Fractional solutions for capacitated NTU-games, with applications to stable matchings[*]

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Abstract. In this paper we investigate some new applications of Scarf's Lemma. First, we introduce the notion of fractional core for NTU-games, which is always nonempty by the Lemma. Stable allocation is a general solution concept for games where both the players and their possible cooperations can have capacities. We show that the problem of finding a stable allocation, given a finitely generated NTU-game with capacities, is always solvable by the Scarf's Lemma. Then we consider an even more general setting where players' contributions in a joint activity may be different. We show that a stable allocation can be found by the Scarf algorithm in this case as well. Finally we describe the interpretation of these results for stable matching problems, and in particular, for the hospitals resident problem with couples.

Keywords: Scarf lemma, NTU-games, core, stable matching, stable allocation, hospitals residents problem, couples
JEL classification: C61, C71, C78

1 Introduction

Complex social and economic situations can be described as games where the players may cooperate with each other. Most studies in cooperative game theory focus on the issue of how the participants form disjoint coalitions, and sometimes also on the way the members of coalitions share the utilities of their cooperations among themselves (in case of games with transferable utility). However, in reality, an agent in the market (or any individual in some social situation) may be involved in more than one cooperation at a time, moreover, a cooperation may be performed with different intensities. For instance, an employer can have several employees and their working hours can be different (but within some reasonable limits). In this case we may assume that a part-time contract has the same terms as a full-time contract, only the working hours and the salary (and other duties and benefits) are set proportionally. Finally, a cooperation might require

different contributions from different players. As an example, consider the coalition of a PhD student and her supervisor. The student is very likely to spend all of her working ours (say 40h per week) on this project, whilst the supervisor might spend 3 hours with this student a week.

Scarf [40] proved that every balanced NTU-game (i.e, cooperative game with non-transferable utilities) has a nonempty core. Informally, a solution is in the core of an NTU-game (or it is *stable*) if there is no deviating coalition whose members could all get strictly better off by forming a new cooperation among themselves. Scarf's theorem was based on a lemma, which became known as the Scarf Lemma, that constructively proved the existence of a fractional solution for any NTU-game. If a game is balanced then this fractional solution is guaranteed to be integral and thus equivalent to a core element, he showed.

The essence of our paper is to describe the notion of fractional stable solutions in a variety of NTU-games, starting from simple coalition formation games to a rather complicated model where players have capacities and the cooperations can be performed with different contributions by the players. Surprisingly, the original Scarf Lemma implies the existence of a stable fractional solutions in all cases, and it can be computed by Scarf's original algorithm. We show that we may use these fractional solutions in practical applications as well, such as the resident allocation problem with couples, where the integral stable solutions correspond to stable matchings. The Scarf algorithm can therefore serve as a heuristic to solve difficult (sometimes NP-hard) stable matching or coalition formation problems, just like as an integer program can be tackled by solving its LP relaxation. Our new models, where the players contribute differently in coalitions, may also provide new descriptions of further economic or social applications. For instance, in some countries (e.g. Hungary, France and Portugal) the research grant applications must contain the contributions of the participants in the proposed projects in terms of their devoted working time, just like in our most general model, the stable schedule problem.

In this paper, we give a new interpretation of the fractional solutions which are obtained by the Scarf algorithm for different settings. First we consider the original setting of the Lemma for finitely generated NTU-games, and we describe the meaning of the output in terms of *fractional core*. We show the correspondence between this notion and the concept of *fractional stable matchings for hypergraphs*. We conclude Section 2 by explaining how the normality of a hypergraph implies the nonemptiness of the core for the corresponding NTU-games. In Section 3, we define the stable allocation problem for hypergraphs, which corresponds to the problem of finding a fractional core for NTU-games where the players can be involved in more than one coalition and the joint activities can be performed at different intensity levels (up to some capacity constrains). We show that, by an appropriate extension of the Scarf Lemma implies the existence of the latter solution. Then, we consider an even more general setting where the contributions of the players may differ in an activity, called *stable allocation problem with contributions*, or *stable schedules*. We show that the Scarf algorithm can find a stable allocation for this case as well. In Section 4, we apply these results for matching problems and we derive some well-known theorems in this context. We also illustrate the usage of Scarf algorithm as a heuristic for the hospitals residents problem with couples (the complete description of this experiment can be found in our follow-up paper [10]). Finally, in Section 5, we present some important open problems and new research directions.

Related literature

The notion of a fractional core can be easily defined by the seminal paper of Scarf [40]. However, to the best of our knowledge, there are not many papers on this notion. Aharoni

and Fleiner [1] described the connection between the fractional core element and fractional stable matchings for the roommates problem and for coalition formation games as well. Besides other results, they demonstrated how the Scarf Lemma implies the existence of stable half-integral solutions for the stable roommates problem, a theorem originally proved by Tan [43].

Stable matching problems have been extensively studied in the last half century, since the paper by Gale and Shapley [22] has been published, by economists (see e.g. [38]), computer scientists (see e.g. [23]), and in other disciplines as well.[1] The main reason of the increased attention on this subject is the large number of important practical applications. Perhaps the earliest centralised matching scheme is the US resident allocation program, called NRMP, where the Gale-Shapley algorithm has been in operation since 1952. One of the special features of this program (and its counterparts in other countries, such as in Scotland [47]) is that couples may submit joint applications for pair of positions. This feature makes the problem challenging to solve both in theory and practice (see more in a recent survey on this problem [12]). At the end of our paper we illustrate how the Scarf algorithm can be used as a heuristic to solve this problem.

We are not aware of any paper on NTU-games where both the players and their cooperations have capacities, apart from the papers on the stable allocation problem for matching markets ([5], [2], [8] and [20]). However, there is a closely related paper to ours on TU-games with overlapping coalitions [16]. In the latter paper the general model introduced is essentially the TU version of our stable allocation problem with contributions (or stable schedules), that we study in Section 4. So our last problem can be seen as the extension of their TU-model for NTU-games.

Our contribution

The main contribution of our paper is that we extend the result of Scarf and show the existence of a fractional stable solution for NTU-games with capacities (Theorem 12) and NTU-games with contributions (Theorem 13). We also give alternative proofs for known theorems. We show that whenever the basic coalitions of a NTU-game form a normal hypergraph then the core is nonempty, originally proved by Vasin and Gurvich [44], and by Kaneko and Wooders [26] (Theorem 10). For bipartite matching games this boils down to the result of Kaneko [25], Crawford-Knoer [17] (Theorem 15). For non-bipartite matching games we obtain a slight generalisation of Tan's [43] theorem, namely the existence of a stable half-matching[2]. For nonbipartite capacitated matching games we retrieve the result of Biró and Fleiner [8] on the existence of half-integral stable allocations (Theorem 18), which implies the existence of an integral stable allocation for bipartite graphs, a theorem of Baïou and Balinski [5]. Finally, we demonstrate with examples that finding stable fractional solutions can be done efficiently by the Scarf algorithm for settings that are coming from real applications, such as the matching with couples problem, and thus this approach can serve as a new heuristic to provide fair solutions in centrally coordinated markets.

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[1]The 2012 Nobel-Prize in Economic Sciences has been awarded to Alvin Roth and Lloyd Shapley for the theory of stable allocations and the practice of market design.

[2]Note that very recently Alkan and Tuncay [3] showed similar results for non-bipartite matching games. They proved the existence of stable half-matching (under the name of semistable allocations) for a special case where the Pareto-frontier of the utility spaces are continuous decreasing functions.

2   Fractional core - fractional stable matchings

In this section, first we describe the Scarf Lemma and we give a new interpretation of the fractional results obtained by the Scarf algorithm.

2.1   Definitions, preliminaries

We recall the definition of *n-person games with nontransferable utility* (NTU-game for short).

Definition 1 *An* NTU-game *is given by a pair (N, V), where $N = \{1, 2, \ldots, n\}$ is the set of players and V is a mapping of a set of feasible utility vectors, a subset $V(S)$ of $\mathbb{R}^S$ to each coalition of players, $S \subseteq N$, such that $V(\emptyset) = \emptyset$, and for all $S \subseteq N$, $S \neq \emptyset$:*

a) $V(S)$ is a closed subset of $\mathbb{R}^S$

b) $V(S)$ is comprehensive, i.e. if $u^S \in V(S)$ and $\tilde{u}^S \leq u^S$ then $\tilde{u}^S \in V(S)$

c) The set of vectors in $V(S)$ in which each player in S receives no less than the maximum that he can obtain by himself is a nonempty, bounded set.

One of the most important solution concepts is the core.

Definition 2 *A utility vector $u^N \in V(N)$ is in the core of the game, if there exists no coalition $S \subseteq N$ with a feasible utility vector $\tilde{u}^S \in V(S)$ such that $u_i^N < \tilde{u}_i^S$ for every player $i \in S$. Such a coalition is called blocking coalition.*

An NTU-game (N, V) is *superadditive* if $V(S) \times V(T) \subseteq V(S \cup T)$ for every pair of disjoint coalitions S and T. In what follows, we restrict our attention to superadditive games.

Partitioning games are special superadditive games. Given a set of *basic coalitions* $\mathcal{B} \subseteq 2^N$, that contain all singletons (i.e. every single player has the right not to cooperate with the others), a partitioning game $(N, V, \mathcal{B})$ is defined as follows: if $\Pi_\mathcal{B}(S)$ denotes the set of partitions of S into basic coalitions, then $V(S)$ can be generated as:

$$V(S) = \{u^S \in \mathbb{R}^S | \exists \pi = \{B_1, B_2, \ldots, B_k\} \in \Pi_\mathcal{B}(S) : u^S \in V(B_1) \times V(B_2) \times \cdots \times V(B_k)\}$$

This means that u^S is a feasible utility vector of S if there exist a partition π of S into basic coalitions such that each utility vector $u^S|_{B_i}$ can be obtained as a feasible utility vector by basic coalition B_i in π.

Given an NTU-game (N, V), let $U(S)$ be the set of *Pareto optimal* utility vectors of the coalition S, i.e. $u^S \in U(S)$ if there exists no $\tilde{u}^S \in V(S)$, where $u^S \neq \tilde{u}^S$ and $u^S \leq \tilde{u}^S$.

A utility vector $u^S \in V(S)$ is *separable* if there exist a proper partition π of S into subcoalitions $S_1, S_2, \ldots, S_k$ such that $u^S|_{S_i}$ is in $V(S_i)$ for every $S_i \in \pi$. A utility vector that is non-separable, Pareto-optimal and in which each player receives no less than the maximum that he can obtain by himself is called an *efficient* vector. A coalition S is *essential* if $V(S)$ contains an efficient utility vector. In other words, a coalition S is essential, if its members can obtain an efficient utility vector that is not achievable independently by its subcoalitions. The set of essential coalitions is denoted by $\mathcal{E}(N, V)$.

We say that a coalition S is not *relevant* if for every utility vector $u^S \in V(S)$ there exists a proper subcoalition $T \subset S$ such that $u^S|_T$ is in $V(T)$. The set of relevant coalitions is denoted by $\mathcal{R}(N, V)$. The idea behind this notion is that if a non-relevant coalition S is blocking with a utility vector u^S, then one of its subcoalitions, say T_1, must be also

blocking with utility vector $u^{T_1} = u^S|_{T_1}$. Moreover, if T_1 is not relevant or u^{T_1} is separable, then we can find another coalition $T_2 \subset T_1$, such that $u^{T_2} = u^{T_1}|_{T_2} = u^S|_{T_2}$, an so on. Continuing this argument, it is clear that there must be a relevant coalition $T_i \subset S$, that is blocking with a non-separable vector $u^{T_i} = u^S|_{T_i}$. This observation implies the following Proposition:

Proposition 3 *A utility vector $u^N \in V(N)$ is in the core if and only if it is not blocked by any relevant coalition with an efficient utility vector.*

Obviously, if a coalition is not essential, then it cannot be relevant either. In a partitioning game, the set of essential coalitions must be a subset of the basic coalitions by definition.

Proposition 4 *For every partitioning game $(N, V, \mathcal{B})$, $\mathcal{R}(N, V, \mathcal{B}) \subseteq \mathcal{E}(N, V, \mathcal{B}) \subseteq \mathcal{B}$ holds.*

Scarf [40] observed that the previously introduced notions are purely ordinal in character: they are invariant under a continuous monotonic transformation of the utility function of any individual. Hence, without loss of generality, we may assume that $U^{\{i\}} = \{0\}$ for every singleton, and all the efficient utility vectors are nonnegative. Moreover, the discussion can be carried out on an abstract level with the outcomes for each individual represented by arbitrary ordered sets, as we describe this in detail below.

Suppose that in order to obtain a particular non-separable vector $u^{S,k}$ in $U(S)$, the members of S have to perform a joint activity, say $\mathfrak{a}^{S,k}$. Let $\mathcal{A}^S$ denote the set of activities that yield efficient utility vectors in $U(S)$. The preference of a player over the possible activities in which he can be involved is determined by the utilities that he obtains in these activities. Formally, we suppose that $\mathfrak{a}^{S,k} \leq_i \mathfrak{a}^{T,l} \iff u_i^{S,k} \leq u_i^{T,l}$ for any pair of activities $\mathfrak{a}^{S,k}$ and $\mathfrak{a}^{T,l}$, where $i \in S$ and $i \in T$.

Considering an efficient utility vector $u^{N,l}$ of the grandcoalition N, the non-separability implies that $u^{N,l}$ corresponds to a joint activity $\mathfrak{a}^{N,l}$ of the entire set of players. Otherwise, if $u^{N,l}$ is separable, then $u^{N,l}$ can be obtained as a direct sum of independent efficient utility vectors of essential subcoalitions that form a partition of the grandcoalition. This can be considered as a set of independent activities of the subcoalitions. An *outcome of the game*, denoted by X then can be regarded as a partition π of the players and a set of activities $\mathcal{A}^\pi$ performed independently by the coalitions in π, so let $X = (\pi, \mathcal{A}^\pi)$. An outcome X is judged by a player i according to the activity he is involved in, denoted by $\mathfrak{a}_i(X)$. An outcome is in the core of the game, or in other words, it is *stable* if there exist no blocking coalition S with an activity $\mathfrak{a}^{S,k}$ that is strictly preferred by all of its members, i.e., $\mathfrak{a}^{S,k} >_i \mathfrak{a}_i(X)$ for every $i \in S$. (This is equivalent to the blocking condition $u_i^{N,l} < \tilde{u}_i^S$, if the outcome X corresponds to the utility vector $u^{N,l}$.)

An NTU-game is *finitely generated* if for every essential coalition S, $U(S)$ contains a finite number of vectors. Here, the preference order of a player over the set of activities, in which he can be involved, can be represented by preference lists. As Scarf observed in [40] and [41], a general NTU-game can be approximated by a finitely generated NTU-game (see an illustration in Figure 1). In this paper, for simplicity, we state all our theorems for finitely generated games. However, we shall note that our results imply the existence of ϵ-stable (fractional) solutions (similarly defined as the ϵ-core, see e.g. Wooders [45]) for any positive ϵ. Therefore our results are essentially valid without the assumption of the NTU-game being finitely generated.

If for every essential coalition S, in a given NTU-game, $U(S)$ contains only one single vector, u^S, then an outcome of the game is simply a partition, since each essential coalition
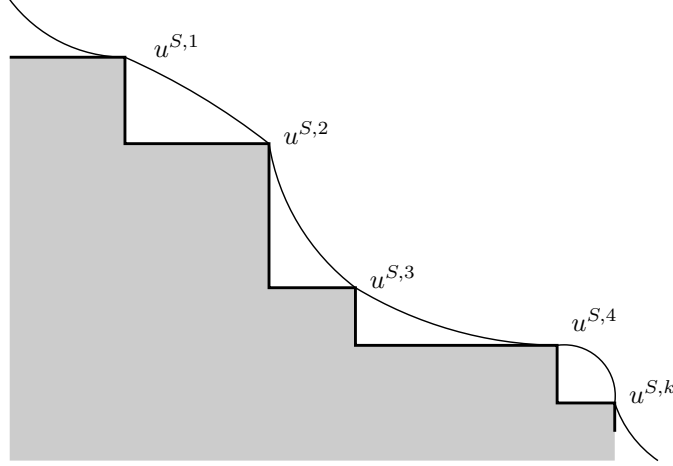
Figure 1: Approximation with finite number of efficient utility vectors.

has only one activity to perform. So here, instead of activities, each player has a preference order over the essential coalitions in which he can be a member. These games are called *(hedonic) coalition formation games* (CFG for short) [7] and [14], and an outcome that is in the core of the game is called a *core-partition*. The following example illustrates a CFG.

Example 1.

Suppose that we are given 6 players: A, B, C, D, E and F, and 4 possible basic coalitions with corresponding joint activities. The first activity, b (bridge) can be played by A, B, C and D, the second one, p (poker) can be played by C, D and E. Finally, B can play chess with C (denoted by c_1) and D can play chess with F (denoted by c_2). The preferences of the players over the joint activities are as follows.
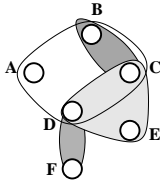
Activities	Participants	Players	Preference lists
$b:$	$\{A, B, C, D\}$	$B:$	$b\ c_1$
$p:$	$\{C, D, E\}$	$C:$	$p\ b\ c_1$
$c_1:$	$\{B, C\}$	$D:$	$b\ p\ c_2$
$c_2:$	$\{D, F\}$		

Here, $\{p, \{A\}, \{B\}, \{F\}\}$ is a core-partition, since b is not blocking because C prefers his present coalition p to b, similarly, c_1 is not blocking because C prefers playing poker with D and E to playing chess with B, and c_2 is not blocking because D also prefers playing poker to playing chess with F. One can easily check that $\{b, \{E\}, \{F\}\}$ is also a core-partition, but the partition $\{c_1, c_2, \{A\}, \{E\}\}$ is not in the core, since p and b are both blocking coalitions.

2.2 Fractional core by Scarf's Lemma

First, we present Scarf's Lemma [40] and then we introduce the notion of fractional core. The following description of the Lemma is due to Aharoni and Fleiner [1] (here $[n]$ denotes the set of integers $1, 2, \ldots, n$, and $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise).

Theorem 5 (Scarf [40]) *Let n, m be positive integers, and b be a vector in $\mathbb{R}_+^n$. Also*

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let $A = (a_{i,j}), C = (c_{i,j})$ be matrices of dimension $n \times (n + m)$, satisfying the following three properties: the first n columns of A form an $n \times n$ identity matrix (i.e. $a_{i,j} = \delta_{i,j}$ for $i, j \in [n]$), the set $\{x \in \mathbb{R}_+^{n+m} : Ax = b\}$ is bounded, and $c_{i,i} < c_{i,k} < c_{i,j}$ for any $i \in [n], i \neq j \in [n]$ and $k \in [n + m] \setminus [n]$.

Then there is a nonnegative vector x in $\mathbb{R}_+^{n+m}$ such that $Ax = b$ and the columns of C that correspond to $\text{supp}(x)$ form a dominating set, that is, for any column $i \in [n + m]$ there is a row $k \in [n]$ of C such that $c_{k,i} \leq c_{k,j}$ for any $j \in \text{supp}(x)$.

Let the columns of A and C correspond to the efficient utility vectors (or equivalently to some activities) of the essential coalitions in a finitely generated NTU-game as follows. If the k-th columns of A and C correspond to the utility vector $u^{S,l}$, then let $a_{i,k}$ be 1 if $i \in S$ and 0 otherwise, (so the k-th column of A is the *membership vector* of coalition S). Furthermore, let $c_{i,k} = u_i^{S,l}$ if $i \in S$ and $c_{i,k} = M$ otherwise, where M is a sufficiently large number. We set $c_{i,i} = u_i^{\{i\}} = 0$ and $c_{i,j} = 2M$ if $i \neq j \leq n$. Finally, let $b = 1^N$, which implies that a solution x satisfying $Ax = b$ is a so-called *balanced collection of coalitions*. By applying Scarf's Lemma for this setting, we obtain a solution x that we call a *fractional core element* of the game. We refer to the set of fractional core elements as the *fractional core* of the game.

What is the meaning of a fractional core element? Let us suppose first, that a fractional core element x is integer, so $x_i \in \{0, 1\}$ for all i. In this case we show that x gives a utility vector u^N that is in the core of the game. Let u^N be the utility vector of N received by summing up those independent essential utility vectors for which $x(u^{S,k}) = 1$, then u^N is in $V(N)$ by superadditivity. To prove that u^N must be in the core of the game, let $u^{S,k}$ be an essential utility vector, with $x(u^{S,k}) = 0$. By the statement of Scarf's Lemma, there must be a player i and an essential utility vector $u^{T,l}$, such that $i \in T$, $x(u^{T,l}) = 1$ and $u_i^{S,k} \leq u_i^{T,l}$, so S cannot be a blocking coalition with the efficient utility vector $u^{S,k}$.

In other words, the $Ax = 1^N$ condition of the solution says that x represents a partition π of N and a set of activities $\mathcal{A}^\pi$ that are performed. (We say that $\mathfrak{a}^{S,k}$ is *performed* if $x(u^{S,k}) = 1$, i.e. $\mathfrak{a}^{S,k} \in \mathcal{A}^\pi$, where that S is a coalition in partition π). Let $X = (\pi, \mathcal{A}^\pi)$ be the corresponding outcome, and let $\mathfrak{a}^{S,k}$ be an activity not performed, (i.e. $\mathfrak{a}^{S,k} \notin \mathcal{A}^\pi$). Then, by Scarf's Lemma there must be a player i of S for which the performed activity, $\mathfrak{a}_i(X)$ he is involved in is not worse than $\mathfrak{a}^{S,k}$, i.e., $\mathfrak{a}^{S,k} \leq_i \mathfrak{a}_i(X)$, thus S cannot be a blocking coalition with activity $\mathfrak{a}^{S,k}$.

In the non-integer case, we shall regard $x(u^{S,k})$ as the *intensity* at which the activity $\mathfrak{a}^{S,k}$ is performed by coalition S. The $Ax = 1^N$ condition means that each player participates in activities with total intensity 1, including maybe the activity that this player performs alone. The domination condition says that for each activity, which is not performed with intensity 1, there exists a member of the coalition who is not interested in increasing the intensity of this activity, since he is satisfied by some other preferred activities that fill his remaining capacity. Formally, if $x(u^{S,k}) < 1$ then there must be a player i in S such that $\sum_{a^{T,l} \geq_i a^{S,k}} x(u^{T,l}) = 1$.

In Example 1, $x(p) = \frac{1}{3}$, $x(b) = \frac{2}{3}$ is a fractional core element, since for each activity there is at least one player who is not interested in increasing the intensity of that activity. We illustrate with the following example that the fractional core of a game may admit a unique fractional core element where the intensities of certain activities can be arbitrary small nonnegative values.

Example 2.

In this example we have 6 players with 6 possible joint activities and the following preferences:
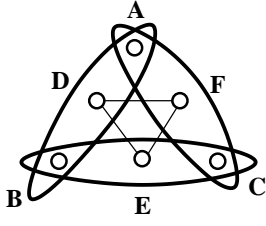
Activities	Participants	Preference	lists
$p_1:$	$\{A, D, B\}$	$A:$	$p_1\ p_3$
$p_2:$	$\{B, E, C\}$	$B:$	$p_2\ p_1$
$p_3:$	$\{C, F, A\}$	$C:$	$p_3\ p_2$
$c_1:$	$\{D, E\}$	$D:$	$p_1\ c_1\ c_2$
$c_2:$	$\{E, F\}$	$E:$	$p_2\ c_2\ c_3$
$c_3:$	$\{F, D\}$	$F:$	$p_3\ c_3\ c_1$

It can be verified that the only fractional core element here is x, where $x(p_1) = x(p_2) = x(p_3) = \frac{1}{2}$ and $x(c_1) = x(c_2) = x(c_3) = \frac{1}{4}$. One can easily extend this construction for $3n$ players, where some activities in the unique fractional core element have intensity 2^{-n}.

2.3 Fractional stable matching for hypergraphs

For a finitely generated NTU-game, the problem of finding a stable outcome is equivalent to the *stable matching problem* (SM for short) *for a hypergraph*, as defined by Aharoni and Fleiner [1]. Here, the vertices of the hypergraph correspond to the players, the edges correspond to the efficient vectors (or to activities being performed by the players concerned), and the preference of a vertex over the edges it is incident with comes from the preference of the corresponding player over the activities he can be involved in. This is called a *hypergraphic preference system*. A *matching* corresponds to a set of joint activities performed by certain coalitions that form a partition of the grandcoalition together with the singletons (i.e. with the vertices not covered by the matching). A matching M is *stable* if there exist no *blocking edge*, i.e. an edge $e \notin M$ such that for every vertex v covered by e, either v is unmatched in M or strictly prefers e to the edge that covers v in M. The corresponding set of activities gives a stable outcome, since there exist no blocking coalition with an activity that is strictly preferred by all of its members. Note that different activities performed by the same players are represented by *multiple edges* in the corresponding hypergraph. A hypergraph which represents the efficient utility vectors of a CFG is simple (i.e, does not contain multiple edges and loops). [3]

The notion of a *fractional stable matching* for an instance of SM for a hypergraph was defined by Aharoni and Fleiner [1] as follows. A function x assigning non-negative weights to edges of the hypergraph is called a *fractional matching* if $\sum_{v \in h} x(h) \leq 1$ for every vertex v. A fractional matching x is called *stable* if every edge e contains a vertex v such that $\sum_{v \in h, e \leq_v h} x(h) = 1$. The existence of a fractional stable matching can be verified by Scarf's Lemma just like the existence of a fractional core element. Actually, these two notions are basically equivalent.

To show the equivalence formally, we consider the polytope of intensity vectors $\{x | Ax = 1^N, x \geq 0\}$ on the one hand, where A is the membership-matrix of the efficient utility vectors (or the corresponding activities) of dimension $n \times (n+m)$ as defined by Scarf's Lemma. On the other hand, the fractional matching polytope is $\{x | Bx \leq 1^N, x \geq 0\}$, where B

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[3]We shall note that Aharoni and Fleiner [1] supposed in their model that the preferences of the players are strict (i.e., no player is indifferent between any pair of activities). In the literature on stable matching, the setting where players may have *ties* in their lists is referred to as *stable matching problem with ties*. In this case, a matching M is (weakly) stable if it does not admit a blocking edge (where the definition of a blocking edge is the same as described above). However, an instance of SM with ties can be always derived to another instance of SM (with no ties) by simply breaking the ties arbitrary, and a matching that is stable for the derived instance is (weakly) stable for the original one. The same applies for the core and fractional core in the context of NTU-games. In fact, the Scarf algorithm starts with a perturbation of matrix C in the case that any player is indifferent between two activities she may be involved in (i.e., when her utilities in these two activities are the same for her).

is the vertex-edge incidence matrix of the hypergraph of dimension $n \times m$. Therefore, $A = (I_n|B)$, so the difference is only the $n \times n$ identity matrix, i.e. the membership-matrix of the singletons. So, there is a natural one-to-one correspondence between the elements of the two polytopes: if x^m is a fractional matching of dimension m, then let $\bar{x}^v = 1^N - Ax^m$ be a vector of dimension n, that gives the unfilled intensities of the players (or in other words, the intensities of the single activities). The direct sum of these two independent vectors, x is an intensity vector of dimension $n + m$, and vice versa. The stability condition is equivalent to the domination condition of Scarf's Lemma.

Aharoni and Fleiner [1] showed that a fractional stable matching can be assumed to be an extremal point of the fractional matching polytope. This fact comes from a statement similar to the following Proposition:

Proposition 6 *If x is a fractional core element of a finitely generated NTU-game, and $x = \sum \alpha_i x^i$, where $\alpha_i > 0$ for all i, $\sum \alpha_i = 1$ and x^i satisfies the $Ax^i = 1^N$ and $x^i \geq 0$ conditions, then each x^i must be a fractional core element.*

The proof of this Proposition is straightforward, since $supp(x^i) \subseteq supp(x)$, that implies the dominating property of the fractional core element.

Corollary 7 *For any finitely generated NTU-game, there exists a fractional core element that is an extremal point of the polytope $\{x|Ax = 1^N, x \geq 0\}$.*

Corollary 7 implies that if, for a given finitely generated NTU-game, all the extremal points of the above polytope are integers (or, in other words, the polytope has the *integer property*) then the game has a nonempty core.

2.4   Normality implies the nonemptiness of the core

The definition of a normal hypergraph is due to Lovász [32]. If H is a hypergraph and H' is obtained from H by deleting edges, then H' is called a *partial hypergraph* of H. The *chromatic index* $\chi_e(H)$ of a hypergraph H is the least number of colours sufficient to colour the edges of H so that no two edges with the same colour have a vertex in common. Note that the maximum degree, $\Delta(H)$ (that is, the maximum number of edges containing some one vertex) is a lower bound for the chromatic index. A hypergraph H is *normal* if every partial hypergraph H' of H satisfies $\chi_e(H') = \Delta(H')$. It is easy to see that the normality is preserved by adding or deleting multiple edges or loops. The following theorem of Lovász [32] gives an equivalent description of normal hypergraphs.

Theorem 8 (Lovász) *The fractional matching polytope of a hypergraph H has the integer property if and only if H is normal.*

Suppose that for a finitely generated NTU-game the set of essential coalitions forms a normal hypergraph. The hypergraph of the corresponding SM must be also normal, since it is obtained by adding multiple edges and by removing the loops. By Theorem 8, the fractional matching polytope, $\{x|Bx \leq 1^N, x \geq 0\}$ has the integer property, and so has the polytope of intensity vectors, $\{x|Ax = 1^N, x \geq 0\}$ as it was discussed previously. This argument and Corollary 7 verify the following Lemma 9.

Lemma 9 *If, for a finitely generated NTU-game, the set of essential coalitions, $\mathcal{E}(N,V)$ forms a normal hypergraph, then the core of the game is nonempty.*

By Lemma 9 and Proposition 4 the following holds.

Theorem 10 *If the set of basic coalitions, $\mathcal{B}$ forms a normal hypergraph, then every finitely generated NTU-game $(N, V, \mathcal{B})$ has a nonempty core.*

Let $A^{\mathcal{B}}$ denote the membership-matrix of the set of basic coalitions $\mathcal{B}$. The fact that the integer property of the polytope $\{x | A^{\mathcal{B}}x = 1^N, x \geq 0\}$ implies the nonemptiness of every NTU-game $(N, V, \mathcal{B})$ was proved first by Vasin and Gurvich [44], and independently, by Kaneko and Wooders [26]. Later, Le Breton *et al.* [31], Kuipers [30] and Boros and Gurvich [15] observed independently that the integer property of the polytope $\{x | A^{\mathcal{B}}x = 1^N, x \geq 0\}$ is equivalent to the integer property of the matching polytope $\{x | A^{\mathcal{B}}x \leq 1^N, x \geq 0\}$, and to the normality of the corresponding hypergraph.

We note that the converse of Theorem 10 is also true: Boros and Gurvich [15] proved that if the set of basic coalitions, $\mathcal{B}$ has the property that the polytope $\{x | A^{\mathcal{B}}x = 1^N, x \geq 0\}$ has a non-integer extremal point (thus the corresponding hypergraph is not normal), then there always exist a finitely generated NTU-game $(N, V, \mathcal{B})$ with an empty core.

3   Fractional solutions for capacitated NTU-games

There are many economic and social situations where the agents may be involved in more than one coalition at a time. A possible way to describe such situations is by capacitated NTU-games.

3.1   Fractional b-core element

In what follows, we introduce the notion of fractional b-core element as a solution of Scarf's Lemma with the original settings. Let the same matrices A and C of dimension $n \times (n + m)$ correspond to the set of effective utility vectors (or activities) in a finitely generated NTU-game as it was described in the previous section. The only modification is that now b is an arbitrary vector of $\mathbb{R}_+^n$ (instead of 1^N). Let $x \in \mathbb{R}_+^{n+m}$ be referred to as a *fractional b-core element* if x is a solution of the Scarf Lemma for the above setting.

Here, $b(i)$ is an upper bound for the total intensity at which player i is capable to perform activities, since $\sum_{i \in S} x(u^{S,l}) = b(i)$. The domination condition of the Lemma says that for every activity $\mathfrak{a}^{T,k}$, there exists some player i who is not interested in increasing the intensity of $\mathfrak{a}^{T,k}$, because his remaining intensity is filled with better activities, i.e., if $u^{T,k}$ corresponds to activity $\mathfrak{a}^{T,k}$, then $\sum_{u_i^{S,l} \geq u_i^{T,k}} x(u^{S,l}) = b(i)$.

In fact, to produce a fractional core element (in other words, a fractional 1^N-core element) with the algorithm of Scarf, we perturb not just matrix C (in case of indifferences), and also the right-hand side vector 1^N, to avoid the degeneracy. The standard nondegeneracy assumption provides that all variables associated with the n columns of a feasible basis for the equations $A\tilde{x} = \tilde{b} = 1^N + \varepsilon^N$ are strictly positive. Thus, the perturbation uniquely determines the steps of Scarf algorithm. By rounding the final fractional $\tilde{b}$-core element $\tilde{x}$, a fractional core element x is found. The following simple Lemma says that the fractional b-core element has the *scaling property*.

Lemma 11 *Given a finitely generated NTU-game, and a positive constant λ. Suppose that $b' = \lambda \cdot b$, then x is a fractional b-core element if and only if $x' = \lambda \cdot x$ is a fractional b'-core element.*

3.2   Fractional b-core element with capacities: stable allocation

Let us suppose that the intensities of the activities in the finitely generated NTU-game are constrained by *capacities*. Formally, for each joint activity $\mathfrak{a}^{S,l}$ and for the corresponding

utility vector $u^{S,l}$, there may exist a nonnegative capacity $c(u^{S,l})$ for which $x(u^{S,l}) \le c(u^{S,l})$ is required.

The stable allocation problem can be defined for hypergraphs as follows. Suppose that we are given a hypergraph H and for each vertex v a strict preference order over the edges incident with v (again, this corresponds to the preferences of the players over the activities in which they can be involved). Suppose, that we are given nonnegative *bounds* on the vertices $b : V(H) \to \mathbb{R}_+$ and nonnegative *capacities* on the edges $c : E(H) \to \mathbb{R}_+$. A nonnegative function x on the edges, is an *allocation* if $x(e) \le c(e)$ for every edge e and $\sum_{v \in h} x(h) \le b(v)$ for every vertex v. An allocation is *stable* if every *unsaturated* edge e (i.e., every edge e with $x(e) < c(e)$) contains a vertex v such that $\sum_{v \in h, e \le_v h} x(h) = b(v)$. In this case we say that e is *dominated at v*. If every bound and capacity is integral then we refer to this problem as the *integral stable allocation problem*.

Theorem 12 *Every stable allocation problem for hypergraphs is solvable.*

Proof: Suppose that we are given a given a hypergraph H. Let $V(H) = \{v_1, v_2, \ldots, v_n\}$ be the set of vertices and let $E(H) = \{e_1, e_2, \ldots, e_m\}$ be the set of edges. We define the extended membership-matrix A, and the extended preference-matrix C of size $(n+m) \times (n+2m)$ as follows.

The left part of A is an identity matrix of size $(n+m) \times (n+m)$, (i.e. $a_{i,j} = \delta_{i,j}$ for $i, j \in [n+m]$). At the bottom of the right side there is another identity matrix of size $m \times m$, so $a_{n+i,n+m+j} = \delta_{i,j}$ for $i, j \in [m]$. Finally, at the top of the right side we have the vertex-edge incidence matrix of H (i.e. $a_{i,n+m+j} = 1$ if $v_i \in e_j$ and 0 otherwise for $i \in [n]$ and $j \in [m]$).

The top-right part of C correspond to the preference of the vertices (that is the preference of the players over the activities). We require the following two conditions:

- $c_{i,n+m+j} < c_{i,n+m+k}$ whenever $v_i \in e_j \cap e_k$ and $e_j <_{v_i} e_k$;

- $c_{i,n+m+j} < c_{i,n+m+k}$ whenever $v_i \in e_j \setminus e_k$.

Furthermore, suppose that $c_{n+i,n+m+i} < c_{n+i,n+m+j}$ for every $i \ne j \in [m]$ in the bottom-right part of C. Finally, let the left part of C be such that it satisfies the conditions of Scarf's Lemma. The constant vector, $b \in R_+^{n+m}$ is given by the bounds and capacities, so let $b_i = b(v_i)$ for $i \in [n]$ and $b_{n+j} = c(e_j)$ for $j \in [m]$.

We shall prove that the fractional core element x, obtained by Scarf's Lemma, gives a stable allocation, x^e by simply taking the last m coordinates of x. Here, x_j^e is equal to $x^e(e_j)$ that is the weight of the edge e_j (or equivalently, this is the intensity at which the corresponding activity is performed). If $\bar{x}^v$ and $\bar{x}^e$ are the vectors obtained by taking the $[1, \ldots, n]$ and $[n+1, \ldots, n+m]$ coordinates of x, then these vectors correspond to the remaining weights of the vertices and edges (or the remaining intensities of the players and the activities), respectively.

It is straightforward to see that x^e is an allocation by $Ax = b$, since the first n equations preserve the $\sum_{v \in h} x^e(h) \le b(v)$ condition for every vertex v, and the last m equations preserve $x^e(e_j) \le c(e_j)$ for every edge e_j.

To prove stability, let us consider an unsaturated edge e_j and let us suppose that the corresponding dominating row by the Lemma has index i. First we show, that $i \in [n]$. From $Ax = b$, $\bar{x}^e(e_k) + x^e(e_k) = c(e_k)$ for every edge e_k. Since $x^e(e_j) < c(e_j)$, then $\bar{x}^e(e_j) > 0$, thus the assumptions on C imply that $i \ne n+j$, for any other $i \in [n+m] \setminus [n]$ the contradiction is trivial. If $i \in [n]$ then e_j is dominated at v_i, since $\bar{x}^v(v_i) = 0$ by the assumptions on C, and the $Ax = b$ condition for the i-th row together with the statement of the Lemma imply that $\sum_{v_i \in e_k, e_j \le_{v_i} e_k} x^e(e_k) = b(v_i)$. ∎

We illustrate the notion of stable allocation on a stable matching problem in Example 3 of the next section, where we also describe the matrices A and C of the Lemma.

3.3 Stable allocations with contributions

It is possible that the contributions of the participants are not equal in a cooperation. Imagine an internal project of a company where the hours allocated to the employees involved can be different (e.g., a quality manager may have less work load assigned than an engineer in terms of working hours). We can model this extension easily for a stable allocation problem, that we call *stable allocation problem with contributions*. We only need to use *contribution vectors* here rather than membership vectors, as for the simple stable allocation problem.

In addition to the definitions of the stable allocation problem, we introduce a contribution vector $r_e : V(H) \to \mathbb{R}_+$ for each edge e of the hypergraph. We assume that $v \in e$ if and only if $r_e(v) > 0$, that is when agent v contributes to contract e. A nonnegative function x on the edges, is an *allocation* if $x(e) \le c(e)$ for every edge e and $\sum_{v \in e} x(e)r_e(v) \le b(v)$ for every vertex v. An allocation is *stable* if every *unsaturated* edge e (i.e., every edge e with $x(e) < c(e)$) contains a vertex v such that $\sum_{v \in f, e \le_v f} x(f)r_f(v) = b(v)$. If every bound, capacity and contribution is integral then we refer to this problem as the *integral stable allocation problem with contributions*.

Theorem 13 *Every stable allocation problem with contributions is solvable.*

The same proof technique works here as for Theorem 12 with the only difference that here we shall use contribution vectors instead of membership vectors when defining matrix A in Scarf's Lemma. Therefore the proof is omitted.

If we restrict the solution of a stable allocation problem to take any contract with either full or no intensity, i.e., $x(e) = c(e)$ or $x(e) = 0$, then we can create a corresponding stable allocation problem, where each contract has capacity one, and the allocation is binary. This can be done simply by multiplying each agents contribution in contract e by $c(e)$. We call the latter problem, binary stable allocation problem with unit edge-capacities, as *stable schedule problem*.

To illustrate the meaning of this model, let us describe a motivating example. In a department there are different activities, involving students and teachers. A lecture is an activity that requires, say, 6 hours from the professor and 4 hours from each student per week. The supervision of a PhD student is a 2-members activity that typically involves 4 hours supervision by the professor and 40 hours work by the student. A seminar is a one-hour activity by the regular participants and probably 3 hours activity by the organiser. A research project is a more complex activity that involves both professors and students with different contributions and sub-activities. In this situation a schedule is a solution with some activities performed with different contributions where no individual is over-scheduled (i.e. her total contribution in the activities performed is less than or equal to her capacity), and there is no available activity that could be realised, by possibly cancelling some performed activity, such that each individual involved would be strictly better off.

Another interesting real-life problem that can be described by the stable schedule problem is the project contribution system applied by the Hungarian Scientific Research Fund and the French National Research Agency. In Hungary, the research proposal should contain a declaration by each participants on her contribution time in the project, any number between 0 and 1. A participant may be involved in any number of projects, but the total contribution of a full-time researcher may not exceed 1, and cannot be more than 0.7 in case of lecturers. In France and Portugal, the systems are similar, the participants

should declare what percentage of their research time they are willing to spend on each particular project they are involved in. In the planning phase of these grant applications, a researcher might reject to participate in a proposal if she has already used her total contribution in other projects that she prefers.

At the end of the next section we will also describe the hospital residents problem with couples, that can be modelled as a stable schedule problem, and the Scarf algorithm appears to be an efficient heuristic for finding stable solutions in a centralised matching scheme.

4 Matching games

Matching games can be defined as partitioning NTU-games, where the cardinality of each basic coalition is at most 2. For simplicity, in this section we suppose that no player is indifferent between two efficient utility vectors, so their preferences over the joint activities are strict. If a matching game is finitely generated, then the problem of finding an outcome that is in the core is equivalent to a SM for a graphic preference system, where the edges of the graph correspond to efficient utility vectors (and to joint activities).

4.1 Stable matching problem

If the graph of a matching game is simple (i.e, if it contains no multiple edges and loops) then the problem of finding a core-partition for the resulting CFG is called *stable roommates problem*. Otherwise, if the graph has multiple edges then we may refer to SM as *stable roommates problem with multiple activities*.

Let us suppose the set of players N can be divided into two parts, say M and W, such that every two-member basic coalition contains one member from each side (so if $\{m, w\} \in \mathcal{B}$ then $m \in M$ and $w \in W$). In this case, we get a *two-sided matching game* (in the general nonbipartite case the matching game is called *one-sided*).

If a two-sided matching game is finitely generated then the corresponding graphic representation of the SM is bipartite. For bipartite graphs, the following Proposition is well-known.

Proposition 14 *Every bipartite graph is normal.*

Proposition 14 and Theorem 10 imply the following result.

Theorem 15 (Kaneko [25], Crawford-Knoer [17]) *Every finitely generated two-sided matching game has a nonempty core.*

Theorem 15 was proved for every two-sided matching game by Crawford and Knoer [17] and Kaneko [25]. In the latter paper this game was called *central assignment game*. For the corresponding CFG-s, called *stable marriage problems*, this result was proved by Gale and Shapley [22]. The related TU-setting, the *assignment game*, was studied by Shapley and Shubik [42].

A one-sided matching game can have an empty core, even for a CFG, as Gale and Shapley [22] illustrated with an example. However the half-integer property of the fractional matching polytope implies the existence of stable half-integer solutions. In particular, a *half-core element* is a fractional core element x with the half-integer property, i.e. $x_i \in \{0, \frac{1}{2}, 1\}$. The following statement is due to Balinski [6].

Theorem 16 (Balinski [6]) *The fractional matching polytope for every graph has only half-integer extremal points.*

As Aharoni and Fleiner [1] showed, Theorem 16 and Corollary 7 imply that for every matching game there exists a half-core element.

Theorem 17 (Tan [43]) *If a matching game is finitely generated then it always has a half-core element.*

For CFG-s, the fact that for every instance of SM there exists a stable half-matching was proved by Tan [43]. Finally we note that an easy consequence of Theorem 16 and Lemma 11 is that for every finitely generated matching game, there always exists an integer 2^N-core element.

4.2 Stable allocation problem for graphs

The stable allocation problem was introduced by Baïou and Balinski [5] for bipartite graphs. The integer version, where the allocation x is required to be integer on every edge for integer bounds and capacities, was called the *stable schedule problem* by Alkan and Gale [2]. However, they considered a more general model, the case of so-called *substitutable* preferences. We will discuss the differences briefly in the last Section.

Biró and Fleiner [8] generalised the algorithm of Baïou and Balinski [5] for nonbipartite graphs, resulting in a weakly polynomial algorithm that finds a half-integral stable allocation for any given instance of integral stable allocation problem. Dean and Munshi [20] strengthened this result by giving a strongly polynomial algorithm for the same problem. However, as we will prove below, the existence of a stable half-integer allocation is an easy consequence of Theorem 12.

Theorem 18 *For every integral stable allocation problem in a graph there exists a half-integral stable allocation. If the graph is bipartite, then every integral stable allocation problem is solvable.*

Proof: Suppose that we have a stable allocation x that has some weights that are not half-integers. We create another stable allocation x' with half-integer weights as follows. If $x(e)$ is not integer then let v be the vertex where e is dominated. Since $b(v)$ is integer, there must be another edge, f that is incident with v and has non-integer weight. Moreover, f cannot be dominated at v. By this argument, it can be verified that the edges with non-integer weights form vertex-disjoint cycles, moreover, in each such a cycle the fractional parts of the weights are ε and $1 - \varepsilon$ alternately. If a cycle is odd, then ε must be $\frac{1}{2}$. If a cycle is even, then ε can be modified to be 0 (or 1) in such a way that the obtained allocation x' remains stable and has only half-integer weights.

If the graph is bipartite, thus has no odd cycle, then x' has only integer weights, so x' is an integral stable allocation . ∎

We note that the scaling property holds also for the stable allocation problem. By this, if every bound and capacity is even, then the existence of an integral stable allocation is straightforward by Theorem 18.

Finally, we illustrate the integral stable allocation problem for a graph and show how a half-integer stable allocation can be obtained by the Scarf algorithm.

Example 3.

We are given 6 agents and 7 possible pairwise activities. The bounds of the agents are: $b(a_1) = 4$, $b(a_2) = 1$, $b(a_3) = 1$, $b(a_4) = 2$, $b(a_5) = 2$ and $b(a_6) = 4$. The capacities of the possible activities are: $c(\{a_1, a_2\}) = 1$, $c(\{a_1, a_3\}) = 2$, $c(\{a_1, a_5\}) = 2$, $c(\{a_1, a_6\}) = 3$, $c(\{a_2, a_5\}) = 1$, $c(\{a_3, a_6\}) = 2$ and $c(\{a_4, a_6\}) = 1$. The preference lists of the agents are

the following:
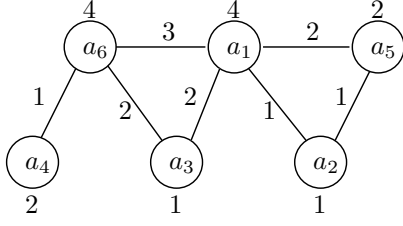
Players	Preference lists
a_1 :	$a_6\ a_3\ a_2\ a_5$
a_2 :	$a_5\ a_1$
a_3 :	$a_6\ a_1$
a_4 :	a_6
a_5 :	$a_1\ a_2$
a_6 :	$a_4\ a_3\ a_1$

The extended membership-matrix A is:

$$
A =
\left[
\begin{array}{cccccc|ccccccc|ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{array}
\right]
$$

Matrix C, that represents the preference of the players, can be generated as follows:

$$
C =
\left[
\begin{array}{cccccc|ccccccc|ccccccc}
0 & 24 & 23 & 22 & 21 & 20 & 19 & 18 & 17 & 16 & 15 & 14 & 13 & 3 & 4 & 2 & 5 & 8 & 7 & 6 \\
25 & 0 & 23 & 22 & 21 & 20 & 19 & 18 & 17 & 16 & 15 & 14 & 13 & 4 & 11 & 10 & 9 & 5 & 7 & 6 \\
25 & 24 & 0 & 22 & 21 & 20 & 19 & 18 & 17 & 16 & 15 & 14 & 13 & 12 & 4 & 10 & 9 & 8 & 5 & 6 \\
25 & 24 & 23 & 0 & 21 & 20 & 19 & 18 & 17 & 16 & 15 & 14 & 13 & 12 & 11 & 10 & 9 & 8 & 7 & 5 \\
25 & 24 & 23 & 22 & 0 & 20 & 19 & 18 & 17 & 16 & 15 & 14 & 13 & 12 & 11 & 5 & 9 & 4 & 7 & 6 \\
25 & 24 & 23 & 22 & 21 & 0 & 19 & 18 & 17 & 16 & 15 & 14 & 13 & 12 & 11 & 10 & 3 & 8 & 4 & 5 \\
\hline
25 & 24 & 23 & 22 & 21 & 20 & 0 & 18 & 17 & 16 & 15 & 14 & 13 & 1 & 11 & 10 & 9 & 8 & 7 & 6 \\
25 & 24 & 23 & 22 & 21 & 20 & 19 & 0 & 17 & 16 & 15 & 14 & 13 & 12 & 1 & 10 & 9 & 8 & 7 & 6 \\
25 & 24 & 23 & 22 & 21 & 20 & 19 & 18 & 0 & 16 & 15 & 14 & 13 & 12 & 11 & 1 & 9 & 8 & 7 & 6 \\
25 & 24 & 23 & 22 & 21 & 20 & 19 & 18 & 17 & 0 & 15 & 14 & 13 & 12 & 11 & 10 & 1 & 8 & 7 & 6 \\
25 & 24 & 23 & 22 & 21 & 20 & 19 & 18 & 17 & 16 & 0 & 14 & 13 & 12 & 11 & 10 & 9 & 1 & 7 & 6 \\
25 & 24 & 23 & 22 & 21 & 20 & 19 & 18 & 17 & 16 & 15 & 0 & 13 & 12 & 11 & 10 & 9 & 8 & 1 & 6 \\
25 & 24 & 23 & 22 & 21 & 20 & 19 & 18 & 17 & 16 & 15 & 14 & 0 & 12 & 11 & 10 & 9 & 8 & 7 & 1 \\
\end{array}
\right]
$$

In the algorithm, we used the following perturbation on b: we set $\tilde{b}_i = b_i + \varepsilon_i =$

$b_i + 1/p_{101-i}$, where p_i is the i-th prime number. By rounding the output $\tilde{x}$ of the Scarf algorithm, we get the following fractional b-core element x, and stable allocation x^e:
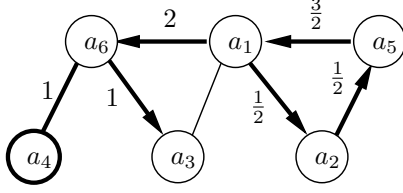
$$x = [\bar{x}^v | \bar{x}^e | x^e], \text{ where } x^e = [\frac{1}{2}, 0, \frac{3}{2}, 2, \frac{1}{2}, 1, 1]$$

$$\bar{x}^v = [0, 0, 0, 1, 0, 0] \text{ and } \bar{x}^e = [\frac{1}{2}, 2, \frac{1}{2}, 1, \frac{1}{2}, 1, 0]$$

In the above figure, we oriented each unsaturated edge to its endvertex, where it is dominated.

4.3   Stable allocation problems with contributions

Dean and Swar [19] showed that the extension of the Dean-Munshi [20] algorithm can solve the stable allocation problem with contributions for bipartite graphs, called as the *generalized stable allocation problem*, in strongly polynomial time. A particular integral version was studied by Dean et al. [18] and by Manlove and McDermid [33]. Here, the items on one side, say jobs, have sizes and they should be allocated with machines with capacities. This is a particular setting since each job has the same size with regard to all machines. Dean et al. [18] called the problem *unsplittable stable marriage problem* and they showed that a stable solution always exists if the quotas can be violated by the largest size of any job. Manlove and McDermid defined the *stable marriage problem with sizes* in a similar way, but they considered the capacities of the machines to be rigid. They showed that the problem of finding a stable solution is NP-hard, even if jobs have sizes of 1 and 2 only, a case which is also known as a special setting of the hospital residents problem with couples (see more details about this problem in the next section).

4.4   The hospitals residents problem with couples

Strictly speaking this problem does not fell into the category of matching games, as it concerns coalitions of size three, involving a couple (formed by two residents) and a pair of hospitals where the couple is seeking positions. The US resident allocation program, now called National Resident Matching Program, has been using the Gale-Shapley algorithm [22] to allocate junior doctors to hospitals since 1952, see details in [36]. However, the coordinators of the scheme realised that married couples tend not to participate in the program, typically in order to avoid getting assigned to hospital geographically far from each other. Therefore the scheme has been redesigned in the late nineties by Roth and Peranson, as described in [37].

Unfortunately stable solution may not exist any more in this setting [36], and the problem of finding one is NP-hard [35]. Therefore heuristics are used in all the real applications. Biró et al. [11] compared some heuristics used in the American and Scottish applications, called NRMP and SFAS respectively. In our follow-up paper [10] we showed how we can reduce this problem to a stable schedule problem (i.e. binary integral stable allocation problem with contributions). Therefore, we can use the Scarf algorithm as a new heuristic to find stable solutions.

To illustrate the usage of the Scarf algorithm, here we include the description of an experiment described in our follow-up paper [10]. We implemented and tested this method and compared its performance with the two algorithms that are used in NRMP [37] and in SFAS [11]. The results of a particular experiment for same random samples used in [11] are shown in Table 1.

In this computational experiment, there were 500 residents and the proportion of

	Number of couples										
Algorithm	12	25	50	75	100	125	150	175	200	225	250
R–P (1999)	952	897	701	547	395	277	170	83	41	9	3
B–I–S (2011)	976	958	911	870	811	752	682	546	281	71	10
Scarf by B–F–I (2013)	895	813	649	532	426	356	316	261	202	174	158

Table 1: Randomly created matching markets with couples for 500 residents.

couples varied from 5% to 100%. For each proportion of couples, 1000 random instances were generated and we counted how many instances each variant could solve. The table shows that the heuristics due to [11] obtained a much better success ratio than the Roth-Peranson heuristics, especially for high proportions of couples. But surprisingly, the Scarf-algorithm was even much better than the others when (almost) all the applicants form couples. Note that a similar situation occurs in another applications, the Hungarian higher education matching scheme [46], where applicants may apply for pairs of teacher's studies (e.g. to become a teacher in maths and physics). More details on this experiment can be found in [10].

Finally, we note that in the case when each hospital has one place only, we get an NTU-game (which is almost a coalition formation game, except that a couple can apply to a pair of hospitals in two possible ways). If the hospitals have capacities, but no couple may apply for a pair of positions in the same hospital, then the problem can be reduced to the integral stable allocation problem, where the stable matchings corresponds to the integral stable allocations. However, if a couple can submit combined applications for a pair of positions in the same hospital, then we shall use contributions. A combined application can be described as a contract where the contributions of the couple is 1 and the contribution of the hospital is 2. As we prove in [10], this most general version of the hospital residents problem with couples can be reduced to the stable schedule problem, although here the correspondence hold only in one direction, a stable schedule translates to a stable matching of the original problem, but not the other way. Nevertheless, the Scarf algorithm can be used as a heuristic to solve this problem as well, whenever the algorithm returns an integral solution, we found a stable matching.

5 Further notes

The meaning of stability. In our stability definition we implicitly suppose that the deviating coalition may keep all of their existing cooperations with both the members of the deviating coalition and the outsiders. This restriction is not always realistic, and therefore considering alternative definitions of blocking coalitions and corresponding alternative stability concepts is a natural question to investigate. In fact, this approach has been already studied for TU-games with overlapping coalitions [16] (subsection 5.2), where three core definitions have been provided. As it is described in detail in [16] (subsection 8.2), the concept of fuzzy games (see e.g. Aubin [4]) is similar in nature to TU games with overlapping coalitions, but not quite the same.

In our models, the agents have linear preferences over the possible contracts, which means that their preferences over the set of contracts are responsive and thus also substitutable (see e.g. [36]). This is obviously not always the case in real markets, where complementarities can exist. Furthermore, the choice function of the agents over the sets of fractional contracts is also very restricted by our definition. For instance, our stability condition implies that if a resident was asked to choose between two half-time jobs and one full time job at the worse hospital then she would always choose the two half-time

jobs. This is not always a realistic assumption in practical applications.

Guarantees for solvability. The original goal of Scarf [40] was to give a necessary condition for the nonemptiness of the core for general NTU-games (and this condition was the balancedness of the game). As we described in Section 3, if the coalition structure of an NTU-game can be represented with a normal hypergraph then the core of the game is always nonempty (regardless of the players' preferences). The bipartite graph is an easy example for normal hypergraphs, and so every two-sided matching game has nonempty core. But what other games have this property? Our claim is that certain network games also have a coalition structure where the underlying hypergraph is normal. Another way of guaranteing solvability of NTU-games can be to make assumptions on the players' preferences.

Understanding Scarf's results. Scarf proved his Lemma in an algorithmic way. Is there some deeper reasons for the correctness of the Lemma (and a more general interpretation of the algorithm)? What is the relation of this result to other fundamental theorems, such as the Sperner Theorem? There are some recent papers [1, 28, 21] attempting to answer this question, but yet, there still are many open problems regarding this issue.

Also, it would be interesting to know how the Scarf algorithm works for special games. For instance, does the Scarf algorithm run in polynomial time for matching games?

At the beginning of the Scarf algorithm we perturb matrix C and vector b. By doing so, the steps in the algorithm and the final output are fully determined. Can we output every core element of a given NTU-game by using a suitable perturbation? How does the perturbation effect the solution, we obtain by the algorithm? For stable marriage problem we observed that using small epsilons for men and larger epsilons for women we always get the man-optimal stable matching. Can we output each stable matching by a suitable perturbation? Is that true that the smaller epsilon we give to a woman the better partner she is going to get in the resulting stable matching? Is a similar statement true for general NTU-games?

Further theoretical applications. In this paper we demonstrated the usage of Scarf Lemma in stable matching theory. However, there are still many related open questions. For instance, the stable matching problem with contributions has not been studied yet to the best of our knowledge (only some special cases, such as the hospitals residents problem with couples). A natural question is whether can we find a stable integral allocation in polynomial time for two-sided or one-sided matching games with contributions?

Furthermore, there can be other family of NTU-games where the concept of stable allocations may be worth to study, e.g. network games.

Practical applications. The problem of finding a core element for an NTU-game can be NP-hard for some family of games or stable matching problems, but we may well need to solve such problems in practical applications. In section 4, we illustrated how the Scarf algorithm can be used as a heuristic to solve the hospitals residents problem (detailed descriptions on our experiments are given in our follow-up paper [10]). This suggests that perhaps the same technique can be used in other applications as well.

A further example for practical applications, where the underlying problem is NP-hard and where the Scarf algorithm could be potentially used as a heuristic, is the kidney exchange problem with bounded length exchange cycles. Centralised programs for organising exchanges among incompatible patient-donor pairs have been established in several countries, such as the US, UK, Netherlands, etc. Stable exchanges as solution concept for this application have been suggested by Roth et al. [39]. However, as Biró and McDermid showed [13], if the length of the exchange cycles is bounded to be at most three (as it is the case in all of the above mentioned applications), a stable solution might not exist and the problem of finding a stable exchange is NP-complete.

Note that the usage of the Scarf algorithm is similar to the usage of LP-relaxations in solving integer programming problems. The latter techniques proved to be very efficient in practice in the last decades, and so we believe that the Scarf algorithm will also be found useful with regard to this aspect, too. As an example, we can mention a very recent paper [34] where the authors prove that in certain settings the existence of a stable integral solution can be guaranteed for slightly adjusted quotas by rounding a fractional stable solution of the original instance.

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