On the relationship between standard intersection cuts, lift-and-project cuts, and generalized intersection cuts

Egon Balas[1] and Tamás Kis[2]

[1]Tepper School of Business, Carnegie Mellon University, Pittsburgh, Pennsylvania 15213, eb17@andrew.cmu.edu
[2]Institute for Computer Science and Control, H-1518 Budapest, Hungary, kis.tamas@sztaki.mta.hu

revised 08.10.15

Abstract

We examine the connections between the classes of cuts in the title. We show that lift-and-project (L&P) cuts from a given disjunction are equivalent to generalized intersection cuts (GICs) from the family of polyhedra obtained by taking positive combinations of the complements of the inequalities of each term of the disjunction. While L&P cuts from split disjunctions are known to be equivalent to standard intersection cuts (SICs) from the strip obtained by complementing the terms of the split, we show that L&P cuts from more general disjunctions may not be equivalent to any SIC. In particular, we give easily verifiable necessary and sufficient conditions for a L&P cut from a given disjunction D to be equivalent to a SIC from the polyhedral counterpart of D. Irregular L&P cuts, i.e. those that violate these conditions, have interesting properties. For instance, unlike the regular ones, they may cut off part of the corner polyhedron associated with the LP solution from which they are derived. Furthermore, they are not exceptional: their frequency exceeds that of regular cuts. A numerical example illustrates some of the above properties.

Keywords: integer programming, intersection cuts, lift-and-project cuts, generalized intersection cuts, corner polyhedra

1 Introduction

Consider a mixed integer program

$$\min\{cx : x \in P_I\} \tag{MIP}$$

and its linear programming relaxation

$$\min\{cx : x \in P\}, \tag{LP}$$

where

$$\begin{aligned} P &:= \{x \in \mathbb{R}^N : Ax \geq b, x \geq 0\} \\ &= \{x \in \mathbb{R}^N : \tilde{A}x \geq \tilde{b}\}, \quad N = \{1, \ldots, n\}, \end{aligned}$$

A is $m \times n$ and

$$P_I := P \cap \{x \in \mathbb{R}^N : x_j \in \mathbb{Z}, j \in N' \subseteq N\}.$$

All data are assumed to be rational.

The *integer hull* of P is $\operatorname{conv} P_I$. Next we extend the vector x by m new components representing the surplus variables, i.e. $x_{n+i} = \sum_{j \in N} a_{ij} x_j - b_i$. If $\bar{x}$ is an optimal solution of (LP) with basic and nonbasic index sets I and J, respectively, then the optimal simplex tableau can be written as

$$\begin{aligned} x_i &= \bar{a}_{i0} - \sum_{j \in J} \bar{a}_{ij} x_j, \qquad && i \in I \\ x_j &\geq 0 && j \in I \cup J. \end{aligned}$$

When it is not clear from the context, we will denote by x_N the set of variables $x_1, \ldots, x_n$, and by x_J the set of nonbasic variables from x_1 through x_{n+m}.

The LP cone $C(J)$ is the projection onto the space of structural variables x_N of the pointed polyhedral cone in $\mathbb{R}^{n+m}$ with apex $\bar{x}$ and n extreme rays with direction vectors r^j, $j \in J$, where $r^j_i = -\bar{a}_{ij}$ for $i \in I$, $r^j_i = 0$ for $i \in J \setminus \{j\}$, and $r^j_j = 1$. The convex hull of integer points in $C(J)$ is known as the *corner polyhedron*, denoted corner(J) [16]. The relationship between the various relaxations of P_I introduced above can be summarized by $C(J) \supset P \supset \operatorname{conv} P_I$ and $C(J) \supset \text{corner } (J) \supset \operatorname{conv} P_I$.

One well known way of generating valid cutting planes for (MIP) is to intersect the n extreme rays of $C(J)$ with the boundary of some convex set S whose interior contains $\bar{x}$ but no feasible integer point. We will call such a set P_I*-free*. If the extreme rays $\bar{x} + r^j \lambda_j$, $\lambda_j \geq 0$, $j \in J$, intersect the boundary of S at the points defined by $\lambda_j = \lambda^*_j$, $j \in J$, then the hyperplane through these n points defines the *intersection cut*

$$\sum_{j \in J} \frac{1}{\lambda_j^*} x_j \geq 1 \tag{1}$$

in terms of the nonbasic variables J, shortly denoted by $\pi_J x_J \geq 1$, valid for P_I [3]. When expressed in terms of the structural variables, the same cut will be denoted $\pi_N x_N \geq \pi_0$. Intersection cuts defined as above, i.e. as cuts obtained by intersecting the extreme rays of the cone $C(J)$ with the boundary of some P_I-free set S, were introduced in the early 1970's [3, 4]. More recently, intersection cuts became the focus of renewed interest in the context of cut generation from multiple rows of the simplex tableau (see [1], [15], [13]). However, this more recent literature uses a narrower definition of intersection cuts, namely as cuts obtained by intersecting the extreme rays of $C(J)$ with the boundary of some convex set S' whose interior contains no integer point (feasible or not). Such a set is called *lattice-free.* This definition is narrower than the original one, as it excludes cuts from P_I-free sets that are not lattice-free. In the sequel we will call intersection cuts from P_I-free sets standard intersection cuts (SIC's), and those from lattice-free sets, restricted intersection cuts (RIC's). See [8] for a discussion of the relationship between the two. If S and S' are two lattice-free convex sets and $S \subset S'$, then the intersection cut from S' will dominate the one from S. Furthermore, it is well known that any inclusion-maximal lattice-free set is polyhedral [18]. In the sequel we will always consider convex sets that are inclusion-maximal, hence polyhedral.

When the set S used to generate the intersection cuts is a strip of the form $S := \{x \in \mathbb{R}^N : \lfloor \bar{a}_{k0} \rfloor \leq x_k \leq \lceil \bar{a}_{k0} \rceil\}$ for some fractional $\bar{a}_{k0}$, where $k \in N_1 \subseteq I$, then (1) is the mixed integer Gomory cut (when the nonbasic variables are continuous), also known as the simple disjunctive cut $\gamma x_J \geq \gamma_0$ from the condition $x_k \leq \lfloor \bar{a}_{k0} \rfloor \vee x_k \geq \lceil \bar{a}_{k0} \rceil$, whose coefficients are

$$\gamma_j = \max\left\{\frac{\bar{a}_{kj}}{f_{k0}}, \frac{-\bar{a}_{kj}}{1 - f_{k0}}\right\}, j \in J,$$

where $f_{k0} = \bar{a}_{k0} - \lfloor \bar{a}_{k0} \rfloor$.

More generally, an intersection cut from a *polyhedral* set

$$S := \{x \in \mathbb{R}^N : d^t x \leq d_{t0}, t \in T\} \tag{2}$$

whose interior contains no feasible integer point, is equivalent to a *disjunctive* cut, i.e. one from a disjunction

$$\{x \in \mathbb{R}^N : \vee_{t \in T}(d^t x \geq d_{t0})\} \tag{3}$$

whose terms are the weak complements of the inequalities defining S. Conversely, cuts from a disjunction of the type (3), i.e. with a single inequality in each term, called *simple* disjunctive cuts, can be viewed as (are equivalent to) intersection cuts from the polyhedron (2) defined by the weak complements of the inequalities in (3). This straightforward connection between intersection cuts and disjunctive cuts, however, breaks down in the case of cuts from more general disjunctions, like

$$\{x \in \mathbb{R}^N : \vee_{t \in T}(D^t x \geq d_0^t)\} \tag{4}$$

where the disjunctive set is a union of polyhedra more general than single halfspaces: a cut from (4) cannot be obtained as an intersection cut by complementing the inequalities of (4). However, every cut obtained from (4) can also be obtained from some simple disjunction of the form

$$\{x \in \mathbb{R}^N : \vee_{t \in T}(\delta^t x \geq \delta_{t0})\} \tag{5}$$

where for each $t \in T$, $\delta^t x \geq \delta_{t0}$ is a nonnegative linear combination of the inequalities $D^t x \geq d_{t0}$; and so the collection of disjunctive cuts from (4) is equivalent to the collection of intersection cuts from all members of the family of polyhedra obtainable by taking nonnegative combinations of the inequalities of each term of (4) and complementing the resulting inequalities. Such a family of polyhedra may be thought of as a *parametric* polyhedron, where the parameters are the weights assigned to the inequalities of each $D^t x \geq d_{t0}$.

Returning now to the simple disjunction (3), its application to the linear program with feasible set P gives rise to the condition

$$\{x \in P : \vee_{t \in T}(d^t x \geq d_{t0})\} = \left\{ x \in \mathbb{R}^N : \vee_{t \in T} \begin{pmatrix} \tilde{A}x & \geq & \tilde{b} \\ d^t x & \geq & d_{t0} \end{pmatrix} \right\}, \tag{6}$$

and while (6) is clearly valid for all $x \in P$ satisfying (3), introducing the constraints of P into each term of the disjunction (3) yields a stronger condition: the disjunctive cuts derived from (6) dominate those from (3). Cuts derived from a disjunction of the type (6), i.e. a disjunction amended by introducing into each term the set of constraints valid for each of them, are known as *lift-and-project* (L&P) cuts [6]. This term was originally introduced for the case of 0-1 programs, where the inequalities $d^t x \geq d_{t0}$ of the disjunction are of the form $x_t \leq 0$ or $x_t \geq 1$, but here we use the term for the more general case of disjunctions of the form (6). The name reflects the fact that members of such a family of cuts can be obtained as projected solutions of a higher-dimensional linear program describing the convex hull of the union of polyhedra defined by (6) [5].

In particular, each facet $\alpha x \geq \beta$ of this convex hull is given by the (α, β)-component of a basic feasible solution to a system of the form

$$\begin{array}{rclcl} \alpha - u^t\tilde{A} & - & u_0^t d^t & = & 0 \\ -\beta + u^t\tilde{b} & + & u_0^t d_{t0} & = & 0 \\ \sum_{t\in T}(u^t e & + & u_0^t) & = & 1 \\ & & u^t, u_0^t & \geq & 0, \end{array} \quad \begin{array}{l} t \in T \\ \\ \\ t \in T \end{array} \tag{7}$$

where $e = (1, \ldots, 1)^T$ and the last equation is a normalization constraint. Minimizing some linear form $p\alpha - \beta$ over (7), i.e. solving a cut generating linear program (CGLP) yields an inequality $\alpha x \geq \beta$ that cuts off p by a maximum amount.

In the case when the constraint (3) underlying (6) is a 2-term split disjunction, i.e. one of the form $\pi x \leq \pi_0 \vee \pi x \geq \pi_0 + 1$, where $\pi \in \mathbb{R}^n$ with $\pi_j \in \mathbb{Z}$, $j \in N'$, $\pi_j = 0$, $j \in N \setminus N'$, and $\pi_0 \in \mathbb{Z}$, the family of L&P cuts is known to be equivalent to the family of intersection cuts (or simple disjunctive cuts, or Gomory mixed-integer cuts). More precisely, there is a many-to-one correspondence between basic feasible solutions to the CGLP and basic (feasible or infeasible) solutions to the LP (cf. [10]). However, this equivalence does not carry over to cuts derived from non-split disjunctions. As we will show and discuss in detail in this paper, in the case of cuts from multiple-term disjunctions, or non-split two-term disjunctions, the above correspondence breaks down: there are basic feasible solutions to the CGLP that have no counterpart in any feasible or infeasible basic solution to LP. This fact has far reaching consequences with regard to the nature of cuts from these more general disjunctions. For one thing, intersection cuts derived from an LP cone and a lattice-free convex set have been shown [12] to always be valid for the associated corner polyhedron; hence they can never cut off any part of the latter. This of course is also true for their L&P counterparts. By contrast, as we will show, L&P cuts from disjunctions corresponding to lattice-free convex sets may cut off parts of the associated corner polyhedron, i.e. may have no intersection cut counterpart.

The last class of cuts that we will discuss is that of generalized intersection cuts (GIC's), introduced in [9]. These are all cuts valid for $P \setminus \operatorname{int} S$ for some maximal P_I-free set S. They are called GIC's because they can be obtained by intersecting the boundary of S with appropriate subsets of edges of P, in a way analogous to intersection cuts.

Let $\tilde{A}_K$ be a full column rank submatrix of $\tilde{A}$ with rows indexed by K, $|K| > n$, and let $C_K := \{x \in \mathbb{R}^n : \tilde{A}_K x \geq \tilde{b}_K\}$. Further, let E be the set of edges $E_j := (v, w)$ of C_K with $v \in \operatorname{int} S$, $w \notin \operatorname{int} S$, and let Q be the set of intersection points $p^j := E_j \cap \operatorname{bd} S$. Then any solution to either of the systems

$$\alpha p^j \geq \beta, \quad p^j \in Q, \tag{8}$$

for $\beta \in \{1, -1, 0\}$ such that $\alpha z < \beta$ for some vertex z of C_K yields a valid cut for $\operatorname{conv} P_I$, called a GIC (see [9] for a proof and details).

There are rules for generating proper sets Q short of producing all the intersection points, based on starting with the edges of some LP cone $C(J)$ and obtaining additional edges by successively activating the hyperplanes defining P. It is easy to see that if $C_K = P$, i.e. K is the row index set of $\tilde{A}$, then every GIC from S defines a facet of $\operatorname{conv}(P \setminus \operatorname{int} S)$; and conversely, every facet of $\operatorname{conv}(P \setminus \operatorname{int} S)$ that is not a facet of P is defined by a GIC from S.

It then follows that the set of all GIC's from a P_I-free set S defines all the facets of $\operatorname{conv}(P \setminus \operatorname{int} S)$ that are not facets of P.

*

In the next section we discuss the relationship of GIC's to L&P cuts and show that the two classes are equivalent. Section 3 deals with the relationship of SIC's and L&P cuts and it shows that the correspondence established in [10] between SIC's of a special type, namely split cuts, and L&P cuts from an associated higher dimensional CGLP (cut generating linear program), does not carry over to SIC's that are not split cuts. Theorem 7 establishes a correspondence between SIC's (whether split or not) for the LP relaxation, on the one hand, and L&P cuts from a particular class of basic feasible solutions to the corresponding CGLP, on the other. Theorem 9 and 10 establish sufficient conditions for a L&P cut from a general disjunction to have as a counterpart a SIC from some LP basis, while Theorems 11 and 12 define the situations in which these sufficient conditions are also necessary. The upshot of these results is that L&P cuts that do not meet the above necessary and sufficient conditions, called irregular (as opposed to those that meet them, called regular), have some highly desirable properties. In particular, they cut off the optimal LP solution by more than any of the SIC's. Furthermore, as discussed in section 4, the irregular L&P cuts from disjunctions representing multiple (as opposed to simple) splits cut off parts of the corner polyhedron associated with the optimal LP basis. The following section (5) deals with the frequency of irregular cuts, while the concluding section (6) gives a numerical example illustrating the properties of irregular cuts.

2 GIC's and L&P cuts

Given a disjunction of the form (6), the family of L&P cuts $\alpha x \geq \beta$ from (6) is given by the solution set to the system (7). In particular, we have the following

Proposition 1. *In any basic feasible solution to (7) that yields an inequality $\alpha x \geq \beta$ not dominated by the constraints of (LP), $u_0^t > 0$ for all $t \in T$.*

Proof. If $u_0^t = 0$ for some $t \in T$, then $\alpha = u^t \tilde{A}$, $\beta = u^t \tilde{b}$; i.e., $\alpha x \geq \beta$ is a nonnegative linear combination of the inequalities of $\tilde{A}x \geq \tilde{b}$. $\square$

It follows from Proposition 1 that the only basic feasible solutions to the CGLP associated with a disjunction of type (6) that yield actual cuts, i.e. inequalities that cut off some part of P, are those with $u_0^t > 0$ for all $t \in T$.

Theorem 2. *Let*

$$S := \{x \in \mathbb{R}^N : d^t x \leq d_{t0}, \quad t \in T\}$$

be a maximal polyhedron such that $P_I \cap \operatorname{int} S = \emptyset$. The family of GIC's from S is equivalent to the family of those L&P cuts associated with basic feasible solutions to (7) such that $u_0^t > 0$ for all $t \in T$.

Proof. The family of L&P cuts from (7) is known to define the convex hull of the corresponding union of polyhedra; and the family of L&P cuts coming from basic feasible solutions to (7) such that $u_0^t > 0$ for all $t \in T$ is known to define all the facets of this convex hull that do not belong to the constraints defining P. Since the collection of these facets defines $\operatorname{conv}(P \setminus \operatorname{int} S)$, it implies all GIC's.

Conversely, the family of GIC's from S defines $\operatorname{conv}(P \setminus \operatorname{int} S)$. To see this, note that for $C_K = P$ in the definition of (8), Q is the set of all intersection points of edges of P with $\operatorname{bd} S$; and thus every basic solution to (8) defines a facet of $\operatorname{conv}(P \setminus \operatorname{int} S)$, and in turn every facet of $\operatorname{conv}(P \setminus \operatorname{int} S)$ corresponds to such a basic solution. $\square$

Consider now a disjunction of type (4) and its L&P counterpart

$$\left\{ x \in \mathbb{R}^N : \vee_{t \in T} \left(\begin{array}{ccc} \tilde{A}x & \geq & \tilde{b} \\ D^t x & \geq & d_0^t \end{array} \right) \right\} \tag{9}$$

The inequalities $\alpha x \geq \beta$ defining the convex hull of the union of polyhedra (9) are given by the (α, β)-components of basic feasible solutions to the system

$$\begin{array}{rcrcl} \alpha - u^t \tilde{A} & - & v^t D^t & = & 0 \\ -\beta + u^t \tilde{b} & + & v^t d_0^t & = & 0 \\ \sum_{t \in T} (u^t e & + & v^t e) & = & 1 \\ & & u^t, v^t \geq 0, & & t \in T \end{array} \qquad t \in T \tag{10}$$

where e is the summation vector with the required number of components.

A generalization of Proposition 1 for this case reads

Proposition 3. *In any basic feasible solution to (10) that yields an inequality $\alpha x \geq \beta$ not dominated by the constraints of (LP), v^t has at least one positive component for each $t \in T$.*

Proof. Same as for Proposition 1. $\square$

Theorem 4. *Let $\bar{\alpha}x \geq \bar{\beta}$ be the L&P cut associated with the basic feasible solution $(\bar{\alpha}, \bar{\beta}, \{\bar{u}^t, \bar{v}^t\}_{t\in T})$ to (10), where $\bar{v}^t e > 0$ for all $t \in T$. Then $\bar{\alpha}x \geq \bar{\beta}$ is a GIC from the maximal P_I-free polyhedron*

$$S(\bar{v}) := \{x \in \mathbb{R}^N : (\bar{v}^t D^t)x \leq \bar{v}^t d_0^t, t \in T\}.$$

Proof. Clearly, $\operatorname{int} S(\bar{v})$ contains no point satisfying the disjunction (9). Now denote $\delta^t := \bar{v}^t D^t$, $\delta_{t0} := \bar{v}^t d_0^t$, $t \in T$. Then $S(\bar{v})$ becomes

$$S(\bar{v}) = \{x \in \mathbb{R}^N : \delta^t x \leq \delta_{t0}, t \in T\},$$

and (10) becomes

$$\begin{array}{rcll} \alpha \quad - u^t\tilde{A} & - & v_0^t\delta^t = 0 & \\ -\beta + u^t\tilde{b} & + & v_0^t\delta_{t0} = 0 & \quad t \in T \\ \sum_{t\in T}(u^t e & + & v_0^t) = 1 & \\ & & u^t, v_0^t \geq 0, \quad t \in T & \end{array} \tag{10'}$$

which is of the form (7). From Theorem 2, the family of GIC's from $S(\bar{v})$ is equivalent to the family of those L&P cuts associated with basic feasible solutions to (10′) such that $v_0^t > 0$ for all $t \in T$. Theorem 4 then follows. $\square$

Note that while in the case of a simple disjunction of the form (6), a L&P inequality from (7) (based on (6)) corresponds to a GIC from the P_I-free polyhedron S obtained by complementing the inequalities of (6), in the case of a disjunction (9) with multiple inequalities per term, a L&P inequality from (10) (based on (9))corresponds to a GIC from a P_I-free polyhedron obtained by complementing a particular nonnegative combination of the inequalities of each term of the disjunction, where the weights of the combination are given by the particular solution of (10) which yields the L&P cut. Thus a single disjunction gives rise in this case to a multitude of P_I-free polyhedra, one for each basic solution to (10).

An important special case of (9) is the one where the disjunction (4) is simply the disjunctive normal form of a set of simultaneously applied split disjunctions like $x_j \leq \lfloor \bar{x}_j \rfloor$ or $x_j \geq \lceil \bar{x}_j \rceil$. If such a condition is applied to a mixed 0-1 program $(\mathrm{MIP})_{0-1}$ with p 0-1 variables, then (9) is just a restatement in another form of the constraint set of $(\mathrm{MIP})_{0-1}$; hence the set of L&P cuts from (10) is the set of GIC's from all maximal lattice-free convex sets.

The family of GIC's that is the object of Theorems 2 and 4 is that of valid cuts obtainable from intersecting all edges of the LP relaxation P with the boundary of S. However, GIC's can be generated from a small fraction of such intersection points. Starting with the LP cone $C(J) = \cap_{i \in J} H_i^+$ with apex at a basic solution $\bar{x}$ and facets H_i^+, $i \in J$, corresponding to the inequalities tight at $\bar{x}$, one can successively activate additional boundary-hyperplanes H_i of P [9]. At any stage of the procedure, GIC's can be generated by intersecting with $\mathrm{bd}\, S$ (respectively $\mathrm{bd}\, S(\bar{v})$) edges of a relaxation of P of the form $P_K := \cap_{i \in K} H_i^+ = \{x \in \mathbb{R}^N : \tilde{A}_K x \geq \tilde{b}_K\}$, where $K \supset J$, and the matrix $\tilde{A}_K$, consisting of the rows of $\tilde{A}$ indexed by K, contains the $n \times n$ nonsingular submatrix $\tilde{A}_J$. Then we have

Corollary 5. *Let S (or $S(\bar{v})$) be as in Theorem 2 (or 4). Then the family of GIC's from S (or $S(\bar{v})$) and $P_K := \cap_{i \in K} H_i^+$ is equivalent to the family of L&P cuts associated with basic feasible solutions to the system (7′) (or 10′) obtained from (7) (respectively (10)) by substituting $(\tilde{A}_K, \tilde{b}_K)$ for $(\tilde{A}, \tilde{b})$, such that $u_0^t > 0$ (or $v^t e > 0$) for all $t \in T$.*

Proof. Obvious, by applying Theorem 2 (or 4) to P_K instead of P. □

The above Theorems and Corollary fall short of establishing the precise correspondence between a given GIC and some equivalent L&P cut, and vice-versa, as done in [10] for simple disjunctive cuts (equivalent to SIC's) and L&P cuts. This task is made difficult by the fact that, unlike in the case of SIC's, GIC's are generated from the intersection points with the boundary of S (or $S(\bar{v})$) of rays of the form $r^{ij} = v^i - \bar{a}_j x_j$ originating at several vertices v^i instead of just one. So a typical GIC is a basic solution to a system of the form (8), where the points $p \in Q$ do not have the simple structure they have in the case of SIC's. However, the following connection to L&P cuts can be proved.

Proposition 6. *A GIC defined by a basic solution to a system of the form (8) is equivalent to a L&P cut defined by a basic feasible solution to the corresponding system (7) (or (10)) such that $u_0^t > 0$ (respectively $v^t e > 0$) for all $t \in T$ and vice-versa.*

Proof. Let $(\bar{\alpha}, \bar{\beta})$ be a basic solution to (8), with $\bar{\alpha} p^j = \bar{\beta}$ for n intersection points $p^j \in Q$, let their set be Q'. From Theorem 2 (or 4), the cut $\bar{\alpha} x \geq \bar{\beta}$ is implied by some positive combination of L&P cuts coming from basic feasible solutions to (7) (or (10)) such that $u_o^t > 0$ (or $v^t e > 0$) for all $t \in T$. Let such a positive combination be $(\sum_i \gamma^i \lambda_i) x \geq \sum_i \beta_i \lambda_i$ with $\sum_i \gamma^i \lambda_i \leq \bar{\alpha}$ and $\sum_i \beta_i \lambda_i = \bar{\beta}$, $\lambda_i > 0$, for all i, $\gamma^{i_1} \neq \gamma^{i_2}$ for all $i_1 \neq i_2$. But then at least one of the inequalities $\gamma^i x \geq \beta_i$ cuts off one of the n intersection points $p^j \in Q'$, contrary to Theorem 2 (or 4) which states that the two families are equivalent. Hence there can be only one inequality $\gamma x \geq \bar{\beta}$ implying $\bar{\alpha} x \geq \bar{\beta}$. A similar argument shows that a L&P cut defined by a basic feasible solution to (7) (or (10)) such that $u_0^t > 0$ (or $v^t e > 0$) for all $t \in T$ is implied by a unique GIC defined by a basic solution to (8). $\square$

3 SIC and L&P cuts

As mentioned in Section 1, in the case of split cuts, standard intersection cuts are equivalent to lift-and-project cuts. More specifically, every intersection cut from an LP basis and a P_I-free convex set S is equivalent to a lift-and-project cut from a basic feasible solution to the CGLP associated with the disjunction corresponding to S, and every L&P cut from a basic feasible solution to such a CGLP is equivalent to a standard intersection cut from a (feasible or infeasible) basis of (LP) and the convex set S [10]. However, this equivalence does not extend beyond the realm of cuts from two-term disjunctions of the form $\pi x \leq \pi_0 \vee \pi x \geq \pi_0 + 1$.

In this section we examine the general case. First we show that any standard intersection cut from a convex polyhedral set S can be represented as a solution of the CGLP (7) corresponding to S, and then we give a sufficient condition for an L&P cut to correspond to an intersection cut (1) for some LP basis (I, J).

First, we derive the intersection cut (1) as a simple disjunctive cut. Recall the definition of set S. For $j \in J$, let λ^*_{tj} be the value of λ for which the ray $\rho^j := \bar{a}_0 - \lambda \bar{a}_j$ hits the hyperplane defined by $d^t x_N = d_{t0}$, and let $\lambda^*_{tj} = \infty$ if ρ^j does not cross this hyperplane. Now we can derive the intersection cut (1) by the following formula: $\pi_j := \max_{t \in T} \pi^t_j$, where $\pi^t_j := 1/\lambda^*_{tj}$, and then

$$\sum_{j \in J} \pi_j x_j \geq 1. \tag{11}$$

We can compute λ^*_{tj} by substituting r^j into $d^t x_N = d_{t0}$:

$$d^t(\bar{a}_0 - \lambda \bar{a}_j) = d_{t0}.$$

Rearranging terms gives

$$\lambda^*_{tj} = \frac{d_{t0} - d^t \bar{a}_0}{-d^t \bar{a}_j} = \frac{\bar{d}_{t0}}{\bar{d}^t_j}, \tag{12}$$

where the second equation follows from Proposition 1 in [17]. Since $\bar{a}_0$ is an interior point of $S = \{x \ \in \mathbb{R}^N \mid d^t x \leq d_{t0}, t \in T\}$, we see that the numerator in (12) is strictly positive. However, the denominator is non-positive if the angle between the ray r^j and the hyperplane $d^t x = d_{t0}$ is between 90° and 270°.

In order to simplify notation, we define $\hat{A} := \tilde{A}_J$, and $\hat{b} := \tilde{b}_J$ as the rows of $\tilde{A}$ and $\tilde{b}$, respectively, indexed by the nonbasic variables J.

Theorem 7. *Let J be the set of nonbasic variables in some basis of the LP relaxation of (MIP), and suppose $\bar{x} = \bar{a}_0$ is an interior point of set S defined by (2). Then the intersection cut $\pi x_J \geq 1$ derived from S and the corresponding simplex tableau is equivalent to the L&P cut $\alpha x_N \geq \beta$ from a basic solution to (7) in which, for each $t \in T$, all but one of the variables u^t_j with $j \in J$ are basic, and all the variables u^t_j with $j \notin J$ are non-basic, except the u^t_0,* *which are all basic and positive. The solution of (7) is given by*

$$\theta\alpha := \pi\hat{A}$$

$$\theta\beta := \pi\hat{b} + 1$$

$$\theta u_J^t := \pi - \pi^t, \quad t \in T$$

$$\theta u_0^t := 1/(-\bar{d}_{t0}), \quad t \in T,$$

where $\theta > 0$ is a scaling factor.

Proof. First we verify that $\alpha = u_J^t\hat{A} + u_0^t d^t$. That is,

$$\theta(\alpha - u_J^t\hat{A} - u_0^t d^t) = \pi\hat{A} - (\pi - \pi^t)\hat{A} + \frac{1}{\bar{d}_{t0}}d^t = \pi^t\hat{A} + \frac{1}{\bar{d}_{t0}}d^t = \sum_{j\in J}\frac{d^t\bar{a}_j}{\bar{d}_{t0}}\hat{A}_j + \frac{1}{\bar{d}_{t0}}d^t$$

$$= \frac{1}{\bar{d}_{t0}}\left(\sum_{j\in J, i\in N} d_i^t(\bar{a}_{ij})\hat{A}_j + d^t\right) = \frac{1}{\bar{d}_{t0}}\left(\sum_{i\in N} d_i^t \sum_{j\in J}(-e_i^T\hat{A}^{-1})_j\hat{A}_j + d^t\right) = 0,$$

where we used the fact that $-(e_i^T\hat{A}^{-1})_j = \bar{a}_{ij}$ [10], and $\sum_{j\in J}(e_i^T\hat{A}^{-1})_j\hat{A}_j = e_i^T$. Here θ is well defined, as both sides of the equation are of the same sign. Since $\bar{a}_0 \in \operatorname{int} S$, each u_0^t is positive. One may similarly verify that $\beta = u_J^t\hat{b} + u_0^t d_{t0}$. The scaling factor θ is used to ensure that $\sum_{t\in T}(u_0^t + \sum_{j\in J} u_j^t) = 1$.

Now we show that the two cuts are equivalent. That is, using the equations already proved, and the fact that $\hat{A}x_N - \hat{b} = x_J$ [10], we have

$$\theta(\alpha x_N - \beta) = \pi\hat{A}x_N - (\pi\hat{b} + 1) = \pi x_J - 1.$$

Finally, we prove that the solution of CGLP constructed above is basic. First, we remove from (7) the variables u_j^t corresponding to $j \notin J$, and eliminate the α and β variables to obtain

$$\begin{array}{rcrcrcrcll} u_J^1\hat{A} & - & u_J^t\hat{A} & + & u_0^1 d^1 & - & u_0^t d^t & = & 0 & \multirow{2}{*}{$t \in T \setminus \{1\}$} \\ u_J^1\hat{b} & - & u_J^t\hat{b} & + & u_0^1 d_{10} & - & u_0^t d_{t0} & = & 0 & \\ & & & & \sum_{t\in T}(u^t e & + & u_0^t) & = & 1 & \\ & & & & & & u^t, u_0^t & \geq & 0, & t \in T. \end{array} \tag{13}$$

Now we construct a basis of (13). Let $M_t' = \{j \in J \; : \; (\pi - \pi^t)_j > 0\}$. Notice that no $j \in J$ may belong to all the sets M_t', $t \in T$, since $\pi_j = \max_{t\in T}\pi_j^t$, and thus for each $j \in J$

there exists $t \in T$ with $(\pi - \pi^t)_j = 0$. Now if some $j \in J$ belongs to less than $|T| - 1$ of the sets M'_t (which may occur if $\pi_j = \pi^t_j$ for more than one $t \in T$), then we assign each such j arbitrarily to some of the sets M'_t so that finally we obtain the sets M_t, $t \in T$, and each $j \in J$ occurs exactly in $|T| - 1$ of these sets.

After these preparations, we claim that the variables $G = \{u^t_0 : t \in T\} \cup \big(\bigcup_{t\in T}\{u^t_j : j \in M_t\}\big)$ constitute a basis of (13). To prove our claim, we derive a new system of equations from (13) as follows. Since $\hat{A}$ is nonsingular, we can multiply the first equation by $\hat{A}^{-1}$ from the right to get

$$u^1_J \;-\; u^t_J \;-\; u^1_0\bar{d}^1 \;+\; u^t_0\bar{d}^t \;=\; 0, \qquad t \in T \setminus \{1\}, \tag{14}$$

where $\bar{d}^1 = -d^1\hat{A}^{-1}$ and $\bar{d}^t = -d^t\hat{A}^{-1}$ as shown in [7, 17]. By substituting this into the second equation of (13), we obtain

$$0 = (u^1_0\bar{d}^1 - u^t_0\bar{d}^t)\hat{b} + u^1_0 d_{10} - u^t_0 d_{t0} = -u^1_0\bar{d}_{10} + u^t_0\bar{d}_{t0}, \quad t \in T \setminus \{1\} \tag{15}$$

where we used $\bar{d}_{t0} = d^t\hat{A}^{-1}\hat{b} - d_{t0}$ from [7, 17]. Now, the set of variables G does not contain any of the u^t_j with $j \in J \setminus M_t$, $t \in T$. Consequently, (14) can be rewritten as

$$u^1_{M_1} \;-\; u^t_{M_t} \;-\; u^1_0\bar{d}^1 \;+\; u^t_0\bar{d}^t \;=\; 0, \qquad t \in T \setminus \{1\}.$$

This implies

$$u^k_{M_k} \;-\; u^t_{M_t} \;-\; u^k_0\bar{d}^k \;+\; u^t_0\bar{d}^t \;=\; 0, \qquad k \neq t \in T. \tag{16}$$

By using (16) we can express $u^t_{M_t}$ as a combination of the vectors $\bar{d}^t$ and $\bar{d}^k$ as follows. Since each $j \in J$ occurs in exactly $|T| - 1$ sets M_t, for each $t \in T$ and for each $j \in M_t$ there exists a unique $k \in T$ with $j \notin M_k$. Hence, we have

$$u^t_{M_t} = \sum_{k \in T\setminus\{t\}} (u^t_0\bar{d}^t_{M_t\setminus M_k} - u^k_0\bar{d}^k_{M_t\setminus M_k}), \quad t \in T.$$

Therefore, using the last equation of (13), we obtain

$$1 = \sum_{t\in T}(u^t_0 + u^t_{M_t}e) = \sum_{t\in T} u^t_0(1 + \sum_{k\in T\setminus\{t\}} (\bar{d}^t_{M_t\setminus M_k} - \bar{d}^t_{M_k\setminus M_t})e). \tag{17}$$

Observe that the system consisting of the equation (17) and the equations

$$-u_0^1\bar{d}_{10} + u_0^t\bar{d}_{t0} = 0, \quad t \in T \setminus \{1\} \tag{18}$$

involves only the variables u_0^t, $t \in T$. It suffices to show that it has a unique solution, because then the value of the variables $u_{M_t}^t$ is uniquely defined. To prove this, suppose that the coefficient matrix of (17)-(18) is singular. Using (18) this implies

$$(1 + \sum_{k\in T\setminus\{1\}} (\bar{d}^1_{M_1\setminus M_k} - \bar{d}^1_{M_k\setminus M_1})e) + \sum_{t\in T\setminus\{1\}} \frac{\bar{d}_{10}}{\bar{d}_{t0}}(1 + \sum_{k\in T\setminus\{t\}} (\bar{d}^t_{M_t\setminus M_k} - \bar{d}^t_{M_k\setminus M_t})e) = 0.$$

Since $\bar{d}_{10} < 0$ by assumption, we can divide through the last equation by it, and after rearranging terms we get

$$\sum_{t\in T}\left(\frac{1}{\bar{d}_{t0}} + \sum_{k\in T\setminus\{t\}}\left(\frac{1}{\bar{d}_{t0}}\bar{d}^t_{M_t\setminus M_k} - \frac{1}{\bar{d}_{k0}}\bar{d}^k_{M_t\setminus M_k}\right)e\right) = 0$$

However, this last expression is nothing else but -1 times

$$\sum_{t\in T}(1/(-\bar{d}_{t0}) + (\pi - \pi^t)e).$$

But this number is positive, so we have encountered a contradiction. □

Note that Theorem 7 is valid for either of the two definitions of an intersection cut, as long as the convex set S used to derive the intersection cut is the same (whether P_I-free or lattice-free) as the one used in the definition of the CGLP.

Next we turn to the converse direction, and provide a sufficient condition for a L&P cut to represent an intersection cut (1). Notice that in the proof of Theorem 7, all the CGLP variables with positive values have subscripts indexed by the set J, where J corresponds to the nonbasic variables in a simplex tableau of the LP relaxation of (MIP). Moreover, for each $j \in J$, there exists $t \in T$ with $u_j^t = 0$. We will prove that the second one of these conditions holds for all basic solutions to (7), whereas the first one is sufficient for a L&P cut to correspond to an intersection cut (1).

Let M denote the row index set of $\tilde{A}$.

Proposition 8. *In any basic solution to (7), for every $i \in M$ there exists $t \in T$ such that $u_i^t = 0$.*

Proof. By contradiction. Let $w = (\alpha, \beta, \{u^t, u_0^t\}_{t \in T})$ be a basic feasible solution to (7), and suppose there exists some $i \in M$ such that $u_i^t > 0$ for all $t \in T$. Let

$$u_i^{t^*} = \min_{t \in T} u_i^t$$

and define a new solution $\bar{w}$ by setting

$$\begin{aligned}
\bar{u}_h^t &= \begin{cases} u_h^t - u_i^{t^*} & h = i \\ u_h^t & h \in M \setminus \{i\} \end{cases} \qquad t \in T \\
\bar{u}_0^t &= u_0^t, \quad t \in T \\
\bar{\alpha} &= \alpha - u_i^{t^*} \tilde{a}_i \quad \text{(where } \tilde{a}_i \text{ denotes row } i \text{ of } \tilde{A}\text{)} \\
\bar{\beta} &= \beta - u_i^{t^*} \tilde{b}_i
\end{aligned}$$

Clearly $\bar{w}$ satisfies all constraints of (7) except for the normalization constraint. We remedy this by rescaling $\bar{w}$ so that in the resulting solution $\tilde{w}$ the sum of the variables $\tilde{u}_h^t$ and $\tilde{u}_0^t$ for all h and t is 1. Now $\tilde{w}$ has one less nonzero component than w, which contradicts the fact that w is a basic solution. □

Theorem 9. *Let $(\alpha, \beta, \{u^t, u_0^t\}_{t \in T})$ be a basic feasible solution to (7) such that $u_0^t > 0$, $t \in T$. If there exists a nonsingular $n \times n$ submatrix $\tilde{A}_J$ of $\tilde{A}$ such that $u_j^t = 0$ for all $j \notin J$ and $t \in T$, then the L&P cut $\alpha x \geq \beta$ is equivalent to the intersection cut $\pi x_J \geq 1$ from S defined by (2) and the LP simplex tableau with nonbasic set J.*

Proof. Suppose the condition of the Theorem is satisfied. Recall the first set of constraints of (7): $\alpha - u^t \tilde{A} - u_0^t d^t = 0$ for $t \in T$. Since there exists a nonsingular $\tilde{A}_J$ such that $u_j^t = 0$ for all $j \notin J$, we can restrict the set of variables to u_J^t, $t \in T$. Therefore, we can infer that the solution of (7) satisfies the following set of equations:

$$\alpha - u_J^t \hat{A} - u_0^t d^t = 0, \quad t \in T,$$

where $\hat{A} = \tilde{A}_J$. Since $\hat{A}$ is invertible, we can multiply this equation from the right by $\hat{A}^{-1}$ to obtain

$$\alpha\hat{A}^{-1} = u^t_J + u^t_0 d^t \hat{A}^{-1}, \quad t \in T.$$

Since for each $j \in J$ there exists $t_j \in T$ with $u^{t_j}_j = 0$ by Proposition 8, we have

$$(\alpha\hat{A}^{-1})_j = u^{t_j}_0 (d^{t_j}\hat{A}^{-1})_j = u^{t_j}_0 \sum_{i\in J} d^{t_j}_i (e^T_i \hat{A}^{-1})_j = u^{t_j}_0 \sum_{i\in J} d^{t_j}_i (-\bar{a}_{ij}) = u^{t_j}_0 d^{t_j}(-\bar{a}_j).$$

We introduce a scaling factor $\theta > 0$ to be chosen later. Let $\pi_j := \frac{1}{\theta}(\alpha\hat{A}^{-1})_j$, and $\pi^t_j := \frac{1}{\theta} u^t_0 d^t(-\bar{a}_j)$ for $t \in T$. Then the above derivation, and $u^t_j \geq 0$ imply that $\pi_j = \max_{t\in T} \pi^t_j$.

Now we use the second equation of (7): $\beta - u^t\tilde{b} - u^t_0 d_{t0} = 0$, $t \in T$. Again, since $u^t_j = 0$ for $j \notin J$ we can rewrite this equation as

$$\beta = u^t_J \hat{b} + u^t_0 d_{t0}, \quad t \in T.$$

Next we substitute for u^t_J the expression $\alpha\hat{A}^{-1} - u^t_0 d^t\hat{A}^{-1}$ to obtain

$$\beta = (\alpha\hat{A}^{-1} - u^t_0 d^t \hat{A}^{-1})\hat{b} + u^t_0 d_{t0} = \alpha\hat{A}^{-1}\hat{b} + u^t_0(d_{t0} - d^t\hat{A}^{-1}\hat{b}), \quad t \in T. \tag{19}$$

Since $\hat{A}^{-1}\hat{b} = \bar{a}_0$, we deduce

$$\beta - \alpha\bar{a}_0 = u^t_0 d_{t0} - u^t_0 d^t \bar{a}_0, \quad t \in T.$$

Here, the left hand side is positive since $\alpha x \geq \beta$ cuts off $\bar{a}_0$, and the right hand side is positive since $\bar{a}_0 \in \operatorname{int} S$. Now we choose θ as $\beta - \alpha\bar{a}_0$, hence

$$u^t_0 = \theta/(d_{t0} - d^t\bar{a}_0), \quad t \in T.$$

However, this implies that

$$\pi^t_j = \frac{1}{\theta} u^t_0 d^t(-\bar{a}_j) = \frac{d^t(-\bar{a}_j)}{d_{t0} - d^t\bar{a}_0}, \quad t \in T, j \in J.$$

This agrees with our former definition of $\pi^t_j := 1/\lambda^*_{tj}$ by (12). Consequently, $\pi x_J \geq 1$ is the intersection cut from S and the basic solution corresponding to the nonbasic set J.

Finally, using the definition of π, we see that $\frac{1}{\theta}\alpha = \pi\hat{A}$, and $\frac{1}{\theta}\beta = \pi\hat{b} + 1$ from (19). Hence, we have

$$\frac{1}{\theta}(\alpha x_N - \beta) = \pi x_J - 1,$$

that is, the two cuts are equivalent. $\square$

Now suppose that instead of (7), we have a CGLP of the form (10), corresponding to a disjunction (9) with multiple inequalities per term. Then Theorem 9 generalizes to the following.

Theorem 10. *Let $(\bar{\alpha}, \bar{\beta}, \{\bar{u}^t, \bar{v}^t\}_{t\in T})$ be a basic feasible solution to (10) such that $\bar{v}^t e > 0$ for all $t \in T$. If there exists a nonsingular $n \times n$ submatrix $\tilde{A}_J$ of $\tilde{A}$ such that $\bar{u}^t_j = 0$ for all $j \notin J$ and $t \in T$, then the lift-and-project cut $\bar{\alpha}x \geq \bar{\beta}$ is equivalent to the intersection cut $\pi x_J \geq 1$ from the set*

$$S(\bar{v}) := \{x \in \mathbb{R}^N : (\bar{v}^t D^t)x \leq \bar{v}^t d^t_0, t \in T\}$$

and the LP simplex tableau with nonbasic set J.

Proof. Analogous to that of Theorem 9. $\square$

Theorems 9 and 10 give sufficient conditions for a lift-and-project cut from a certain disjunction to correspond to an equivalent intersection cut from a convex polyhedron S "complementary" to that disjunction in a well-defined sense. We recall that in this context it does not matter whether the intersection cut in question is from a P_I-free S or just a lattice-free S, as long as the same S is used in the definition of the CGLP as in that of the intersection cut. In the case of Theorem 9, i.e. of a simple disjunction of the form (6), S is the polyhedron obtained by complementing (reversing) each of the inequalities of (6) other than those of P; hence S is the same for any solution of the CGLP (7). On the contrary, in the case of Theorem 10, i.e. of a disjunction of the form (9) with multiple inequalities per term, $S(\bar{v})$ is different for different solutions of the CGLP (10). To be more specific, $S(\bar{v})$ is the polyhedron obtained by complementing (reversing) each of the combined inequalities

$(\bar{v}^t D^t) \geq \bar{v}^t d_0^t$, $t \in T$, where the weights of the combination are part of the CGLP solution. In other words, the intersection cut that corresponds to a given solution of (10), comes from a set $S(\bar{v})$ that is itself a function of that solution. Another way of looking at this is to say that the family of intersection cuts corresponding to the family of CGLP solutions comes from a parametric polyhedron S whose parameters are set by the given CGLP solution. While the cuts are valid for any nonnegative parameter values, they are facet defining only for parameter values corresponding to basic feasible solutions of the CGLP.

Given a L&P cut $\alpha x_N \geq \beta$ obtained from a basic feasible solution to the CGLP system (7) (or (10)), Theorem 9 (respectively 10) gives a sufficient condition for the existence of an intersection cut $\pi x_J \geq 1$ equivalent to $\alpha x_N \geq \beta$. Next we examine the conditions under which this sufficient condition is also necessary.

An inequality $\gamma^1 x \geq \gamma_{10}^1$ is said to dominate the inequality $\gamma^2 x \geq \gamma_{20}$ on P if every $x \in P$ that satisfies $\gamma^1 x \geq \gamma_{10}$ also satisfies $\gamma^2 x \geq \gamma_{20}$.

Theorem 11. *Let $\bar{w} := (\bar{\alpha}, \bar{\beta}, \{\bar{u}^t, \bar{u}_0^t\}_{t \in T})$ be a basic feasible solution to (7) such that $\bar{u}_0^t > 0$, $t \in T$.*

If $\bar{w}$ does not satisfy the (sufficient) condition of Theorem 9, and there is no basic feasible solution $\tilde{w}$ to (7) with $(\tilde{\alpha}, \tilde{\beta}) = \mu(\bar{\alpha}, \bar{\beta})$ for some $\mu > 0$ that satisfies the condition of Theorem 9, then there exists no intersection cut from S equivalent to $\bar{\alpha} x_N \geq \bar{\beta}$. Furthermore, if $(\bar{\alpha}, \bar{\beta})$ uniquely minimizes $\alpha \bar{x}_N - \beta$ over (7), then $\bar{\alpha} \bar{x}_N - \bar{\beta} < \tilde{\alpha} \bar{x}_N - \tilde{\beta}$ for any L&P cut $\tilde{\alpha} x \geq \tilde{\beta}$ equivalent to an intersection cut from S and there exists no intersection cut from S whose L&P equivalent dominates $\bar{\alpha} x \geq \bar{\beta}$ on P.

Proof. Suppose LP admits a basis with nonbasic variables J such that the intersection cut $\pi x_J \geq 1$ derived from S is equivalent to $\bar{\alpha} x_N \geq \bar{\beta}$, i.e., there is a scaling factor μ such that $\mu(\bar{\alpha} x_N - \bar{\beta}) = \pi x_J - 1$. Then by Theorem 7 there exists a basic feasible solution $(\tilde{\alpha}, \tilde{\beta}, \{\tilde{u}^t, \tilde{u}_0^t\}_{t \in T})$ of CGLP which gives rise to a L&P cut $\tilde{\alpha} x_N \geq \tilde{\beta}$ equivalent to $\pi x_J \geq 1$, i.e., $\theta(\tilde{\alpha} x_N - \tilde{\beta}) = \pi x_J - 1$. But then $(\tilde{\alpha} x_N - \tilde{\beta}) = (\mu/\theta)(\bar{\alpha} x_N \geq \bar{\beta})$, i.e., the two cuts are equivalent. However, $(\tilde{\alpha}, \tilde{\beta}, \{\tilde{u}^t, \tilde{u}_0^t\}_{t \in T})$ satisfies the conditions of Theorem 9, which contradicts the assumption of the theorem.

As for the last statement, if $(\bar{\alpha}, \bar{\beta})$ uniquely minimizes $\alpha\bar{x} - \beta$ over (7), then it is a vertex of W^0, the projection onto the (α, β)-subspace of the feasible set of (7) without the normalization constraint, see page 28 of [5], in particular condition (ii). Therefore, from Theorem 4.5 of [5], $\bar{\alpha}x \geq \bar{\beta}$ defines a facet of $\operatorname{conv} P_D(\bar{x})$, the convex hull of the disjunctive set (6). But a facet of a polyhedron cannot be dominated by any other valid inequality for the polyhedron, hence $\bar{\alpha}x \geq \bar{\beta}$ cannot be dominated by any L&P cut equivalent to an intersection cut from S, as any such cut is valid for $\operatorname{conv} P_D(\bar{x})$. □

An analogous Theorem holds for the sufficient condition of Theorem 10.

Theorem 12. *Let $\bar{w} = (\bar{\alpha}, \bar{\beta}, \{\bar{u}^t, \bar{v}^t\}_{t \in T})$ be a basic feasible solution to (10) such that $\bar{v}^t e > 0$, $t \in T$.*

If $\bar{w}$ does not satisfy the (sufficient) condition of Theorem 10, and there is no basic feasible solution $\tilde{w}$ to (10) with $(\tilde{\alpha}, \tilde{\beta}) = \mu(\bar{\alpha}, \bar{\beta})$ for some $\mu > 0$ that satisfies the condition of Theorem 10, then there exists no intersection cut from any member of the family of polyhedra

$$S(v) := \{x \in \mathbb{R}^N : (v^t D^t)x \leq v^t d_{t0}, t \in T\},$$

where $v \geq 0$, $v \neq 0$, equivalent to $\bar{\alpha}\bar{x}_N \geq \bar{\beta}$. Furthermore, if $(\bar{\alpha}, \bar{\beta})$ uniquely minimizes $\alpha\bar{x}_N - \beta$ over (10), then $\bar{\alpha}\bar{x}_N - \bar{\beta} < \tilde{\alpha}\bar{x}_N - \tilde{\beta}$ for any L&P cut $\tilde{\alpha}x \geq \tilde{\beta}$ equivalent to an intersection cut from $S(v)$ and there exists no intersection cut from $S(v)$ whose L&P equivalent dominates $\bar{\alpha}x \geq \bar{\beta}$ on P.

Proof. The statement follows from the same argument as in the proof of Theorem 11. □

A feasible basis for the CGLP system (7) (or (10)) and the associated solution will be called *regular* if the cut that it defines is equivalent to an intersection cut, i.e. if it satisfies the condition of Theorem 9 (respectively 10), *irregular* otherwise. In the sequel we discuss the properties of irregular CGLP bases and solutions. A cut defined by an irregular solution

w is *irregular*, unless there exists a regular solution w' with the same (α, β)-component as that of w, in which case the cut is *regular*.

4 L&P cuts, disjunctive hulls and corner polyhedra

At this point we would like to mention some connection with earlier work. K. Andersen, G. Cornuéjols and Y. Li [2], in extending the results of [10] from split disjunctions of the form $x_k \leq 0$ or $x_k \geq 1$ to more general split disjunctions, have used the relationship (in our notation)

$$\text{conv}\,(P \setminus \text{int}\, S) = \cap_{J \in \mathcal{N}} \text{conv}\,(C(J) \setminus \text{int}\, S)$$

where $\mathcal{N}$ is the collection of all J corresponding to feasible bases, to prove that the split closure is polyhedral. They then explored the question of whether this relationship generalizes to cuts from split disjunctions to cuts from non-split disjunctions (with the corresponding set S), and reached the negative conclusion that cases where the equality in the above equation is replaced by strict inclusion cannot be excluded. They illustrate their finding with a 2-dimensional counterexample. An analogous conclusion was reached and exemplified for non-split two-term disjunctions in [17].

Our results in section 3 amplify this conclusion and make it more specific, by using the lift-and-project representation to pinpoint the gap between the two sides of the above equation.

The same paper [2] then explores the possibility of filling the gap that arises between the two sides of the above equation in the case of cuts from non-split disjunctions by intersecting the expression on the righthand side with sets of the form $\text{conv}\,(C \setminus \text{int}\, S)$, where C is some relaxation of P consisting of subsets of inequalities other than those defining bases. This direction of investigation is further pursued in [14].

Another connection with previous results has to do with the difference between SIC's and RIC's as they relate to corner polyhedra. Restricted intersection cuts have recently been shown [12, 13] to dominate all valid inequalities for corner polyhedra. More specifically, every nontrivial minimal valid inequality for a nonempty corner polyhedron is an intersection cut from some lattice-free set (Theorem 1 of [12]). In particular, every nontrivial facet defining inequality of a corner polyhedron is an intersection cut from a lattice-free set. As pointed out in [8], this property is not shared by SIC's, i.e. intersection cuts according to their original definition, generated from P_I-free as opposed to lattice-free convex sets. Indeed, SIC's from $C(J)$ and some P_I-free convex set can cut off parts of corner(J).

Furthermore, they have a much stronger property.

Theorem 13. *[8] Every facet of* $\operatorname{conv} P_I$ *that cuts off some vertex of* P *is defined by a standard intersection cut.*

Proof. Let F be a facet of $\operatorname{conv} P_I$ defined by the inequality $\varphi x \geq \varphi_0$ satisfied by all $x \in P_I$, but violated by some $x \in P$. Then F contains $\dim \operatorname{conv} P_I$ affinely independent integer points of $\operatorname{conv} P_I$, and

$$\{x \in \mathbb{R}^n : \varphi x < \varphi_0\} \cap P_I = \emptyset.$$

Hence the interior of the set $S := \{x \in \mathbb{R} : \varphi x \leq \varphi_0\}$ contains no point of P_I, i.e. S is a P_I-free convex set. On the other hand, $\operatorname{int} S$ contains some vertex v of P with associated nonbasic index set $J(v)$ cut off by F. Hence the standard intersection cut from $C(J(v))$, the cone associated with the basis B defining the vertex v, is precisely $\varphi x \geq \varphi_0$. $\square$

It then follows that every vertex of the corner polyhedron that is not a vertex of P_I is cut off by some SIC.

Thus *regular* L&P cuts from a CGLP based on a disjunction corresponding to a P_I-free convex set S may cut off some parts of corner(J), whereas regular cuts from a CGLP based on a disjunction corresponding to a lattice-free convex set S are valid for the corresponding corner polyhedron.

Next we address and answer a related question. As $\operatorname{conv} P_I$ is contained in every corner polyhedron, we have

$$\operatorname{conv} P_I \subseteq \cap_{J \in \mathcal{N}} \text{corner}(J), \tag{20}$$

where $\mathcal{N}$ denotes the set of co-bases (whether feasible or not) of the LP. But are there cases when equality holds?

Theorem 14. *Suppose* $\operatorname{conv} P_I$ *is an* n*-dimensional polyhedron. Then the inclusion (20) holds at equality if and only if every facet defining inequality for* $\operatorname{conv} P_I$ *is facet defining for* $\operatorname{corner}(J)$ *for some* $J \in \mathcal{N}$*.*

Proof. Sufficiency. Let $fx \leq f_0$ be a facet defining inequality for $\operatorname{conv} P_I$. If $fx \leq f_0$ for all $x \in \operatorname{corner}(J)$ for some $J \in \mathcal{N}$, then $fx \leq f_0$ for all $x \in \cap_{J \in \mathcal{N}} \operatorname{corner}(J)$, since the latter is a subset of the former. But if every facet defining inequality for $\operatorname{conv} P_I$ is valid for $\cap_{j \in \mathcal{N}} \operatorname{corner}(J)$, then (20) holds at equality.

Necessity. Suppose equality holds in (20), and let $fx \leq f_0$ be a facet defining inequality for $\operatorname{conv} P_I$. We will show that it is facet defining for $\operatorname{corner}(J)$ for some $J \in \mathcal{N}$. Let $F := \{x \in \mathbb{R}^n : fx = f_0\} \cap \operatorname{conv} P_I$ be the facet of $\operatorname{conv} P_I$ induced by $fx \leq f_0$. Then $F \subset \operatorname{corner}(J)$, $F \neq \operatorname{corner}(J)$ for all $J \in \mathcal{N}$, since each corner polyhedron is n-dimensional, and F is $(n-1)$-dimensional. Since each $\operatorname{corner}(J)$ is convex, if F is not contained in a facet of any corner polyhedron, then $fx \leq f_0$ is not valid for any corner polyhedron. Let v be a point in the relative interior of F. Since F is not contained in a facet of any corner polyhedron, v is an interior point of each corner polyhedron. But then there exists a ball B around v contained in every corner polyhedron. Clearly, there exists $y \in B$ cut off by $fx \leq f_0$. Since $y \in B \subset \operatorname{corner}(J)$ for all $J \in \mathcal{N}$, $fx \leq f_0$ is not valid for $\cap_{j \in \mathcal{N}} \operatorname{corner}(J)$, which contradicts our assumption that equality holds in (20).

So suppose F is contained in a facet of $\operatorname{corner}(J)$ for some $J \in \mathcal{N}$. Since F is $(n-1)$-dimensional, and every facet of any corner polyhedron is also $(n-1)$-dimensional, $fx \leq f_0$ is facet defining for corner)J). ☐

In general, i.e. without assuming anything about the dimension of $\operatorname{conv} P_I$, we can prove the following:

Proposition 15. *The inclusion (20) holds at equality if every facet defining inequality for* *conv P_I is valid for* $\mathrm{corner}(J)$ *for some* $J \in \mathcal{N}$*. It holds as strict inclusion if conv P_I has a facet defining inequality that is not valid for* $\cap_{j\in\mathcal{N}}\mathrm{corner}(J)$*.*

Proof. Let $fx \leq f_0$ be a facet defining inequality for conv P_I. If $fx \leq f_0$ for all $x \in \mathrm{corner}(J)$ for some $J \in \mathcal{N}$, then $fx \leq f_0$ for all $x \in \cap_{j\in\mathcal{N}}\mathrm{corner}(J)$, since the latter is a subset of the former. But if every facet defining inequality for conv P_I is valid for $\cap_{j\in\mathcal{N}}\mathrm{corner}(J)$, then (20) holds at equality.

On the other hand, if conv P_I has a facet defining inequality $fx \leq f_0$ invalid for $\cap_{j\in\mathcal{N}}\mathrm{corner}(J)$, then there exists $x \in \cap_{J\in\mathcal{N}}\mathrm{corner}(J) \setminus \mathrm{conv}\, P_I$, i.e., (20) holds as strict inclusion. □

In the sequel we address the question of when do *irregular* solutions to a CGLP based on a disjunction corresponding to a lattice-free convex set S cut off part of the corresponding corner polyhedron.

In Section 1 we defined P_I as $P_I = \{x \in P : x_j \in \mathbb{Z} \text{ for } j \in Q \subset N\}$, where $Q = N'$. In this section we will specialize the disjunction (4) to the disjunctive normal form of the expression $\{x \in \mathbb{R}^n : x_j \leq \lfloor \bar{x}_j \rfloor \text{ or } x_j \geq \lceil \bar{x}_j \rceil, j \in Q\}$, which is

$$\left\{ x \in \mathbb{R}^n : \bigvee_{t\in T} \begin{pmatrix} x_j \leq \lfloor \bar{x}_j \rfloor & j \in Q_t^- \\ x_j \geq \lceil \bar{x}_j \rceil & j \in Q_t^+ \end{pmatrix} \right\} \tag{$\overline{4}$}$$

where $Q_t^+ \cup Q_t^- = Q$ and T indexes the set of $2^{|Q|}$ bipartitions of Q. To simplify things, we will preserve the notation (4) with the proviso that in this section (4) is specialized to ($\overline{4}$). Also, we will denote by ($\overline{9}$) and ($\overline{10}$) the expressions (9) and (10) in which (4) is specialized to ($\overline{4}$).

The corner polyhedron [16] is a relaxation of P_I associated with every basis of P. In this section we will also consider a different relaxation of P_I associated with every basic feasible solution $\bar{x}$ of P, namely, the *disjunctive hull* of P at $\bar{x}$, conv $P_D(\bar{x})$, where $P_D(\bar{x}) := \{x \in P : x \text{ satisfies } (\overline{4})\}$.

At this point it will be useful to introduce the concept of the *parametric cross-polytope* [11] associated with ($\overline{4}$). Let $K(\bar{x})$ be the $|Q| = q$-dimensional unit hypercube centered at $x_j = \lfloor \bar{x}_j \rfloor + \frac{1}{2}$, $j \in Q$, i.e. $K(\bar{x}) := \{x \in \mathbb{R}^Q : \lfloor \bar{x}_j \rfloor \leq x_j \leq \lceil \bar{x}_j \rceil,\ j \in Q\}$. For the sake of simplicity, we will move the origin of the coordinate system to $\lfloor \bar{x} \rfloor$, i.e. we will take $K(\bar{x})$ to be $\{x \in \mathbb{R}^Q : 0 \leq x_j \leq 1, j \in Q\}$. Let $K^*(\bar{x})$ be the q-dimensional cross-polytope (octahedron) circumscribing $K(\bar{x})$, which can be written as

$$K^*(\bar{x}) := \{x \in \mathbb{R}^Q : \sum_{k \in Q_t^+} x_k - \sum_{k \in Q_t^-} x_k \leq |Q_t^+|,\ t = 1, \ldots, 2^q\}$$

where (Q_t^+, Q_t^-) is one of the 2^q bipartitions of Q.

While $K^*(\bar{x})$ circumscribes $K(\bar{x})$, i.e. contains in its boundary every vertex of $K(\bar{x})$, it is just one of all possible convex polyhedra with this property. In order to get an adequate representation of the entire family, we need to parametrize $K^*(\bar{x})$. Introducing the parameters v_{tk}, $t = 1, \ldots, 2^q$, $k = 1, \ldots, q$, we obtain the system

$$\sum_{k \in Q_t^+} v_{tk} x_k - \sum_{k \in Q_t^-} v_{tk} x_k \leq \sum_{k \in Q_t^+} v_{tk}, \quad t = 1, \ldots, 2^q \tag{21}$$

where $v_{tk} \geq 0$ for all t, k. Since the inequalities of (21) are homogeneous, the system can be normalized (see [11] for a convenient way of doing this).

Let $\tilde{K}^*(v)$ denote the parametric cross-polytope defined by (21). Clearly, for any fixed set of v_{tk}, $\tilde{K}^*(v)$ is a convex polyhedron that contains in its boundary all x with integer components x_j for $j \in Q$, hence it is suitable for generating intersection cuts. While $\tilde{K}^*(v)$ is defined in the subspace $\mathbb{R}^Q$, the corresponding set in $\mathbb{R}^N$ is the parametric cylinder $\tilde{K}^*(v) \times \mathbb{R}^{N-Q}$. Clearly, this cylinder is a lattice-free set.

Theorem 16. *$\operatorname{conv} P_D(\bar{x})$ is defined by the family of all L&P cuts from ($\overline{10}$). The family of (restricted) intersection cuts from $\tilde{K}^*(v)$ is equivalent to the family of regular L&P cuts from ($\overline{10}$).*

Proof. The first statement follows from the basic theorem of disjunctive programming [5]. The second one follows from Theorem 12 of section 3 above. □

Proposition 17. *Those facets of $\operatorname{conv} P_D(\bar{x})$ which contain n vertices of the hypercube centered at $\bar{x} + \frac{1}{2}e$ are facets of $\operatorname{conv} P_I$.*

Proof. Suppose $\gamma x_N = \gamma_0$ defines a facet of $\operatorname{conv} P_D(\bar{x})$ which contains n vertices of the hypercube centered at $\bar{x} + \frac{1}{2}e$. Since all these n points belong to P_I, $\gamma x_N = \gamma_0$ is a facet of $\operatorname{conv} P_I$ as well. □

We now turn to the relationship between irregular L&P cuts and the corner polyhedron. Since regular L&P cuts from ($\overline{10}$) are equivalent to restricted intersection cuts from $\tilde{K}^*(\bar{x})$, they cannot cut off any point of $\operatorname{corner}(\bar{x})$.

Theorem 18. *Let $\bar{x}$ be a basic feasible solution of the LP relaxation, and $(\bar{\alpha}, \bar{\beta}, \{\bar{u}^t, \bar{v}^t\}_{t\in T})$ a basic feasible solution of ($\overline{10}$) such that $\bar{\alpha}x \geq \bar{\beta}$ cuts off $\bar{x}$. Suppose there exist a point $\bar{y} \in \operatorname{corner}(\bar{x}) \setminus P$, and a $t^* \in T$ such that $\bar{v}^{t^*} D^{t^*} \bar{y} = \bar{v}^{t^*} d^{t^*}$ and the set of indices $\{i \mid \bar{u}_i^{t^*} > 0\}$ can be partitioned into two subsets (F^+, F^-) with F^- nonempty and F^+ possibly empty, such that*

- *$\bar{y}$ satisfies all the inequalities $\tilde{A}_i x \geq \tilde{b}_i$ with $i \in F^+$ at equality and*
- *$\bar{y}$ violates all the inequalities $\tilde{A}_i x \geq \tilde{b}_i$ with $i \in F^-$.*

Then $\bar{\alpha}x \geq \bar{\beta}$ is an irregular L&P cut and $\bar{y}$ is a point in $\operatorname{corner}(J)$ *which is cut off by $\bar{\alpha}x \geq \bar{\beta}$.*

Proof. Since $(\bar{\alpha}, \bar{\beta}, \{\bar{u}^t, \bar{v}^t\}_{t\in T})$ is a basic feasible solution of ($\overline{10}$), we have $\bar{\alpha} = \sum_i \bar{u}_i^{t^*} \tilde{A}_i + \bar{v}^{t^*} D^{t^*}$, and $\bar{\beta} = \sum_i \bar{u}_i^{t^*} \tilde{b}_i + \bar{v}^{t^*} d^{t^*}$, where $\bar{u}, \bar{v} \geq 0$. Therefore, we deduce that $\bar{\alpha}\bar{y} < \bar{\beta}$, since

$$\bar{\alpha}\bar{y} = (\sum_i \bar{u}_i^{t^*} \tilde{A}_i + \bar{v}^{t^*} D^{t^*})\bar{y} = \sum_{i\in F^+} \bar{u}_i^{t^*} \tilde{A}_i \bar{y} + \sum_{i\in F^-} \bar{u}_i^{t^*} \tilde{A}_i \bar{y} + \bar{v}^{t^*} D^{t^*} \bar{y} < \sum_{i\in F^+} \bar{u}_i^{t^*} \tilde{b}_i + \sum_{i\in F^-} \bar{u}_i^{t^*} \tilde{b}_i + \bar{v}^{t^*} d^{t^*} = \bar{\beta},$$

where the inequality follows from the conditions of the theorem.

Finally, since $\bar{\alpha}x \geq \bar{\beta}$ is not valid for $\operatorname{corner}(\bar{x})$, and $\bar{v}^{t^*} \neq 0$ since $\bar{\alpha}x \geq \bar{\beta}$ is not valid for P, it is an irregular L&P cut. □

The conditions of this theorem are easy to meet. Suppose, e.g., that $C(J)$ admits an extreme ray containing a point $y \in K(\bar{x}) \times \mathbb{R}^{N-Q}$ such that $y \notin P$ and $y_i \in \{\lfloor \bar{x}_i \rfloor, \lceil \bar{x}_i \rceil\}$ for all $i \in Q$. Then $y \in \operatorname{corner}(J) \setminus P$, and if CGLP ($\overline{10}$) admits a basic feasible solution

$(\bar{\alpha}, \bar{\beta}, \{\bar{u}^t, \bar{v}^t\}_{t \in T})$ such that $\bar{v}^t \neq 0$ and $\bar{u}^t_i > 0$ for some of the inequalities determining the extreme ray, and also for an inequality separating y from P, then the conditions of Theorem 18 are met, and the cut $\bar{\alpha}x \geq \bar{\beta}$ is an irregular L&P cut not valid for corner(J). The example in Section 6 gives a simple illustration of this theorem.

Recall that $\bar{\alpha}x \geq \bar{\beta}$ is the irregular L&P cut associated with the solution to $(\overline{10})$ that minimizes $\alpha\bar{x} - \beta$, and that $C(J)$ is the LP cone defined in section 1.

Theorem 19. *If there is an extreme ray of the LP cone $C(J)$ with direction vector r such that $\bar{\alpha}r < 0$, then $\bar{\alpha}x \geq \bar{\beta}$ cuts off some point of* corner(J).

Proof. Suppose $\bar{x} + r\lambda$, $\lambda \geq 0$ is an extreme ray of $C(J)$ such that $\bar{\alpha}r < 0$. Since $\bar{\alpha}\bar{x} < \bar{\beta}$, it follows that $\bar{\alpha}(\bar{x} + r\lambda) < \bar{\beta}$ for all $\lambda \geq 0$, i.e. the hyperplane $\bar{\alpha}x = \bar{\beta}$ does not intersect the ray $\bar{x} + r\lambda$. This means that $C' := C(J) \cap \{x : \bar{\alpha}x < \bar{\beta}\}$ is unbounded. Clearly, C' contains integer points, and they all belong to corner(J) and are cut off by $\bar{\alpha}x \geq \bar{\beta}$. $\square$

Now suppose no such extreme ray of $C(J)$ exists, i.e. $\bar{\alpha}r^j \geq 0$ for all n extreme rays of $C(J)$. Let $H^+ := \{x \in \mathbb{R}^N : \bar{\alpha}x \geq \bar{\beta}\}$, H^- the weak complement of H^+. Then $\bar{\alpha}x \geq \bar{\beta}$ cuts off a part of corner(J) if and only if $(C(J) \cap H^-) \setminus P$ contains some integer point. In fact, $C(J) \cap H^- \setminus P$ is precisely the part of $C(J) \setminus P$ cut off by $\bar{\alpha}x \geq \bar{\beta}$.

The above results make L&P cuts from irregular CGLP solutions look very attractive. If the CGLP optimum is attained for an irregular solution w, then not only does the inequality $\alpha x_N \geq \beta$ cut off the LP optimum $\bar{x}_N$ by a maximum amount, but in many cases it also cuts off part of the corner polyhedron. In section 6 we give an example in which an irregular L&P inequality cuts off part of every corner polyhedron corresponding to the basis cone for any basis (feasible or infeasible) of LP.

5 The frequency of irregular L&P cuts

From Theorems 9-10 and 11-12 it follows that there are two types of basic feasible solutions to the CGLP that have no corresponding SIC in any basic (feasible or infeasible) solution to LP, i.e., that are irregular. Let B be a feasible basis matrix of CGLP, and let $\tilde{A}_K$ be the submatrix of $\tilde{A}$ consisting of those rows that are contained in columns j of B such that u_j^t is basic for some $t \in T$. Then the two types of irregular basic solutions are:

Type 1. The matrix $\tilde{A}_K$ contains a $n \times n$ nonsingular submatrix $\tilde{A}_J$, but $K \setminus J$ is nonempty.

Type 2. The matrix $\tilde{A}_K$ contains no $n \times n$ nonsingular submatrix

For both of these types we will show that they do occur (type 1 only for CGLP from multiple-term disjunctions). In fact, their occurrence is not exceptional, and is no less frequent than that of regular solutions. The results of the previous section show that irregular CGLP solutions have characteristics that make the cutting planes they yield particularly attractive.

We illustrate the above by showing how to construct from a regular CGLP basis irregular bases of both types.

Consider a MIP with constraint set $Ax \geq b$, $x_j \geq 0$, $j \in N$, represented as $\tilde{A}x \geq \tilde{b}$, and $x_j \in \mathbb{Z}$ for $j \in N' \subseteq N$, and suppose we want to generate L&P cuts from the 4-term disjunction

$$\begin{pmatrix} x_k & \leq & 0 \\ x_\ell & \leq & 0 \end{pmatrix} \bigvee \begin{pmatrix} x_k & \geq & 1 \\ x_\ell & \leq & 0 \end{pmatrix} \bigvee \begin{pmatrix} x_k & \leq & 0 \\ x_\ell & \geq & 1 \end{pmatrix} \bigvee \begin{pmatrix} x_k & \geq & 1 \\ x_\ell & \geq & 1 \end{pmatrix}$$

or

$$\begin{pmatrix} \sigma_k^1 x_k & \geq & \rho_k^1 \\ \sigma_\ell^1 x_\ell & \geq & \rho_\ell^1 \end{pmatrix} \bigvee \begin{pmatrix} \sigma_k^2 x_k & \geq & \rho_k^2 \\ \sigma_k^2 x_\ell & \geq & \rho_\ell^2 \end{pmatrix} \bigvee \begin{pmatrix} \sigma_k^3 x_k & \geq & \rho_k^3 \\ \sigma_\ell^3 x_\ell & \geq & \rho_\ell^3 \end{pmatrix} \bigvee \begin{pmatrix} \sigma_k^4 x_k & \geq & \rho_k^4 \\ \sigma_\ell^4 x_\ell & \geq & \rho_\ell^4 \end{pmatrix}$$

where each term contains, in addition to the above pair of inequalities, the constraint set $\tilde{A}x \geq \tilde{b}$. The constraint set of the CGLP is then

$$\begin{array}{rcrcrcrcl} \alpha & & -\tilde{A}^T u^t & - & \sigma_k^t e_k v_t & - & \sigma_\ell^t e_\ell w_t & = & 0 \\ & \beta & -\tilde{b}^T u^t & - & \rho_k^t v_t & - & \rho_\ell^t w_t & = & 0 \qquad t = 1, \ldots, 4 \\ & & \sum_{t=1}^4 e u^t & + & \sum_{t=1}^4 (v_t & + & w_t) & = 1 & \\ & & u^t, & & v_t, & & w_t & \geq & 0, \qquad t = 1, \ldots, 4 \end{array}$$

or, after eliminating α and β by using the $n+1$ equations corresponding to $t=1$,

$$\begin{array}{lclclclclclclcll}
\tilde{A}^T u^1 & - & \tilde{A}^T u^t & + & \sigma_k^1 e_k v_1 & - & \sigma_k^t e_k v_t & + & \sigma_\ell^1 e_\ell w_1 & - & \sigma_\ell^t e_\ell w_t & = & 0 & \\
\tilde{b}^T u^1 & - & \tilde{b}^T u^t & + & \rho_k^1 v_1 & - & \sigma_k^t v_t & + & \rho_\ell^1 w_1 & - & \rho_\ell^t w_t & = & 0 & t=2,\ldots,4 \\
\displaystyle\sum_{t=1}^4 e u^t & & & + & \multicolumn{7}{l}{\displaystyle\sum_{t=1}^4 (v_t + w_t)} & = & 1 & t=1,\ldots,4 \\
\multicolumn{14}{c}{u^t, v_t, w_t \geq 0.}
\end{array} \tag{22}$$

This system has $3n+4$ equations and $4(m+n)+8$ variables, where $m+n$ is the number of rows of $\tilde{A}$.

A basis for the system (22) is a nonsingular $(3n+4)\times(3n+4)$ matrix which can be, for instance, of the form

$$B = \begin{pmatrix} R & U \\ V & Z \end{pmatrix}$$

where R is $3n \times 3n$, V is $4 \times 3n$, U is $3n \times 4$ and Z is 4×4.

In order for B to be nonsingular, it suffices for the $3n \times 3n$ matrix S and the 4×4 matrix $W := Z - VR^{-1}U$ to be nonsingular. For instance, R may be of the form

$$R = \begin{pmatrix} A_1 & -A_2 & & \\ A_1 & & -A_3 & \\ A_1 & & & -A_4 \end{pmatrix}$$

where each A_i is the transpose of a matrix consisting of some subset of the rows of $\tilde{A}$ such that R is nonsingular. One possible way of satisfying this requirement is to have R derived from a nonsingular $n \times n$ submatrix $\tilde{A}_J$ of $\tilde{A}$, namely by setting $A_3 = A_4 = \tilde{A}_J^T$ and $(-A_1, A_2) = \tilde{A}_J^T$, i.e. having $-A_1$ and A_2 be transposes of two matrices formed by a bipartition of the row set of $\tilde{A}_J$. It is then easy to see that R is nonsingular, since each of the three $n \times n$ matrices $(A_1, -A_2)$, $-A_3$ and $-A_4$ is nonsingular. Note that in this case the columns of R correspond to $3n$ of the $4m$ variables u_j^t, $t=1,\ldots,4$, $j=1,\ldots,m$, but since the only rows of $\tilde{A}$ that appear in the columns of R are those indexed by J, we have the condition $u_j^t = 0$, $j \notin J$ satisfied.

The matrix U may consist of the first $3n$ entries of 4 of the 8 columns corresponding to the variables (v_t, w_t), $t=1,\ldots,4$, in which case the $3n \times 4$ matrix U is of the form (assuming w.l.o.g. that the 4 basic components of (v_t, w_t) are v_t, $t = 1, \ldots, 4$)

$$U = \begin{pmatrix} \sigma^1 e_k & -\sigma^2 e_k & & \\ \sigma^1 e_k & & -\sigma^3 e_k & \\ \sigma^1 e_k & & & -\sigma^4 e_k \end{pmatrix}.$$

Finally, V is the $4 \times 3n$ matrix

$$V = \begin{pmatrix} b_1^T & -b_2^T & & & \\ b_1^T & & -b_3^T & & \\ b_1^T & & & & -b_4^T \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

where the b_i^T, $i = 1, \ldots, 4$, are the subvectors of $\tilde{b}^T$ associated with the row indices of $\tilde{A}_J$ defining the corresponding components A_i, $i = 1, \ldots, 4$; and Z is the 4×4 matrix

$$Z = \begin{pmatrix} \rho_k^1 & -\rho_k^2 & & \\ \rho_k^1 & & -\rho_k^3 & \\ \rho_k^1 & & & -\rho_k^4 \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

As mentioned above, a matrix B of the above form is nonsingular if R is nonsingular and $W := Z - VS^{-1}U$ is nonsingular. Our matrix R is nonsingular by construction, and the 4×4 matrix W could be singular only for some very specific values of the components of $\tilde{b}$.

In conclusion, given any $n \times n$ nonsingular submatrix $\tilde{A}_J$ of J, it is straightforward to construct from it a basis of the system (22) that has $u_j^t = 0$ for all $j \notin J$. If such a basis is also feasible, i.e. such that $(u^t, v_t, w_t) \geq 0$ for $t = 1, \ldots, 4$, then it satisfies the regularity condition of Theorem 10. Although B is feasible if and only if the last column of B^{-1} is nonnegative (since the condition of feasibility is $B^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \geq 0$), there is no simple way to guarantee that B satisfies this condition. Nevertheless, we will show that if B is feasible, hence regular, then its neighborhood (in terms of pivoting sequences) contains several irregular relatives, mostly of type 1, but also of type 2.

(a) Irregular matrices of type 1. Given a regular basis B of the above form, all that is needed to make it irregular is to replace one of the submatrices A_3 or A_4 of R with a $n \times n$

nonsingular submatrix $\tilde{A}_K$ of $\tilde{A}$ such that $K \neq J$. Then the resulting basis B' will contain variables u_j^t for $j \in J \cup K$, and if any of those u_j^t with $j \in K \setminus J$ are positive, then B' is irregular. It is easy to see that for any regular basis B, there are multiple irregular bases of type 1 obtainable from B by a few pivots.

More generally, given any regular basic feasible solution to CGLP whose basis contains a set of columns associated with the rows of a $n \times n$ nonsingular matrix $\tilde{A}_J$, there are $|T| \cdot m$ nonbasic variables u_j^t, $j \notin J$, such that if any of them is pivoted into the basis, the solution becomes irregular of type 1. Note that the number of such variables increases linearly with the number of terms of the disjunction underlying the CGLP.

(b) Irregular bases of type 2. Starting from a regular basis B, we can construct an irregular one of type 2 as follows. Let $\tilde{A}_J$ be a *singular* $n \times n$ submatrix of $\tilde{A}$ of rank $n-1$, and such that the rank of $(\tilde{A}_J, \tilde{b}_J)$ is n. Such submatrices $\tilde{A}_J$ always exist and they are quite numerous, since $\tilde{A}$ contains for every 0-1 variable x_j the rows e_j and $-e_j$. Let B' be a $(3n+4) \times (3n+4)$ matrix of the form

$$B' = \begin{pmatrix} R' & U' \\ V' & Z' \end{pmatrix}$$

where R' is the $(3n+3) \times (3n+3)$ matrix

$$R' = \begin{pmatrix} A_1' & -A_2' & & \\ A_1' & & -A_3' & \\ A_1' & & & -A_4' \end{pmatrix},$$

with

$$A_3' = \begin{pmatrix} \tilde{A}_J & \tilde{b}_J \\ -\sigma^3 e_k & -\rho_k^3 \end{pmatrix}^T, \; A_4' = \begin{pmatrix} \tilde{A}_J & \tilde{b}_J \\ -\sigma^4 e_k & -\rho_k^4 \end{pmatrix}^T, \text{ and } (A_1', -A_2') = \begin{pmatrix} \tilde{A}_J & \tilde{b}_J \\ -\sigma^2 e_k & -\rho^2 \end{pmatrix}^T,$$

each of the 3 matrices A_3', A_4' and $(A_1', -A_2')$ being $(n+1) \times (n+1)$, nonsingular. Further, let V' be the $1 \times (3n+3)$ matrix $(1, \ldots, 1)$, let U' be the $(3n+3) \times 1$ matrix $(\sigma^1 e_k^T, \rho_k^1, \sigma^1 e_k^T, \rho_k^1, \sigma^1 e_k^T, \rho_k^1)^T$, and let $Z' = (1)$. Then B' is nonsingular (i.e. a basis) if R' and $Z' - V'(R')^{-1}U'$ are nonsingular. Now R' is nonsingular since each of its 3 blocks A_3', A_4' and $(A_1', -A_2')$ is nonsingular, and $Z' - V'(R')^{-1}U'$ is nonsingular whenever $(1, \ldots, 1)(R')^{-1}U' \neq 1$.

It follows that B' can be a basis of (22) even though the submatrix $\tilde{A}_J$ of $\tilde{A}$ whose rows correspond to the u_i^t represented in B' is singular, i.e. even though it is irregular of type 2.

The above construction suggests that for every regular CGLP basis one can construct multiple irregular bases. In fact, the number of such (irregular) bases grows with m and with $|T|$, the number of terms in the disjunction underlying the CGLP.

6 Numerical Example

Next we give a numerical example that illustrates the occurrence of irregular optimal solutions to the CGLP, resulting in a cut that is violated by the optimal LP solution by more than any SIC from any basis. Furthermore, this irregular cut cuts off part of every corner polyhedron associated with P_I.

Consider the MIP

$$\min y \tag{23}$$

such that

$$y - 1.1x_1 + x_2 \geq -0.15 \tag{24}$$

$$y + x_1 - 1.1x_2 \geq -0.2 \tag{25}$$

$$y + x_1 + x_2 \geq 0.6 \tag{26}$$

$$x_1, x_2 \in \{0, 1\}, y \geq 0 \tag{27}$$

The convex body of the feasible solutions is depicted in Figure 1. The optimal solution of the LP relaxation is $x_1^* = 23/105, x_2^* = 8/21, y^* = 0$. The LP in standard form (with surplus

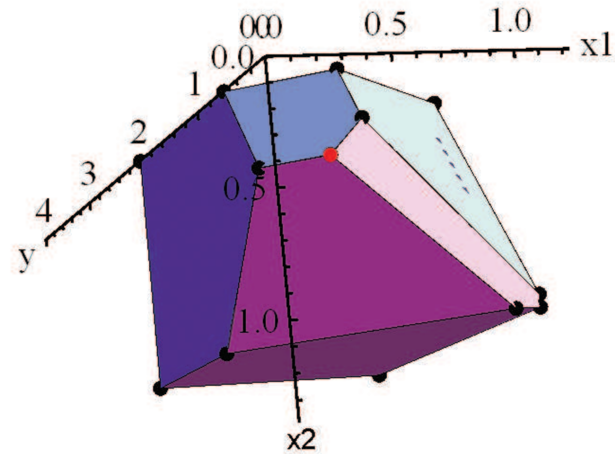

Figure 1: The convex body of feasible points of the LP relaxation. The optimum LP solution is marked by a red dot.

variables) is:

$$\min y \tag{28}$$

such that

$$y - 1.1x_1 + x_2 - s_1 = -0.15 \tag{29}$$

$$y + x_1 - 1.1x_2 - s_2 = -0.2 \tag{30}$$

$$y + x_1 + x_2 - s_3 = 0.6 \tag{31}$$

$$x_1, x_2 \in [0,1], s_1, s_2, s_3, y \geq 0 \tag{32}$$

The simplex tableau for one of the optimal solutions is

s_1	x_1	x_2	y	s_2	s_3	RHS
1			21/10	1	1/10	29/100
	1		1	$-10/21$	$-11/21$	23/105
		1	0	10/21	$-10/21$	8/21

(33)

We will formulate a CGLP with respect to the 3-term disjunction

$$-x_1 \geq 0 \ \vee \ -x_2 \geq 0 \ \vee \ x_1 + x_2 \geq 2$$

corresponding to the lattice-free polyhedron S in $\mathbb{R}^3$ defined by triangle $\{x \in \mathbb{R}^3 : x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 2\}$. Using this, the CGLP is

$$\min \frac{29}{100}u_1^1 + \frac{23}{105}u_5^1 + \frac{82}{105}u_6^1 + \frac{8}{21}u_7^1 + \frac{13}{21}u_8^1 - \frac{23}{105}v_1 \tag{34}$$

$$\begin{pmatrix} \tilde{A}^T \\ \tilde{b}^T \\ \tilde{A}^T \\ \tilde{b}^T \\ e^T \end{pmatrix} u^1 + \begin{pmatrix} -\tilde{A}^T \\ -\tilde{b}^T \\ 0 \\ 0 \\ e^T \end{pmatrix} u^2 + \begin{pmatrix} 0 \\ 0 \\ -\tilde{A}^T \\ -\tilde{b}^T \\ e^T \end{pmatrix} u^3 + \begin{pmatrix} -e_{x_1} \\ 0 \\ -e_{x_1} \\ 0 \\ 1 \end{pmatrix} u_0^1 + \tag{35}$$

$$\begin{pmatrix} e_{x_2} \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u_0^2 + \begin{pmatrix} 0 \\ 0 \\ -e_{x_1,x_2} \\ -2 \\ 1 \end{pmatrix} u_0^3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{36}$$

where $\tilde{A}$ and $\tilde{b}$ are defined by

$$\tilde{A} = \begin{pmatrix} 1 & -1.1 & 1 \\ 1 & 1 & -1.1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \tilde{b} = \begin{pmatrix} -0.15 \\ -0.20 \\ 0.60 \\ 0 \\ 0 \\ -1 \\ 0 \\ -1 \end{pmatrix},$$

and e_{x_i} is a unit vector with 1 in the row corresponding to variable x_i (c.f. $\tilde{A}^T$), $i = 1, 2$, and e_{x_1,x_2} has two 1's, in the respective rows. Notice that u_j^t corresponds to row j of $\tilde{A}$, $j = 1, \ldots, 8$, the first 3 rows represent the inequalities (24)-(26), the fourth row corresponds to $y \geq 0$, the fifth and sixth to $x_1 \geq 0$ and $-x_1 \geq -1$, and the last two to $x_2 \geq 0$ and $-x_2 \geq -1$, respectively.

The optimal basis of CGLP consists of the variables $\{u_0^1, u_0^2, u_0^3, u_2^1, u_3^1, u_1^2, u_3^2, u_2^3, u_4^3\}$, and the basic solution is

v^1	u_2^1	u_3^1	v^2	u_1^2	u_3^2	v^3	u_1^3	u_4^3
0.182156	0.0813197	0.124497	0.170771	0.086741	0.119075	0.0296236	0.00542131	0.200395

As can be seen, the basic variables among the u_j^ℓ correspond to 4 distinct constraints of the LP relaxation, whereas the latter has only three nonbasic variables in any basis. The L&P

cut corresponding to the optimal CGLP solution is

$$0.205816y + 0.0236602x_1 + 0.0350449x_2 \geq 0.0584339 \tag{37}$$

The violation of this cut is -0.0399009, and one may verify that as stated in Theorem 11, no intersection cut from any basis of the LP relaxation of MIP has the same violation when represented as a solution of CGLP. Notice that the optimal CGLP basis is irregular, and this inequality cuts into all the corner polyhedra coming from *any* basis of the LP relaxation, which has been verified case by case. For instance, consider the corner polyhedron defined with respect to the optimal simplex tableau (33):

$$\begin{array}{ccccccc} x_1 & & +y & -10/21s_2 & -11/21s_3 & = & 23/105 \\ & x_2 & & +10/21s_2 & -10/21s_3 & = & 8/21 \\ x_1, x_2 \in \mathbb{Z} & & y \geq 0 & s_2 \geq 0 & s_3 \geq 0 & & \end{array} \tag{38}$$

Proposition 20. *The inequality (37) is not valid for the corner polyhedron (38)*

Proof. Consider the point $\bar{w} = (\bar{x}_1 = 1, \bar{x}_2 = 0, \bar{y} = 0, \bar{s}_1 = -\frac{95}{100}, \bar{s}_2 = \frac{6}{5}, \bar{s}_3 = \frac{2}{5})$. Since this point has integral x_1 and x_2 coordinates, and satisfies both of the equality constraints defining the corner polyhedron (38), and all the nonbasic variables, (y, s_2, s_3), are non-negative, it is in the corner polyhedron (38).

Now, we verify that the point $\bar{w}$ just defined is cut off by (37):

$$0.205816\bar{y} + 0.0236602\bar{x}_1 + 0.0350449\bar{x}_2 = 0.0236602 \not\geq 0.0584339.$$

Finally, notice that $\bar{s}_1 < 0$, that is, $\bar{w} \in \text{corner}(\bar{x}) \setminus P$. □

In fact, inequality (37) defines a facet of the convex hull of feasible solutions of the MIP.

Acknowledgments

The research of Egon Balas was supported by NSF grant CMMI-1263239 and ONR contract N000141210032.

The research of Tamás Kis was supported by the János Bólyai research grant BO/00412/12/3 of the Hungarian Academy of Sciences, and by the OTKA grant K112881.

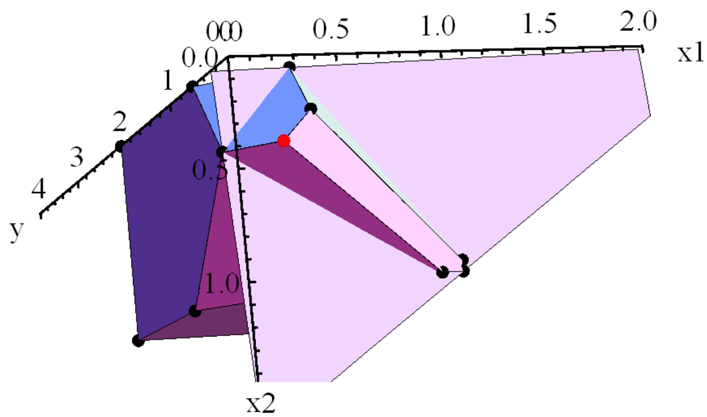

Figure 2: Cut (37).

References

[1] K. Andersen, Q. Louveaux, R. Weismantel and L. Wolsey, Inequalities from two rows of the simplex tableau. M. Fischetti and D. Williamson (eds), *Integer Programming and Combinatorial Optimization*, LNCS 4513, Springer 2007, 1–15.

[2] K. Andersen, G. Cornuéjols and Y. Li, Split closure and intersection cuts. *Math Prog A, 102,* 2005, 457-493.

[3] E. Balas, Intersection Cuts–A New Type of Cutting Planes for Integer Programming. *Operations Research, 19*, 1971, 19-39.

[4] E. Balas, Integer programming and convex analysis: intersection cuts from outer polars. *Math Programming 2,* 1972, 330-382.

[5] E. Balas, Disjunctive Programming: Properties of the Convex Hull of Feasible Points. MSRR No. 348, Carnegie Mellon University, July 1974. Published as an invited paper with a foreword by G. Cornuéjols and W.R. Pulleyblank in *Discrete Applied Mathematics, 89,* 1998, 3-44.

[6] E. Balas, S. Ceria, G. Cornuéjols, A Lift-and-Project Cutting Plane Algorithm for Mixed 0-1 Programs *Mathematical Programming, 58*, 1993, 295-324.

[7] E. Balas, G. Cornuéjols, T. Kis, G. Nannicini, Combining Lift and Project and Reduce and Split, *INFORMS Journal on Computing, 25*, 2013, 475-487. DOI: 10.1287/ijoc.1120.0515

[8] E. Balas and T. Kis, Intersection cuts – standard and restricted. Technical Report, CMU, July 2015.

[9] E. Balas, F. Margot, Generalized intersection cuts and a new cut generating paradigm. *Mathematical Programming A, 135*, 2013, 19-35. DOI: 10.1007/s10107-011-0483-x

[10] E. Balas, M. Perregaard, A Precise Correspondence Between Lift-and-Project Cuts, Simple Disjunctive Cuts, and Mixed Integer Gomory Cuts for 0-1 Programming. *Mathematical Programming B, 94,* 2003, 221-245.

[11] E. Balas, A Qualizza, Intersection cuts from multiple rows: a disjunctive programming approach. *EURO Journal on Computational Optimization, 1,* 2013, 3-49. DOI: 10.1007/s13675-013-0008-x

[12] M. Conforti, G. Cornuéjols, G. Zambelli, Equivalence between intersection cuts and the corner polyhedron. *Operations Research Letters, 38*, 2010, 153-155. DOI: 10.1016/j.orl.2010.02.006

[13] M. Conforti, G. Cornuéjols, G. Zambelli, Corner polyhedron and intersection cuts. *Surveys in Operations Research and Management Science, 16*, 2011, 105-120. DOI: 10.1016/j.sorms.2011.03.001

[14] M. Conforti and A. Del Pia, Disjunctive programming and relaxations of polyhedra. *Math Prog A, 144*, 2014, 307-314.

[15] S. Dey and L. Wolsey, Constrained infinite group relaxations of MIPs. *SIAM J. Optim., 20(6)*, 2010, 2890–2912.

[16] R. Gomory, Some polyhedra related to combinatorial problems. *Linear Algebra and Its Applications, 2,* 1969, 451-558.

[17] T. Kis, Lift-and-project for general two-term disjunctions. *Discrete Optimization, 12*, 2014, 98-114. DOI: 10.1016/j.disopt.2014.02.001

[18] L. Lovász, Geometry of numbers and integer programming. In M. Iri, K. Tanabe (eds.) *Mathematical Programming*. KTK Scientific Publishers, Tokyo, 1989, 177-201.