_v is the geodesic such that $\sigma_v(0) = x$ and $\dot{\sigma}_v(0) = v$. We say that (M, F) has *vanishing mean covariation* if $\mathbf{S}(x, v) = 0$ for every $(x, v) \in TM$, and we denote by $\mathbf{S} = 0$. We notice that any Berwald space has vanishing mean covariation, see Shen [25].

For any $c \leq 0$, we introduce

$$V_{c,n}(\rho) = n\omega_n \int_0^\rho \mathbf{s}_c(t)^{n-1}\mathrm{d}t,$$

where $\mathbf{s}_c$ denotes the unique solution of $y'' + cy = 0$ with $y(0) = 0$ and $y'(0) = 1$, i.e.,

$$\mathbf{s}_c(r) = \begin{cases} r & \text{if} \quad c = 0, \\ \frac{\sinh(\sqrt{-c}r)}{\sqrt{-c}} & \text{if} \quad c < 0. \end{cases}$$

In general, one has for every $x \in M$ that

$$\lim_{\rho \to 0^+} \frac{\mathrm{Vol}_F(B^+(x, \rho))}{V_{c,n}(\rho)} = \lim_{\rho \to 0^+} \frac{\mathrm{Vol}_F(B^-(x, \rho))}{V_{c,n}(\rho)} = 1. \tag{2.6}$$

When $(\mathbb{R}^n, F)$ is a Minkowski space, then on account of (2.5), $\mathrm{Vol}_F(B^+(x, \rho)) = \omega_n \rho^n$ for every $\rho > 0$ and $x \in \mathbb{R}^n$, and $\sigma_F(x)$ =constant. If F is the Randers metric of the form (2.1) on a manifold M, then

$$\mathrm{d}V_F(x) = \left(1 - \|\beta\|_h^2(x)\right)^{\frac{n+1}{2}} \mathrm{d}V_h(x), \tag{2.7}$$

where $\mathrm{d}V_h(x)$ denotes the canonical Riemannian volume form of h on M.

We shall use a Bishop–Gromov volume comparison result; on account of Shen [25], Wu and Xin [30, Theorems 6.1 and 6.3] and Zhao and Shen [33, Theorem 3.6], we recall the following version:

Theorem 2.1 (Volume comparison) *Let (M, F) be an n-dimensional Finsler–Hadamard manifold with $\mathbf{S} = 0$, $\mathbf{K} \leq c \leq 0$ and $x \in M$ fixed. Then the function*

$$\rho \mapsto \frac{\mathrm{Vol}_F(B^+(x, \rho))}{V_{c,n}(\rho)}, \quad \rho > 0,$$

is non-decreasing. In particular, from (2.6) we have

$$\mathrm{Vol}_F(B^+(x, \rho)) \geq V_{c,n}(\rho) \quad for\ all\ \rho > 0. \tag{2.8}$$

If equality holds in (2.8) for some $\rho_0 > 0$, then $\mathbf{K}(\cdot; \dot{\gamma}_y(t)) = c$ for every $t \in [0, \rho_0)$ and $y \in T_xM$ with $F(x, y) = 1$, where γ_y is the constant speed geodesic with $\gamma_y(0) = x$ and $\dot{\gamma}_y(0) = y$.

2.2 Polar and Legendre transforms

We consider the *polar transform* (or, co-metric) of F, defined for every $(x, \alpha) \in T^*M$ by

$$F^*(x, \alpha) = \sup_{y \in T_x M \setminus \{0\}} \frac{\alpha(y)}{F(x, y)}. \tag{2.9}$$

Note that for every $x \in M$, the function $F^*(x, \cdot)$ is a Minkowski norm on T_x^*M. Since $F^{*2}(x, \cdot)$ is twice differentiable on $T_x^*M \setminus \{0\}$, we consider the matrix $g_{ij}^*(x, \alpha) := [\frac{1}{2}F^{*2}(x, \alpha)]_{\alpha^i \alpha^j}$ for every $\alpha = \sum_{i=1}^n \alpha^i \mathrm{d}x^i \in T_x^*M \setminus \{0\}$ in a local coordinate system (x^i).

In particular, if (M, F) is a Randers space of the form (2.1), then

$$F^*(x, \alpha) = \frac{\sqrt{h_x^{*2}(\alpha, \beta) + (1 - \|\beta\|_h^2(x))\|\alpha\|_h^2(x)} - h_x^*(\alpha, \beta)}{1 - \|\beta\|_h^2(x)}, \quad (x, \alpha) \in T^*M, \tag{2.10}$$

where h_x^* denotes the co-metric acting on T_x^*M associated to the Riemannian metric h. Moreover, the symmetrized Finsler metric and its polar transform associated with the Randers metric (2.1) is

$$F_s(x, y) = \sqrt{h_x(y, y) + \beta_x^2(y)}, \; F_s^*(x, \alpha) = \sqrt{\|\alpha\|_h^2(x) - \frac{h_x^{*2}(\alpha, \beta)}{1 + \|\beta\|_h^2(x)}}. \tag{2.11}$$

The *Legendre transform* $J^* : T^*M \to TM$ associates to each element $\alpha \in T_x^*M$ the unique maximizer on T_xM of the map $y \mapsto \alpha(y) - \frac{1}{2}F^2(x, y)$. This element can also be interpreted as the unique vector $y \in T_xM$ with the properties

$$F(x, y) = F^*(x, \alpha) \quad \text{and} \quad \alpha(y) = F(x, y)F^*(x, \alpha). \tag{2.12}$$

In particular, if $\alpha = \sum_{i=1}^n \alpha^i \mathrm{d}x^i \in T_x^*M$, one has

$$J^*(x, \alpha) = \sum_{i=1}^n \frac{\partial}{\partial \alpha_i}\left(\frac{1}{2}F^{*2}(x, \alpha)\right)\frac{\partial}{\partial x^i}. \tag{2.13}$$

2.3 Derivatives, Finsler–Laplace operator

Let $u : M \to \mathbb{R}$ be a differentiable function in the distributional sense. The *gradient* of u is defined by

$$\nabla u(x) = J^*(x, Du(x)), \tag{2.14}$$

where $Du(x) \in T_x^*M$ denotes the (distributional) *derivative* of u at $x \in M$. In local coordinates, one has

$$Du(x) = \sum_{i=1}^n \frac{\partial u}{\partial x^i}(x)\mathrm{d}x^i, \tag{2.15}$$

$$\nabla u(x) = \sum_{i,j=1}^n g_{ij}^*(x, Du(x))\frac{\partial u}{\partial x^i}(x)\frac{\partial}{\partial x^j}.$$

In general, $u \mapsto \boldsymbol{\nabla} u$ is not linear. If $x_0 \in M$ is fixed, then due to Ohta and Sturm [24], one has

$$\begin{aligned} F^*(x, Dd_F(x_0, x)) &= F(x, \boldsymbol{\nabla} d_F(x_0, x)) \\ &= Dd_F(x_0, x)(\boldsymbol{\nabla} d_F(x_0, x)) = 1 \quad \text{for a.e. } \; x \in M. \end{aligned} \tag{2.16}$$

Let X be a vector field on M. In a local coordinate system (x^i), on account of (2.5), the *divergence* is defined by $\text{div}(X) = \frac{1}{\sigma_F} \frac{\partial}{\partial x^i}(\sigma_F X^i)$. The *Finsler–Laplace operator*

$$\boldsymbol{\Delta} u = \text{div}(\boldsymbol{\nabla} u)$$

acts on $W^{1,2}_{\text{loc}}(M)$ and for every $v \in C_0^\infty(M)$,

$$\int_M v \boldsymbol{\Delta} u \,\mathsf{dm}(x) = -\int_M Dv(\boldsymbol{\nabla} u)\,\mathsf{dm}(x), \tag{2.17}$$

see Ohta and Sturm [24] and Shen [27]. Note that in general $\boldsymbol{\Delta}(-u) \neq -\boldsymbol{\Delta} u$, unless (M, F) is reversible. In particular, for a Riemannian manifold $(M, F) = (M, g)$ the Finsler–Laplace operator is the usual Laplace–Beltrami operator $\boldsymbol{\Delta} u = \Delta_g u$, while for a Minkowski space $(\mathbb{R}^n, F)$, by using (2.12), $\boldsymbol{\Delta} u = \Delta_F u = \text{div}(F^*(Du)\nabla F^*(Du)) = \text{div}(F(\nabla u)\nabla F(\nabla u))$ is precisely the Finsler–Laplace operator considered by Cianchi and Salani [10], Ferone and Kawohl [15], Wang and Xia [28,29], and references therein. We shall use the following result from Wu and Xin [30]:

Theorem 2.2 (Laplacian comparison) *Let (M, F) be an n-dimensional Finsler–Hadamard manifold with $\mathbf{S} = 0$. Let $x_0 \in M$ and $c \leq 0$. Then the following statements hold:*

(a) *If $\mathbf{K} \leq c$ then $\boldsymbol{\Delta} d_F(x_0, x) \geq (n-1)\mathbf{ct}_c(d_F(x_0, x))$ for every $x \in M \setminus \{x_0\}$;*
(b) *If $c \leq \mathbf{K}$ then $\boldsymbol{\Delta} d_F(x_0, x) \leq (n-1)\mathbf{ct}_c(d_F(x_0, x))$ for every $x \in M \setminus \{x_0\}$.*

2.4 Reversibility and uniformity constants

Inspired by Rademacher [20], we introduce the *reversibility constant* associated with F,

$$r_F = \sup_{x \in M} r_F(x) \quad \text{where} \quad r_F(x) = \sup_{y \in T_x M \setminus \{0\}} \frac{F(x, y)}{F(x, -y)}. \tag{2.18}$$

It is clear that $r_F \geq 1$ (possibly, $r_F = +\infty$) and $r_F = 1$ if and only if (M, F) is reversible. In the same way, we define the constant r_{F^*} associated with F^* and one has $r_{F^*} = r_F$.

The number

$$l_F = \inf_{x \in M} l_F(x) \quad \text{where} \quad l_F(x) = \inf_{y, v, w \in T_x M \setminus \{0\}} \frac{g_{(x,v)}(y, y)}{g_{(x,w)}(y, y)},$$

is the *uniformity constant of* F which measures how far F and F^* are from Riemannian structures, see Egloff [13]. Indeed, one can see that $l_F \leq 1$, and $l_F = 1$ if and only if (M, F) is a Riemannian manifold, see Ohta [23]. In the same manner, we can define the constant l_{F^*} for F^*, and it follows that $l_{F^*} = l_F$. The definition of l_F in turn shows that

$$F^{*2}(x, t\alpha + (1-t)\beta) \leq tF^{*2}(x, \alpha) + (1-t)F^{*2}(x, \beta) - l_F t(1-t)F^{*2}(x, \beta - \alpha) \tag{2.19}$$

for all $x \in M$, $\alpha, \beta \in T_x^* M$ and $t \in [0, 1]$.

By the above definitions, one can easily deduce that

$$l_F(x) r_F^2(x) \leq 1, \; x \in M. \tag{2.20}$$

For the Randers metric (2.1), a direct computation gives that

$$r_F(x) = \frac{1 + \|\beta\|_h(x)}{1 - \|\beta\|_h(x)} \quad \text{and} \quad l_F(x) = \left(\frac{1 - \|\beta\|_h(x)}{1 + \|\beta\|_h(x)}\right)^2, \quad x \in M, \tag{2.21}$$

see also Yuan and Zhao [31].

Proposition 2.1 *Let (M, F) be a Finsler manifold. Then the following statements hold:*

(a) *If $l_F > 0$ then $r_F < +\infty$;*
(b) *If $r_F < +\infty$, then forward and backward completeness of (M, F) coincide;*
(c) *If (M, F) is of Randers type* [see (2.1)] *with $\mathbf{S} = 0$ then $l_F > 0$.*

Proof (a) follows by (2.20). (b) is a simple consequence of the Hopf–Rinow theorem, since a set in M is forward bounded if and only if it is backward bounded whenever $r_F < +\infty$. (c) If (M, F) is of Randers type with $\mathbf{S} = 0$ and F has the form from (2.1), Ohta [22] proved that β is a Killing form of constant h-length, i.e., there exists $\beta_0 \in (0, 1)$ such that $\|\beta\|_h(x) = \beta_0$ for every $x \in M$. Therefore, by (2.21), one has that $l_F = \left(\frac{1-\beta_0}{1+\beta_0}\right)^2 > 0$. □

3 Reversibility versus Sobolev spaces on non-compact Finsler manifolds

Proof of Theorem 1.1 Due to the convexity of F^{*2}, if $u, v \in W_0^{1,2}(M, F, \mathsf{m})$ then $u + v \in W_0^{1,2}(M, F, \mathsf{m})$. Moreover, since $r_F < \infty$, one also has that $cu \in W_0^{1,2}(M, F, \mathsf{m})$ for every $c \in \mathbb{R}$ and $u \in W_0^{1,2}(M, F, \mathsf{m})$. Consequently, $W_0^{1,2}(M, F, \mathsf{m})$ is a vector space over $\mathbb{R}$.

Note that $\|\cdot\|_{F_s}$ is a norm and $\|\cdot\|_F$ is an asymmetric norm. Moreover, a simple argument based on the definition of the reversibility constant r_F gives that $\|\cdot\|_{F_s}$ and $\|\cdot\|_F$ are equivalent; in particular, one has

$$\left(\frac{1 + r_F^2}{2}\right)^{-1/2} F^*(x, \alpha) \le F_s^*(x, \alpha) \le \left(\frac{1 + r_F^{-2}}{2}\right)^{-1/2} F^*(x, \alpha), \ \forall (x, \alpha) \in T^*M;$$

thus relation (1.2) also yields.

Let

$$L^2(M, \mathsf{m}) = \left\{u : M \to \mathbb{R} : u \text{ is measurable}, \ \|u\|_{L^2(M,\mathsf{m})} < \infty\right\},$$

where

$$\|u\|_{L^2(M,\mathsf{m})} = \left(\int_M u^2(x) \mathsf{d}\mathsf{m}(x)\right)^{1/2}.$$

It is standard that $(L^2(M, \mathsf{m}), \|\cdot\|_{L^2(M,\mathsf{m})})$ is a Hilbert space. Since F^{*2} is a (strictly) convex function, so F_s^{*2}, one can prove that $(W_0^{1,2}(M, F, \mathsf{m}), \|\cdot\|_{F_s})$ is a closed subspace of the Hilbert space $L^2(M, \mathsf{m})$, which concludes the proof. □

Remark 3.1 The statement of Theorem 1.1 remains valid for an arbitrary open domain $\Omega \subset M$ instead of the whole manifold M.

Sharpness of Theorem 1.1. We claim that in general $W_0^{1,2}(M, F, \mathsf{m})$ need not has a vector space structure. In fact, we *cannot* assert that $u \in W_0^{1,2}(M, F, \mathsf{m})$ implies $-u \in$

$W_0^{1,2}(M, F, \mathsf{m})$ whenever the reversibility constant r_F is not finite. A similar phenomenon is already pointed out in [19] for a Funk-type metric.

For completeness, we provide another example on the Finsler–Poincaré disc model. If $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$ are the polar coordinates, let

$$M = B^2(0,2) = \{x = (x_1, x_2) \in \mathbb{R}^2 : r^2 = |x|^2 = x_1^2 + x_2^2 < 4\}$$

and F be the Randers metric given by (2.1) where

$$h = \frac{16}{(4-r^2)^2}(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2) \quad \text{and} \quad \beta = \frac{16r}{16-r^4}\mathrm{d}r,$$

see [5, Section 12.6]. Consequently, if $V = p\frac{\partial}{\partial r} + q\frac{\partial}{\partial\theta} \in T_{(r,\theta)}M$, then we explicitly have

$$F((r,\theta), V) = \frac{4}{4-r^2}\sqrt{p^2 + r^2q^2} + \frac{16pr}{16-r^4}.$$

Note that

$$\|\beta\|_h(x) = \frac{4r}{4+r^2}, \tag{3.1}$$

thus the volume element [see (2.7)] takes the form

$$\mathrm{d}\mathsf{m}(x) = \mathrm{d}V_F(x) = \frac{16r(4-r^2)}{(4+r^2)^3}\mathrm{d}r\mathrm{d}\theta. \tag{3.2}$$

The pair (M, F) is a forward (but not backward) complete Randers space with constant flag curvature $\mathbf{K} = -\frac{1}{4}$ and

$$d_F(\mathbf{0}, x) = \log\left(\frac{4+r^2}{(2-r)^2}\right) \quad \text{and} \quad d_F(x, \mathbf{0}) = \log\left(\frac{(2+r)^2}{4+r^2}\right),$$

where $\mathbf{0} = (0, 0)$.

Due to relations (2.21) and (3.1), one has $r_F(x) = \left(\frac{2+r}{2-r}\right)^2$, where $r = |x|$. Consequently, the reversibility constant

$$r_F = \sup_{x\in M} r_F(x) = \lim_{r\to 2}\left(\frac{2+r}{2-r}\right)^2 = +\infty$$

and

$$l_F = 0.$$

By (2.15), one has

$$Dd_F(\mathbf{0}, x) = \frac{4(2+r)}{(2-r)(4+r^2)}\mathrm{d}r.$$

Therefore, by means of (2.10), a direct computation yields that

$$F^*(x, Dd_F(\mathbf{0}, x)) = 1 \quad \text{and} \quad F^*(x, -Dd_F(\mathbf{0}, x)) = \left(\frac{2+r}{2-r}\right)^2. \tag{3.3}$$

Note that the first relation in (3.3) follows also by (2.16).

Let $u : M \to \mathbb{R}$ be defined by

$$u(x) = -e^{-\frac{d_F(\mathbf{0},x)}{4}}.$$

It is clear that $u \in W^{1,2}_{\text{loc}}(M)$. Since $Du(x) = \frac{1}{4}e^{-\frac{d_F(\mathbf{0},x)}{4}} Dd_F(\mathbf{0}, x)$, by the first relation of (3.3) and (3.2) one has

$$I_+ := \int_M F^{*2}(x, Du(x))\mathsf{dm}(x) = \frac{1}{16}\int_M e^{-\frac{d_F(\mathbf{0},x)}{2}}\mathsf{dm}(x)$$
$$= 2\pi \int_0^2 \frac{r(2-r)^2(2+r)}{(4+r^2)^{\frac{7}{2}}}\mathrm{d}r$$
$$= \frac{\pi}{30},$$

thus $u \in W^{1,2}(M, F, \mathsf{m})$. Furthermore,

$$I := \int_M u^2(x)\mathsf{dm}(x) = \int_M e^{-\frac{d_F(\mathbf{0},x)}{2}}\mathsf{dm}(x) = \frac{8\pi}{15}.$$

Thus,

$$\|u\|^2_F = I_+ + I = \frac{17\pi}{30},$$

so $u \in W^{1,2}_0(M, F, \mathsf{m})$.

However, the second relation of (3.3) and (3.2) imply that

$$I_- := \int_M F^{*2}(x, -Du(x))\mathsf{dm}(x) = \frac{1}{16}\int_M e^{-\frac{d_F(\mathbf{0},x)}{2}} F^{*2}(x, -Dd_F(\mathbf{0}, x))\mathsf{dm}(x)$$
$$= 2\pi \int_0^2 \frac{r(2+r)^5}{(4+r^2)^{\frac{7}{2}}(2-r)^2}\mathrm{d}r$$
$$= +\infty,$$

Therefore,

$$\| -u\|^2_F = I_- + I = +\infty,$$

i.e., $-u \notin W^{1,2}(M, F, \mathsf{m})$ and $-u \notin W^{1,2}_0(M, F, \mathsf{m})$. Moreover, according to (2.11), one has that

$$\|u\|^2_{F_s} = \| -u\|^2_{F_s} = \frac{\pi}{5} + \frac{\pi}{16}\sqrt{-2+2\sqrt{2}}\ln\left(5+4\sqrt{2}+4\sqrt{4-2\sqrt{2}}+6\sqrt{-2+2\sqrt{2}}\right)$$
$$+ \frac{\pi}{8}\sqrt{2+2\sqrt{2}}\arctan\left(\frac{\sqrt{2+2\sqrt{2}}-\sqrt{-2+2\sqrt{2}}}{\sqrt{2}-2}\right)$$
$$\approx 0.1877.$$

Consequently, the norm $\|\cdot\|_{F_s}$ and the asymmetric norm $\|\cdot\|_F$ are not equivalent.

4 Convexity of the singular Hardy–Finsler energy functional

In order to deal with singular problems of type $(\mathcal{P}^{\mu}_{\Omega})$ we first need a Hardy inequality on (not necessarily reversible) Finsler–Hadamard manifold with $\mathbf{S} = 0$. As mentioned before, these spaces include Finsler–Hadamard manifolds of Berwald type (thus, both Minkowski spaces and Hadamard–Riemannian manifolds).

Proposition 4.1 *Let* (M, F) *be an* n*-dimensional* $(n \geq 3)$ *Finsler–Hadamard manifold with* $\mathbf{S} = 0$*, and let* $x_0 \in M$ *be fixed. Then*

$$\int_M F^{*2}(x, -D(|u|)(x))\mathsf{dm}(x) \geq \overline{\mu} \int_M \frac{u^2(x)}{d_F^2(x_0, x)}\mathsf{dm}(x), \quad \forall u \in C_0^\infty(M), \tag{4.1}$$

where the constant $\overline{\mu} = \frac{(n-2)^2}{4}$ *is optimal and never achieved.*

Proof By convexity, we have the following inequality

$$F^{*2}(x, \beta) \geq F^{*2}(x, \alpha) + 2(\beta - \alpha)(J^*(x, \alpha)), \ \forall \alpha, \beta \in T_x^* M. \tag{4.2}$$

Let $x_0 \in M$ and $u \in C_0^\infty(M)$ be arbitrarily fixed and let $\gamma = \sqrt{\overline{\mu}} = \frac{n-2}{2} > 0$. We consider the function $v(x) = d_F(x_0, x)^\gamma u(x)$. Therefore, $u(x) = d_F(x_0, x)^{-\gamma} v(x)$ and one has

$$D(|u|)(x) = -\gamma d_F(x_0, x)^{-\gamma-1}|v| D d_F(x_0, x) + d_F(x_0, x)^{-\gamma} D(|v|)(x).$$

Applying the inequality (4.2) with the choices $\beta = -D|u|$ and $\alpha = \gamma d_F(x_0, x)^{-\gamma-1}|v|$ $Dd_F(x_0, x)$, respectively, one can deduce that

$$\begin{aligned} F^{*2}(x, -D(|u|)(x)) \geq\ & F^{*2}(x, \gamma d_F(x_0, x)^{-\gamma-1}|v(x)| D d_F(x_0, x)) \\ & -2 d_F(x_0, x)^{-\gamma} D(|v|)(x)(J^*(x, \gamma d_F(x_0, x)^{-\gamma-1}|v(x)| D d_F(x_0, x))). \end{aligned}$$

Due to relation (2.16), to the fact that $J^*(x, Dd_F(x_0, x)) = \nabla d_F(x_0, x)$ and $D(|v|)(x) \in T_x^* M$, we obtain

$$\begin{aligned} F^{*2}(x, -D(|u|)(x)) \geq\ & \gamma^2 d_F(x_0, x)^{-2\gamma-2}|v(x)|^2 \\ & -2\gamma d_F(x_0, x)^{-2\gamma-1}|v(x)| D(|v|)(x)(\nabla d_F(x_0, x)). \end{aligned}$$

Integrating the latter inequality over M, it yields

$$\int_M F^{*2}(x, -D(|u|)(x))\mathsf{dm}(x) \geq \gamma^2 \int_M d_F(x_0, x)^{-2\gamma-2}|v(x)|^2 \mathsf{dm}(x) + R_0,$$

where

$$R_0 = -2\gamma \int_M d_F(x_0, x)^{-2\gamma-1}|v(x)| D(|v|)(x)(\nabla d_F(x_0, x))\mathsf{dm}(x).$$

Since $\mathbf{S} = 0$ and $\mathbf{K} \leq 0$, Theorem 2.2 (a) shows that

$$d_F(x_0, x)\boldsymbol{\Delta} d_F(x_0, x) \geq n - 1 \quad \text{for a.e.} \quad x \in M.$$

Consequently, by (2.17), (2.16) and the latter estimate one has

$$\begin{aligned} R_0 &= -\gamma \int_M D(|v|^2)(d_F(x_0, x)^{-2\gamma-1} \nabla d_F(x_0, x))\mathsf{dm}(x) \\ &= \gamma \int_M |v(x)|^2 \mathrm{div}(d_F(x_0, x)^{-2\gamma-1} \nabla d_F(x_0, x))\mathsf{dm}(x) \\ &= \gamma \int_M |v(x)|^2 d_F(x_0, x)^{-2\gamma-2} \left(-2\gamma - 1 + d_F(x_0, x)\boldsymbol{\Delta} d_F(x_0, x)\right) \mathsf{dm}(x) \geq 0, \end{aligned}$$

which completes the first part of the proof.

We now prove that $\gamma^2 = \frac{(n-2)^2}{4}$ is sharp. Fix the numbers $R > r > 0$ and a smooth cutoff function $\psi : M \to [0, 1]$ with $\mathrm{supp}(\psi) = \overline{B^+(x_0, R)}$ and $\psi(x) = 1$ for $x \in B^+(x_0, r)$, and for every $\varepsilon > 0$, let $u_\varepsilon(x) = (\max\{\varepsilon, d_F(x_0, x)\})^{-\gamma}$, $x \in M$.

On the one hand, by (2.16) we have

$$\begin{aligned}
I_1(\varepsilon) &:= \int_M F^{*2}(x, -D(\psi u_\varepsilon)(x))\mathrm{dm}(x) \\
&= \int_{B^+(x_0,r)} F^{*2}(x, -Du_\varepsilon(x))\mathrm{dm}(x) + \int_{B^+(x_0,R)\setminus B^+(x_0,r)} F^{*2}(x, -D(\psi u_\varepsilon)(x))\mathrm{dm}(x) \\
&= \gamma^2 \int_{B^+(x_0,r)\setminus B^+(x_0,\varepsilon)} d_F(x_0, x)^{-2\gamma-2}\mathrm{dm}(x) + \tilde{I}_1(\varepsilon),
\end{aligned}$$

where the quantity

$$\tilde{I}_1(\varepsilon) = \int_{B^+(x_0,R)\setminus B^+(x_0,r)} F^{*2}(x, -D(\psi u_\varepsilon)(x))\mathrm{dm}(x)$$

is finite and does not depend on $\varepsilon > 0$ whenever $\varepsilon < r$. On the other hand,

$$\begin{aligned}
I_2(\varepsilon) &:= \int_M \frac{(\psi u_\varepsilon)^2(x)}{d_F(x_0, x)^2}\mathrm{dm}(x) \\
&\geq \int_{B^+(x_0,r)\setminus B^+(x_0,\varepsilon)} d_F(x_0, x)^{-2\gamma-2}\mathrm{dm}(x) =: \tilde{I}_2(\varepsilon).
\end{aligned}$$

By (2.8), one has

$$\mathrm{Vol}_F(B^+(x_0, \rho)) \geq \omega_n \rho^n, \quad \forall \rho > 0.$$

Therefore, by applying the layer cake representation, we deduce that for $0 < \varepsilon < r$, one has

$$\begin{aligned}
\tilde{I}_2(\varepsilon) &= \int_{B^+(x_0,r)\setminus B^+(x_0,\varepsilon)} d_F(x_0, x)^{-2\gamma-2}\mathrm{dm}(x) = \int_{B^+(x_0,r)\setminus B^+(x_0,\varepsilon)} d_F(x_0, x)^{-n}\mathrm{dm}(x) \\
&\geq \int_{r^{-n}}^{\varepsilon^{-n}} \mathrm{Vol}_F(B^+(x_0, \rho^{-\frac{1}{n}}))\mathrm{d}\rho \\
&\geq \omega_n \int_{r^{-n}}^{\varepsilon^{-n}} \rho^{-1}\mathrm{d}\rho \\
&= n\omega_n(\ln r - \ln \varepsilon).
\end{aligned}$$

In particular, $\lim_{\varepsilon\to 0^+} \tilde{I}_2(\varepsilon) = +\infty$. Thus, from the above relations it follows that

$$\begin{aligned}
\frac{(n-2)^2}{4} &\leq \inf_{u\in C_0^\infty(M)\setminus\{0\}} \frac{\int_M F^{*2}(x, -D(|u|)(x))\mathrm{dm}(x)}{\int_M \frac{u^2(x)}{d_F^2(x_0,x)}\mathrm{dm}(x)} \\
&\leq \lim_{\varepsilon\to 0^+} \frac{I_1(\varepsilon)}{I_2(\varepsilon)} \leq \lim_{\varepsilon\to 0^+} \frac{\gamma^2 \tilde{I}_2(\varepsilon) + \tilde{I}_1(\varepsilon)}{\tilde{I}_2(\varepsilon)} \\
&= \gamma^2 = \frac{(n-2)^2}{4}.
\end{aligned}$$

A standard reasoning shows that this constant is never achieved. □

Remark 4.1 Proposition 4.1 can be proved for an arbitrary open domain $\Omega \subset M$ instead of the whole manifold M with $x_0 \in \Omega$.

In the sequel, we prove the main result of this section.

Theorem 4.1 *Let (M, F) be an n-dimensional $(n \geq 3)$ Finsler–Hadamard manifold with $\mathbf{S} = 0$ and $l_F > 0$. Let $\Omega \subseteq M$ be an open domain and $x_0 \in \Omega$. Then the functional $\mathscr{K}_\mu : W_0^{1,2}(\Omega, F, \mathsf{m}) \to \mathbb{R}$ defined by*

$$\mathscr{K}_\mu(u) = \int_\Omega F^{*2}(x, Du(x))\mathsf{dm}(x) - \mu \int_\Omega \frac{u^2(x)}{d_F^2(x_0, x)}\mathsf{dm}(x)$$

is positive unless $u = 0$ and strictly convex whenever $0 \leq \mu < l_F r_F^{-2}\overline{\mu}$.

Proof Let $0 \leq \mu < l_F r_F^{-2}\overline{\mu}$ and $x_0 \in \Omega$ be fixed arbitrarily. By (2.20), one has $r_F^2 \leq l_F^{-1} < +\infty$. The positivity of $\mathcal{K}_\mu$ follows by Proposition 4.1. Let $0 < t < 1$ and $u, v \in W_0^{1,2}(\Omega, F, \mathsf{m})$, $u \neq v$ be fixed. Then, by (2.19), from the fact that

$$F^*(x, D(v-u)(x)) \geq r_F^{-1} F^*(x, -D(|v-u|)(x)), \quad x \in \Omega,$$

and Proposition 4.1, one has

$$\begin{aligned}
\mathscr{K}_\mu\,(tu + (1-t)v) &= \int_\Omega F^{*2}(x, tDu(x) + (1-t)Dv(x))\mathsf{dm}(x) \\
&\quad -\mu \int_\Omega \frac{(tu + (1-t)v)^2}{d_F(x_0, x)^2}\mathsf{dm}(x) \\
&\leq t \int_\Omega F^{*2}(x, Du(x))\mathsf{dm}(x) + (1-t) \int_\Omega F^{*2}(x, Dv(x))\mathsf{dm}(x) \\
&\quad -l_F t(1-t) \int_\Omega F^{*2}(x, D(v-u)(x))\mathsf{dm}(x) \\
&\quad -\mu \int_\Omega \frac{(tu + (1-t)v)^2}{d_F^2(x_0, x)}\mathsf{dm}(x) \\
&= t\mathscr{K}_\mu\,(u) + (1-t)\mathscr{K}_\mu\,(v) \\
&\quad -t(1-t)l_F \int_\Omega \left(F^{*2}(x, D(v-u)(x)) - \mu l_F^{-1} \frac{(v-u)^2}{d_F^2(x_0, x)} \right)\mathsf{dm}(x) \\
&\leq t\mathscr{K}_\mu\,(u) + (1-t)\mathscr{K}_\mu\,(v) - t(1-t)l_F r_F^{-2} \\
&\quad \int_\Omega \left(F^{*2}(x, -D|v-u|(x)) - \mu l_F^{-1} r_F^2 \frac{(v-u)^2}{d_F^2(x_0, x)} \right)\mathsf{dm}(x) \\
&< t\mathscr{K}_\mu\,(u) + (1-t)\mathscr{K}_\mu\,(v)\,,
\end{aligned}$$

which concludes the proof. □

5 Singular Poisson equations on Finsler–Hadamard manifolds

Let (M, F) be a (not necessarily reversible) complete, n-dimensional $(n \geq 3)$ Finsler manifold, and $\Omega \subset M$ be an open domain, $x_0 \in \Omega$. For $\mu \in \mathbb{R}$, on $W_0^{1,2}(\Omega, F, \mathsf{m})$ we define the *singular Finsler–Laplace operator*

$$\mathcal{L}_F^\mu u = \mathbf{\Delta}(-u) - \mu \frac{u}{d_F^2(x_0, x)}.$$

Proposition 5.1 (Comparison principle) *Let* (M, F) *be an* n*-dimensional* $(n \geq 3)$ *Finsler–Hadamard manifold with* $\mathbf{S} = 0$ *and* $l_F > 0$*. Let* $\Omega \subset M$ *be an open domain. If* $\mathcal{L}_F^{\mu} u \leq \mathcal{L}_F^{\mu} v$ *in* Ω *and* $u \leq v$ *on* $\partial\Omega$*, then* $u \leq v$ *a.e. in* Ω *whenever* $\mu \in [0, l_F r_F^{-2} \overline{\mu})$*.*

Proof Assume that $\Omega_+ = \{x \in \Omega : u(x) > v(x)\}$ has a positive measure. Then, multiplying $\mathcal{L}_F^{\mu} u \leq \mathcal{L}_F^{\mu} v$ by $(u - v)_+$, by (2.17) one obtains

$$\int_{\Omega_+} (D(-v) - D(-u))(\nabla(-v) - \nabla(-u)) \mathrm{dm}(x) - \mu \int_{\Omega_+} \frac{(u - v)^2}{d_F^2(x_0, x)} \mathrm{dm}(x) \leq 0.$$

By (2.14) and the mean value theorem, the definition of l_F yields that for every $x \in \Omega_+$,

$$\begin{aligned} (D(-v) - D(-u))(\nabla(-v) - \nabla(-u)) &\geq l_F F^{*2}(x, D(-v) - D(-u)) \\ &= l_F F^{*2}(x, D(u - v)) \\ &\geq l_F r_F^{-2} F^{*2}(x, -D(u - v)). \end{aligned}$$

Combining these relations with Proposition 4.1, it follows that

$$\left(l_F r_F^{-2} - \frac{\mu}{\overline{\mu}} \right) \int_{\Omega_+} F^{*2}(x, -D(u - v)(x)) \mathrm{dm}(x) \leq 0,$$

which is a contradiction. □

Let $\mu \in [0, l_F r_F^{-2} \overline{\mu})$ and $\kappa \in L^{\infty}(\Omega)$. We consider the singular Poisson problem

$$\begin{cases} \mathcal{L}_F^{\mu} u = \kappa(x) & \text{in} \quad \Omega; \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{$\mathcal{P}_{\Omega}^{\mu,\kappa}$}$$

where $\Omega \subset M$ is an open, bounded domain. We introduce the *singular energy functional* associated with the operator $\mathcal{L}_F^{\mu}$ on $W_0^{1,2}(\Omega, F, \mathrm{m})$, defined by

$$\mathcal{E}_{\mu}(u) = (\mathcal{L}_F^{\mu} u)(u).$$

According to (2.17), we have in fact

$$\mathcal{E}_{\mu}(u) = \int_{\Omega} F^{*2}(x, -Du(x)) \mathrm{dm}(x) - \mu \int_{\Omega} \frac{u^2(x)}{d_F(x_0, x)^2} \mathrm{dm}(x) = \mathscr{K}_{\mu}(-u).$$

Theorem 5.1 *Let* (M, F) *be an* n*-dimensional* $(n \geq 3)$ *Finsler–Hadamard manifold with* $\mathbf{S} = 0$ *and* $l_F > 0$*. Let* $\Omega \subset M$ *be an open, bounded domain and a non-negative function* $\kappa \in L^{\infty}(\Omega)$*. Then problem* $(\mathcal{P}_{\Omega}^{\mu,\kappa})$ *has a unique, non-negative weak solution for every* $\mu \in [0, l_F r_F^{-2} \overline{\mu})$*.*

Proof Let $\mu \in [0, l_F r_F^{-2} \overline{\mu})$ be fixed and consider the energy functional associated with problem $(\mathcal{P}_{\Omega}^{\mu,\kappa})$, i.e.,

$$\mathcal{F}_{\mu}(u) = \frac{1}{2} \mathscr{K}_{\mu}(-u) - \int_{\Omega} \kappa(x) u(x) \mathrm{dm}(x), \quad u \in W_0^{1,2}(\Omega, F, \mathrm{m}).$$

It is clear that $\mathcal{F}_{\mu} \in C^1(W_0^{1,2}(\Omega, F, \mathrm{m}), \mathbb{R})$, and its critical points are precisely the weak solutions of problem $(\mathcal{P}_{\Omega}^{\mu,\kappa})$. Let $R > 0$ and $x_0 \in M$ be such that $\Omega \subset B^+(x_0, R)$. According to Wu and Xin [30, Theorem 7.3], we have

$$\lambda_1(\Omega) = \inf_{u \in W_0^{1,2}(\Omega, F, \mathrm{m}) \setminus \{0\}} \frac{\int_{\Omega} F^{*2}(x, Du(x)) \mathrm{dm}(x)}{\int_{\Omega} u^2(x) \mathrm{dm}(x)} \geq \frac{(n - 1)^2}{4R^2 r_F^2}.$$

Consequently, for every $u \in W_0^{1,2}(\Omega, F, \mathsf{m})$, one has that

$$\int_\Omega F^{*2}(x, Du(x))\mathrm{d}\mathsf{m}(x) \geq \frac{\lambda_1(\Omega)}{1+\lambda_1(\Omega)}\|u\|_F^2.$$

Since $\|\cdot\|_F$ and $\|\cdot\|_{F_s}$ are equivalent [see (1.2)], we conclude that $\mathcal{F}_\mu$ is bounded from below and coercive on the reflexive Banach space $(W_0^{1,2}(\Omega, F, \mathsf{m}), \|\cdot\|_{F_s})$, i.e., $\mathcal{F}_\mu(u) \to +\infty$ whenever $\|u\|_{F_s} \to +\infty$. Due to Theorem 4.1, $\mathcal{F}_\mu$ is strictly convex on $W_0^{1,2}(\Omega, F, \mathsf{m})$, thus the basic result of the calculus of variations implies that $\mathcal{F}_\mu$ has a unique (global) minimum point $u_\mu \in W_0^{1,2}(\Omega, F, \mathsf{m})$ of $\mathcal{F}_\mu$, see Zeidler [32, Theorem 38.C and Proposition 38.15], which is also the unique critical point of $\mathcal{F}_\mu$. Since $\kappa \geq 0$, Proposition 5.1 implies that $u_\mu \geq 0$. □

Remark 5.1 Theorem 1.2 directly follows by Theorem 5.1.

Lemma 5.1 *Let $f \in C^2(0,\infty)$ be a non-increasing function. Then*

$$\mathcal{L}_F^\mu(f(d_F(x_0,x))) = -f''(d_F(x_0,x)) - f'(d_F(x_0,x)) \times \Delta d_F(x_0,x) \\ -\mu\frac{f(d_F(x_0,x))}{d_F^2(x_0,x)}, \quad x \in M \setminus \{x_0\}.$$

Proof Since $f' \leq 0$, the claim follows from basic properties of the Legendre transform. Namely, one has

$$\begin{aligned}\Delta(-f(d_F(x_0,x))) &= \mathrm{div}(\nabla(-f(d_F(x_0,x)))) = \mathrm{div}\left(J^*(x, D(-f(d_F(x_0,x))))\right)\\ &= \mathrm{div}(J^*(x, -f'(d_F(x_0,x))Dd_F(x_0,x)))\\ &= \mathrm{div}(-f'(d_F(x_0,x))\nabla d_F(x_0,x))\\ &= -f''(d_F(x_0,x)) - f'(d_F(x_0,x)) \times \boldsymbol{\Delta} d_F(x_0,x),\end{aligned}$$

which concludes the proof. □

For every $\mu \in [0,\overline{\mu})$, $c \leq 0$ and $\rho > 0$, we recall the ordinary differential equation

$$\begin{cases} f''(r) + (n-1)f'(r)\mathbf{ct}_c(r) + \mu\frac{f(r)}{r^2} + 1 = 0, \ r \in (0,\rho], \\ f(\rho) = 0, \ \int_0^\rho f'(r)^2 r^{n-1}\mathrm{d}r < \infty. \end{cases} \tag{$\mathcal{Q}_{c,\rho}^\mu$}$$

Proposition 5.2 *($\mathcal{Q}_{c,\rho}^\mu$) has a unique, non-negative, non-increasing solution belonging to $C^\infty(0,\rho)$.*

Proof We fix $\mu \in [0,\overline{\mu})$, $c \leq 0$ and $\rho > 0$. Let us consider the Riemannian space form (M, g_c) with constant sectional curvature $c \leq 0$,, i.e., (M, g_c) is isometric to the Euclidean space when $c = 0$, or (M, g_c) is isometric to the hyperbolic space with sectional curvature $c < 0$. Let $x_0 \in M$ be fixed. Since (M, g_c) verifies the assumptions of Theorem 5.1, problem

$$\begin{cases} -\Delta_{g_c} u - \mu\frac{u}{d_{g_c}^2(x_0,x)} = 1 & \text{in} \quad B_{g_c}(x_0,\rho); \\ u = 0 & \text{on} \quad \partial B_{g_c}(x_0,\rho), \end{cases} \tag{$\mathcal{R}_{c,\rho}^\mu$}$$

has a unique, non-negative solution u_0 which is nothing but the unique global minimum point of the energy functional $\mathcal{F}_\mu : W_0^{1,2}(B_{g_c}(x_0,\rho), g_c, \mathsf{m}) \to \mathbb{R}$ defined by

$$\mathcal{F}_\mu(u) = \frac{1}{2}\int_{B_{g_c}(x_0,\rho)} |Du(x)|_{g_c}^2 \mathrm{d}\mathsf{m}(x) - \frac{\mu}{2}\int_{B_{g_c}(x_0,\rho)} \frac{u^2(x)}{d_{g_c}^2(x_0,x)}\mathrm{d}\mathsf{m}(x) \\ - \int_{B_{g_c}(x_0,\rho)} u(x)\mathrm{d}\mathsf{m}(x).$$

In this particular case, dm denotes the canonical Riemannian volume form on (M, g_c).

Let $u_0^\star : B_{g_c}(x_0, \rho) \to [0, \infty)$ be the non-increasing symmetric rearrangement of u_0 in the space form (M, g_c), see Baernstein [4]. Note that Pólya–Szegő and Hardy–Littlewood inequalities imply that

$$\int_{B_{g_c}(x_0,\rho)} |Du_0(x)|^2_{g_c} \mathsf{dm}(x) \geq \int_{B_{g_c}(x_0,\rho)} |Du_0^\star(x)|^2_{g_c} \mathsf{dm}(x),$$

and

$$\int_{B_{g_c}(x_0,\rho)} \frac{u_0^2(x)}{d^2_{g_c}(x_0, x)} \mathsf{dm}(x) \leq \int_{B_{g_c}(x_0,\rho)} \frac{u_0^{\star 2}(x)}{d^2_{g_c}(x_0, x)} \mathsf{dm}(x),$$

respectively. Moreover, by the Cavalieri principle, we also have that

$$\int_{B_{g_c}(x_0,\rho)} u_0(x) \mathsf{dm}(x) = \int_{B_{g_c}(x_0,\rho)} u_0^\star(x) \mathsf{dm}(x).$$

Therefore, we obtain that $\mathcal{F}_\mu(u_0) \geq \mathcal{F}_\mu(u_0^\star)$. Consequently, by the uniqueness of the global minimizer of $\mathcal{F}_\mu$ we have $u_0 = u_0^\star$; thus, its form is $u_0(x) = f(t)$ where $t = d_{g_c}(x_0, x)$ and $f : (0, \rho] \to \mathbb{R}$ is a non-negative and non-increasing function. Clearly, $f(\rho) = 0$ since $u_0(x) = 0$ whenever $d_{g_c}(x_0, x) = \rho$. Moreover, since $u_0 = u_0^\star \in W_0^{1,2}(B_{g_c}(x_0, \rho), g_c, \mathsf{m})$, a suitable change of variables gives that $\int_0^\rho f'(r)^2 r^{n-1} \mathrm{d}r < \infty$. By Lemma 5.1 and Theorem 2.2 it follows that the first part of $(\mathcal{R}^\mu_{c,\rho})$ can be transformed into the first part of $(\mathcal{Q}^\mu_{c,\rho})$; in particular, problem $(\mathcal{Q}^\mu_{c,\rho})$ has a non-negative, non-increasing solution. Standard regularity theory implies that $f \in C^\infty(0, \rho)$, see Evans [14, p. 334]. Finally, if we assume that $(\mathcal{Q}^\mu_{c,\rho})$ has two distinct non-negative, non-increasing solutions f_1 and f_2, then both functions $u_i(x) = f_i(d_{g_c}(x_0, x))$ $(i \in \{1, 2\})$ verify $(\mathcal{R}^\mu_{c,\rho})$, which are distinct global minima of the energy functional $\mathcal{F}_\mu$, a contradiction. □

Proof of Theorem 1.3 Let u be the unique solution of problem $(\mathcal{P}^\mu_\Omega)$. We claim that

$$\begin{cases} \mathcal{L}^\mu_F(\sigma_{\mu,\rho_1,c_1}(d_F(x_0, x))) \leq 1 = \mathcal{L}^\mu_F(u) & \text{in } B^+(x_0, \rho_1); \\ \sigma_{\mu,\rho_1,c_1}(d_F(x_0, x)) = 0 \leq u(x) & \text{on } \partial B^+(x_0, \rho_1), \end{cases}$$

where $\rho_1 = \sup\{\rho > 0 : B^+(x_0, \rho) \subset \Omega\}$. On one hand, since $c_1 \leq \mathbf{K}$, due to Theorem 2.2 (b) and to the fact that σ_{μ,ρ_1,c_1} is non-increasing, by equation $(\mathcal{Q}^\mu_{c_1,\rho_1})$ one has for $x \in B^+(x_0, \rho_1) \setminus \{x_0\}$,

$$\begin{aligned} 1 &= -\sigma''_{\mu,\rho_1,c_1}(d_F(x_0, x)) - (n-1)\sigma'_{\mu,\rho_1,c_1}(d_F(x_0, x))\mathbf{ct}_{c_1}(d_F(x_0, x)) \\ &\quad -\mu \frac{\sigma_{\mu,\rho_1,c_1}(d_F(x_0, x))}{d_F^2(x_0, x)} \\ &\geq -\sigma''_{\mu,\rho_1,c_1}(d_F(x_0, x)) - \sigma'_{\mu,\rho_1,c_1}(d_F(x_0, x))\Delta d_F(x_0, x) - \mu \frac{\sigma_{\mu,\rho_1,c_1}(d_F(x_0, x))}{d_F^2(x_0, x)} \\ &= \mathcal{L}^\mu_F(\sigma_{\mu,\rho_1,c_1}(d_F(x_0, x))). \end{aligned}$$

On the other hand, since u is non-negative in Ω, it follows that $0 = \sigma_{\mu,\rho_1,c_1}(d_F(x_0, x)) \leq u(x)$ on $\partial B^+(x_0, \rho_1)$. It remains to apply the comparision principle (Proposition 5.1), obtaining

$$\sigma_{\mu,\rho_1,c_1}(d_F(x_0, x)) \leq u(x) \quad \text{for a.e.} \quad x \in B^+(x_0, \rho_1).$$

Similarly, by using Theorem 2.2 (a) and $\mathbf{K} \leq c_2$, one can prove that

$$\begin{cases} 1 = \mathcal{L}_F^\mu(u) \leq \mathcal{L}_F^\mu(\sigma_{\mu,\rho_2,c_2}(d_F(x_0,x))) & \text{in} \quad \Omega; \\ u(x) = 0 \leq \sigma_{\mu,\rho_2,c_2}(d_F(x_0,x)) & \text{on} \quad \partial\Omega, \end{cases}$$

where $\rho_2 = \inf\{\rho > 0 : \Omega \subset B^+(x_0,\rho)\}$. In particular, by Proposition 5.1 again we have that

$$u(x) \leq \sigma_{\mu,\rho_2,c_2}(d_F(x_0,x)) \quad \text{for a.e.} \quad x \in \Omega.$$

If $\mathbf{K} = c \leq 0$ and $\Omega = B^+(x_0,\rho)$ for some $\rho > 0$, then $\rho_1 = \rho_2 = \rho$, and from above it follows that $u(x) = \sigma_{\mu,\rho,c}(d_F(x_0,x))$ is the unique weak solution of problem $(\mathcal{P}^\mu_{B^+(x_0,\rho)})$ which is also a pointwise solution in $B^+(x_0,\rho) \setminus \{x_0\}$. □

A simple consequence of Theorem 1.3 is the following

Corollary 5.1 *Let $(M,F) = (\mathbb{R}^n, \|\cdot\|)$ be a Minkowski space and let $\mu \in [0, l_F r_F^{-2}\overline{\mu})$, $x_0 \in \mathbb{R}^n$ and $\rho > 0$ be fixed. Then $u = \sigma_{\mu,\rho,0}(\|\cdot - x_0\|) \in C^\infty(B^+(x_0,\rho) \setminus \{x_0\})$ is the unique pointwise solution to problem $(\mathcal{P}^\mu_{B^+(x_0,\rho)})$ in $B^+(x_0,\rho) \setminus \{x_0\}$.*

Proof $(M,F) = (\mathbb{R}^n, \|\cdot\|)$ being a Minkowski space, it is a Finsler–Hadamard manifold with $\mathbf{S} = 0$, $\mathbf{K} = 0$ and $l_F > 0$. It remains to apply Theorem 1.3. □

Remark 5.2 (i) In addition to the conclusions of Corollary 5.1, one can see that

(a) $\sigma_{\mu,\rho,0} \in C^1(B^+(x_0,\rho))$ if and only if $\mu = 0$, and
(b) $\sigma_{\mu,\rho,0} \in C^2(B^+(x_0,\rho))$ if and only if $\mu = 0$ and $F = \|\cdot\|$ is Euclidean.

(ii) When $(M,F) = (\mathbb{R}^n, \|\cdot\|)$ is a reversible Minkowski space and $\mu = 0$, Corollary 5.1 reduces to Theorem 2.1 from Ferone and Kawohl [15].

In connection with Corollary 5.1 we establish an estimate for the solution of the singular Poisson equation on *backward* geodesic balls on Minkowski spaces. To do this, we assume that $\sigma_{\mu,r_F^{-1}\rho,0}$ is extended beyond $r_F^{-1}\rho$ formally by the same function, its explicit form being given after the problem $(\mathcal{Q}^\mu_{c,\rho})$. Although problem $(\mathcal{P}^\mu_{B^-(x_0,\rho)})$ cannot be solved explicitly in general, the following sharp estimates can be given for its unique solution by means of the reversibility constant r_F.

Proposition 5.3 *Let $(M,F) = (\mathbb{R}^n, \|\cdot\|)$ be a Minkowski space and let $\mu \in [0, l_F r_F^{-2}\overline{\mu})$, $x_0 \in \mathbb{R}^n$ and $\rho > 0$ be fixed. If $\tilde{u}_{\mu,\rho}$ denotes the unique weak solution to problem $(\mathcal{P}^\mu_{B^-(x_0,\rho)})$, then*

$$(\sigma_{\mu,r_F^{-1}\rho,0}(\|x - x_0\|))_+ \leq \tilde{u}_{\mu,\rho}(x) \leq \sigma_{\mu,r_F\rho,0}(\|x - x_0\|) \quad for\ a.e. \quad x \in B^-(x_0,\rho).$$

Moreover, the above two bounds coincide if and only if (M,F) is reversible.

Proof The proof immediately follows by the comparison principle Proposition 5.1, showing that

$$\begin{cases} \mathcal{L}_F^\mu(w^-_{\mu,\rho}) = 1 = \mathcal{L}_F^\mu(w^+_{\mu,\rho}) & \text{in} \quad B^-(x_0,\rho); \\ w^-_{\mu,\rho} \leq 0 \leq w^+_{\mu,\rho} & \text{on} \quad \partial B^-(x_0,\rho), \end{cases}$$

where $w^-_{\mu,\rho}(x) = \sigma_{\mu,r_F^{-1}\rho,0}(\|x - x_0\|)$ and $w^+_{\mu,\rho}(x) = \sigma_{\mu,r_F\rho,0}(\|x - x_0\|)$, respectively. □

Proof of Theorem 1.4 Let $x_0 \in M$ be fixed and we assume that for some $\mu \in [0, l_F r_F^{-2}\overline{\mu})$, the function $u(x) = \sigma_{\mu,\rho,c}(d_F(x_0, x))$ is the unique pointwise solution of $(\mathcal{P}^{\mu}_{B^+(x_0,\rho)})$ on $B^+(x_0, \rho) \setminus \{x_0\}$ for some $\rho > 0$. By Lemma 5.1 and from the fact that $\sigma_{\mu,\rho,c}$ is a solution of $(\mathcal{Q}^{\mu}_{c,\rho})$, it follows that

$$\boldsymbol{\Delta} d_F(x_0, x) = (n-1)\mathbf{ct}_c(d_F(x_0, x)) \text{ in } B^+(x_0, \rho) \setminus \{x_0\}$$

pointwisely. The latter relation and a simple calculation shows that

$$\boldsymbol{\Delta} w_c(d_F(x_0, x)) = 1 \text{ in } B^+(x_0, \rho) \setminus \{x_0\},$$

where

$$w_c(r) = \int_0^r \mathbf{s}_c(s)^{-n+1} \int_0^s \mathbf{s}_c(t)^{n-1} \mathrm{d}t\mathrm{d}s. \tag{5.1}$$

Let $0 < \tau < \rho$ be fixed arbitrarily. The unit outward normal vector to the forward geodesic sphere $S^+(x_0, \tau) = \partial B^+(x_0, \tau) = \{x \in M : d_F(x_0, x) = \tau\}$ at $x \in S^+(x_0, \tau)$ is given by $\mathbf{n} = \boldsymbol{\nabla} d_F(x_0, x)$. Let us denote by $\mathrm{d}\varsigma_F(x)$ the canonical volume form on $S^+(x_0, \tau)$ induced by $\mathsf{dm}(x) = \mathrm{d}V_F(x)$. By Stokes' formula (see [26], [30, Lemma 3.2]) and $g_{(x,\mathbf{n})}(\mathbf{n}, \mathbf{n}) = F(x, \mathbf{n})^2 = F(x, \boldsymbol{\nabla} d_F(x_0, x))^2 = 1$ [see (2.16)], on account of relation (5.1) we have

$$\begin{aligned}
\mathrm{Vol}_F(B^+(x_0, \tau)) &= \int_{B^+(x_0,\tau)} \boldsymbol{\Delta}(w_c(d_F(x_0, x)))\mathsf{dm}(x) \\
&= \int_{B^+(x_0,\tau)} \mathrm{div}(\boldsymbol{\nabla}(w_c(d_F(x_0, x))))\mathsf{dm}(x) \\
&= \int_{S^+(x_0,\tau)} g_{(x,\mathbf{n})}(\mathbf{n}, w_c'(d_F(x_0, x))\boldsymbol{\nabla} d_F(x_0, x))\mathrm{d}\varsigma_F(x) \\
&= w_c'(\tau) \times \mathrm{Area}_F(S^+(x_0, \tau)).
\end{aligned}$$

Therefore,

$$\frac{\mathrm{Area}_F(S^+(x_0, \tau))}{\mathrm{Vol}_F(B^+(x_0, \tau))} = \frac{1}{w_c'(\tau)} = \frac{\mathbf{s}_c(\tau)^{n-1}}{\int_0^\tau \mathbf{s}_c(t)^{n-1}\mathrm{d}t},$$

or equivalently,

$$\frac{\frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{Vol}_F(B^+(x_0, \tau))}{\mathrm{Vol}_F(B^+(x_0, \tau))} = \frac{\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\tau \mathbf{s}_c(t)^{n-1}\mathrm{d}t}{\int_0^\tau \mathbf{s}_c(t)^{n-1}\mathrm{d}t}.$$

Integrating the latter expression on the interval $[s, \rho]$, $0 < s < \rho$, and exploiting (2.6), it follows that

$$\frac{\mathrm{Vol}_F(B^+(x_0, \rho))}{V_{c,n}(\rho)} = \lim_{s \to 0^+} \frac{\mathrm{Vol}_F(B^+(x_0, s))}{V_{c,n}(s)} = 1. \tag{5.2}$$

According to Theorem 2.1, it yields

$$\mathbf{K}(\cdot; \dot{\gamma}_{x_0,y}(t)) = c$$

for every $t \in [0, \rho)$ and $y \in T_{x_0}M$ with $F(x_0, y) = 1$, where $\gamma_{x_0,y}$ is the constant speed geodesic with $\gamma_{x_0,y}(0) = x_0$ and $\dot{\gamma}_{x_0,y}(0) = y$. This concludes the proof. □

Proof of Theorem 1.5 The implications "(a)⇒(b)" and "(c)⇒(a)" are trivial, see Corollary 5.1; it remains to prove "(b)⇒(c)". By the Proof of Theorem 1.4 we know the validity of relation (5.2) for every $\rho > 0$. Let $x \in M$ and $\rho > 0$ be arbitrarily fixed. We have that

$$\begin{aligned}
1 &\leq \frac{\mathrm{Vol}_F(B^+(x,\rho))}{V_{0,n}(\rho)} \quad (\text{see}(2.8)) \\
&\leq \limsup_{r\to\infty} \frac{\mathrm{Vol}_F(B^+(x,r))}{V_{0,n}(r)} \quad (\text{monotonicity from Theorem 2.1}) \\
&\leq \limsup_{r\to\infty} \frac{\mathrm{Vol}_F(B^+(x_0, r+d_F(x_0,x)))}{V_{0,n}(r)} \quad (B^+(x,r) \subset B^+(x_0, r+d_F(x_0,x))) \\
&= \limsup_{r\to\infty} \left(\frac{\mathrm{Vol}_F(B^+(x_0, r+d_F(x_0,x)))}{V_{0,n}(r+d_F(x_0,x))} \times \frac{V_{0,n}(r+d_F(x_0,x))}{V_{0,n}(r)} \right) \\
&= 1, \quad (\text{see } (5.2))
\end{aligned}$$

because one has

$$\lim_{r\to\infty} \frac{V_{0,n}(r+d_F(x_0,x))}{V_{0,n}(r)} = \lim_{r\to\infty} \frac{(r+d_F(x_0,x))^n}{r^n} = 1. \tag{5.3}$$

Consequently,

$$\mathrm{Vol}_F(B^+(x,\rho)) = V_{0,n}(\rho) = \omega_n \rho^n \text{ for all } x \in M \quad \text{and} \quad \rho > 0. \tag{5.4}$$

On account of Theorem 2.1 and relation (5.4), we conclude that $\mathbf{K} = 0$.

Note that every Berwald space with $\mathbf{K} = 0$ is necessarily a locally Minkowski space, see Bao et al. [5, Section 10.5]. Therefore, the global volume identity (5.4) actually implies that (M, F) is isometric to a Minkowski space. □

Acknowledgments Research supported by a Grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project no. PN-II-ID-PCE-2011-3-0241. Cs. Farkas is also supported by the State of Hungary, co-financed by the European Social Fund in the framework of TÁMOP 4.2.4.A/2-11-1-2012-0001 'National Excellence Program' and by Collegium Talentum. The research of A. Kristály is also supported by János Bolyai Research Scholarship.

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