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Semi-inner products and the concept of semi-polarity

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Abstract The lack of an inner product structure in general Banach spaces yields the motivation to introduce a semi-inner product with a more general axiom system, one missing the requirement for symmetry, than the one determing a Hilbert space. We use the semi-inner product on a finite dimensional real Banach space $(\mathbb{X}, \|\cdot\|)$ to define and investigate three concepts. First, we generalize that of *antinorms*, already defined in Minkowski planes, for even dimensional spaces. Second, we introduce *normality maps*, which leads us, in the third part, to the study of *semi-polarity*, a variant of the notion of polarity which makes use of the underlying semi-inner product.

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1 Introduction

Motivated by the lack of inner product in general Banach spaces, Lumer [16] defined semi-inner product spaces, which enabled him to adapt Hilbert space arguments to the theory of Banach spaces. From the viewpoint of functional analysis, real (and complex) semi-inner product spaces have been in the mainstream of scientific research; for references in this regard see the book [7]. Our aim is to examine them for purely geometric purposes. We start with some preliminary definitions.

Let $\mathbb{X}$ be a real vector space. A *semi-inner product* on $\mathbb{X}$ is a real function $[\cdot,\cdot]$ on $\mathbb{X}\times\mathbb{X}$ satisfying the following properties for any $x,y,z\in\mathbb{X}$.

(i) $[x+y,z]=[x,z]+[y,z]$, $[\lambda x,y]=\lambda[x,y]$ for all real λ,
(ii) $[x,x]>0$, when $x\neq 0$,
(iii) $[x,y]^2\leq[x,x][y,y]$.

A real vector space $\mathbb{X}$, equipped with a semi-inner product, is said to be a (real) *semi-inner product space*. It is well-known that any semi-inner product $[\cdot,\cdot]$ on $\mathbb{X}$ induces a norm, by setting $\|x\|=\sqrt{[x,x]}$. Conversely, every *Banach space* $(\mathbb{X},\|\cdot\|)$ can be transformed into a semi-inner product space (see [9, Theorem 1]) in the following way.

Let $\mathbb{S}:=\{x\in\mathbb{X}:\|x\|=1\}$ be the *unit sphere* of $(\mathbb{X},\|\cdot\|)$, and $\mathbb{X}^*$ be the *dual space of* $\mathbb{X}$. On $\mathbb{X}^*$ one can define a norm $\|\cdot\|^*$, called the *dual norm*, in the usual way, i.e.,

$$\|f\|^*:=\sup\{f(x):\|x\|=1\}\quad\text{for}\quad f\in\mathbb{X}^*. \tag{1}$$

If $\mathbb{S}^*$ is the unit sphere of $(\mathbb{X}^*,\|\cdot\|^*)$, then for any $x\in\mathbb{S}$ there exists, by the Hahn-Banach Theorem, at least one functional (exactly one functional if the norm is smooth) $f_x\in\mathbb{S}^*$ with $f_x(x)=1$. For any $\lambda x\in\mathbb{X}$, where $x\in\mathbb{S}$, we choose $f_{\lambda x}\in\mathbb{X}^*$ such that $f_{\lambda x}=\lambda f_x$. Then a semi-inner product $[\cdot,\cdot]$ is defined on $\mathbb{X}$ by

$$[x,y]:=f_y(x). \tag{2}$$

The aim of the paper is to investigate three geometric concepts related to semi-inner products. After collecting the main tools of our examination in Section 2, in Section 3, by means of a symplectic form defined on the space, we introduce the *antinorm* of an even dimensional real Banach space and examine its properties. We remark that for normed planes, this notion was studied, e.g. in [19] and [4]. In Section 4, by means of antinorms, we define and examine *normality maps*. In Section 5, based on the semi-inner product structure of $\mathbb{X}$, we define the notion of *semi-polars* in $\mathbb{X}$ and generalize the properties of polars, known in Euclidean spaces. Finally, in Section 6 we collect our questions and additional remarks.

We note that, whereas in functional analysis the polar of a set in a space $\mathbb{X}$ is a subset of the dual space $\mathbb{X}^*$ (cf. [1]), in geometry polarity is regarded as a correspondence between sets of the same Euclidean space, where linear functionals in $\mathbb{X}^*$ are identified with points in $\mathbb{X}$ via the inner product of the space. Our aim is to define a variant of polarity providing a correspondence between subsets of the same normed space, based on the semi-inner product defined by the norm.

2 Preliminaries

Let $(\mathbb{X}, \|\cdot\|)$ be a *normed space* (i.e., a finite dimensional real Banach space) with *origin* o and *unit ball* $\mathbf{B} = \{x \in \mathbb{X} : \|x\| \le 1\}$, which is a compact, convex subset of $\mathbb{X}$ with boundary $\mathbb{S}$, centered at its interior point o. Let $\mathbf{B}_{\mathbb{E}}$ and $\mathbb{S}_{\mathbb{E}}$ be the unit ball and sphere, respectively, with respect to a *Euclidean norm*, i.e., a norm induced by an inner product on $\mathbb{X}$. A vector $x \neq 0$ is *normal* to a vector $y \neq 0$, denoted by $x \dashv y$, if, for any real λ, the inequality $\|x\| \le \|x+\lambda y\|$ holds; see, e.g., [20, § 6].

For a convex body K, i.e., a compact, convex subset of $\mathbb{X}$ with nonempty interior and $u \neq o$, let $h(K,u)$ be the *support function* in direction u. The *support function of K with respect to the norm* $\|\cdot\|$ is defined by $h_B(K,u) = \dfrac{h(K,u)}{h(\mathbf{B},u)}$. Alternatively, for every $u \neq o$ this normed support function $h_B(K,u)$ can be viewed as the signed distance with respect to $\|\cdot\|$ from the origin o to a supporting hyperplane H of K such that the outer normal of H with respect to K yields a positive inner product with u; see, e.g., [5] or [18, § 2]. This means that the normed support function $h_B(K,u)$ of K can be expressed as $\sup\{[x,u] : x \in K\}$.

We denote the family of all convex bodies, containing the origin o as an interior point, by $\mathfrak{X}_o$. For $K \in \mathfrak{X}_o$, let $g(K,\cdot)$ be the *gauge function* of K, i.e.,

$$g(K,x) := \min\{\lambda \ge 0 : x \in \lambda K\} \quad \text{for} \quad x \in \mathbb{X}.$$

Note that $g(\mathbf{B},x) = \|x\|$ for every $x \in \mathbb{X}$.

From now on, let $(\mathbb{X}, \|\cdot\|)$ be a smooth and strictly convex normed space. We denote by $[\cdot,\cdot]$ the semi-inner product induced by the norm $\|\cdot\|$. If $(\mathbb{X}, \|\cdot\|)$ is an inner product space, i.e., the corresponding semi-inner product is, in addition, symmetric, then we denote this product by $[\cdot,\cdot]_{\mathbb{E}}$. The following properties are proved in [9] (see also [14], [13, § 2.4], and [15]).

(iv) The homogeneity property: $[x,\lambda y] = \lambda[x,y]$ for all $x, y \in \mathbb{X}$ and all real λ.
(v) $[y,x] = 0 \Longleftrightarrow \|x\| \le \|x+\lambda y\|$ for all $\lambda \in \mathbb{R}$.
(vi) The generalized Riesz-Fischer Representation Theorem: To every linear functional $f \in \mathbb{X}^*$ there exists a unique vector $y \in \mathbb{X}$ such that $f(x) = [x,y]$ for all $x \in \mathbb{X}$. Then $[x,y] = [x,z]$ for all $x \in \mathbb{X}$ if and only if $y = z$.
(vii) The dual vector space $\mathbb{X}^*$ is a semi-inner product space by $[f_x, f_y]^* = [y,x]$.

Remark 21 *Property (v) can be written in the form*

(v′) $x \neq 0$, $y \neq 0$ *and* $[y,x] = 0 \Longleftrightarrow x \dashv y$.

Remark 22 *By Property (vi), we have a one-to-one map $F : \mathbb{X} \to \mathbb{X}^*$ with $F : x \mapsto f_x$, where f_x is determined by (2). Property (vii) implies that F is norm-preserving.*

Proposition 21 *The norm defined by (1) is induced by the semi-inner product* $[\cdot,\cdot]^*$ *on* $\mathbb{X}^*$ *defined by (vii).*

Proof First, observe that

$$\sqrt{[f_x, f_x]^*} = \sqrt{[x,x]} = \|x\|, \tag{3}$$

and that $\sup\{[y,o] : \|y\| = 1\} = 0 = \|o\|$.

Let $f_x \in \mathbb{X}^*$. Then $\sup\{f_x(y) : \|y\| = 1\} = \sup\{[y,x] : \|y\| = 1\}$. Since $[y,x]^2 \leq [y,y] \cdot [x,x] = \|x\|^2$ for all y with $\|y\| = 1$, we obtain

$$\sup\{f_x(y) : \|y\| = 1\} \leq \|x\|. \tag{4}$$

On the other hand, if $x \neq o$, then $\sup\{[y,x] : \|y\| = 1\} \geq \left[\frac{1}{\|x\|}x, x\right] = \frac{1}{\|x\|}[x,x] = \|x\|$, which, together with (4), yields $\sup\{f_x(y) : \|y\| = 1\} = \|x\|$.

Proposition 22 *For the map* F *and any* $x, y \in \mathbb{X}$, $\lambda, \mu \in \mathbb{R}$, *we have*

$$\|F(\lambda x + \mu y)\|^* \leq |\lambda|\|Fx\|^* + |\mu|\|Fy\|^*.$$

Proof The definition of F implies

$$\|F(\lambda x+\mu y)\|^* = \|f_{\lambda x+\mu y}\|^* = \|\lambda x+\mu y\| \leq \|\lambda x\|+\|\mu y\| = |\lambda|\|Fx\|^*+|\mu|\|Fy\|^*.$$

3 Antinorms

In this and the next section, we assume that $\mathbb{X}$ is even dimensional. Our main goal is to generalize the notion of antinorm for even dimensional normed spaces, defined in [19] for normed planes, and examine which of their properties remain true.

Let $\langle\cdot,\cdot\rangle$ be a (nondegenerate) bilinear symplectic form on $\mathbb{X}$; that is, a bilinear form satisfying $\langle x,y\rangle = -\langle y,x\rangle$ for all $x, y \in \mathbb{X}$, and the property that $\langle x,y\rangle = 0$ for all $y \in \mathbb{X}$ yields that $x = o$. Then the vector space $\mathbb{X}$ and its dual space $\mathbb{X}^*$ can be identified via

$$G : \begin{matrix} \mathbb{X} \to \mathbb{X}^* \\ x \mapsto g_x \end{matrix}, \quad \text{where} \quad g_x(y) := \langle y, x\rangle; \tag{5}$$

see [19, § 2.3]. It is easy to see that G is an isomorphism.

We note that if $\dim \mathbb{X} = 2$, then every symplectic form $\langle x,y\rangle$ is a scalar multiple of the 2×2 determinant, or geometrically, up to multiplication by a constant, is the signed area of the parallelogram with vertices $o, x, x+y, y$. On the other hand, it is well-known (cf. [6] or [2]) that for spaces of dimension greater than two, there are many (even though symplectically isomorphic) symplectic forms which are not scalar multiples of one another.

From now on we fix a symplectic form on $\mathbb{X}$.

Definition 31 *The* antinorm *of* $(\mathbb{X}, \|\cdot\|)$*, with respect to the symplectic form* $\langle\cdot,\cdot\rangle$*, is defined, for all* $x \in \mathbb{X}$ *as*

$$\|x\|_a := \|Gx\|^* = \|g_x\|^* = \sup\{\langle y, x\rangle : \|y\| = 1\}. \tag{6}$$

For this norm, we denote the map defined in Remark 22 by F_a*, and the unit ball/sphere of the antinorm by* $\mathbf{B}_a$ *and* $\mathbb{S}_a$*, respectively.*

We note that, as it can be simply checked, the antinorm is indeed a norm defined on $\mathbb{X}$. Nevertheless, unlike in the plane, the antinorm relies very much on the symplectic form, i.e., different forms yield different antinorms.

The following theorem was proven in [19] for normed planes. To formulate it, for any $\phi \in \mathbb{X}^*$, we write $x \perp_{\|.\|} \phi$, if $|\phi(x)| = \|\phi\|^*$; that is, if the supporting hyperplane of $\|x\|\mathbf{B}$ at x is a level surface of ϕ.

Theorem 31 *Let* $\|\cdot\|_{a,a}$ *denote the antinorm of* $(\mathbb{X}, \|\cdot\|_a)$ *with respect to the symplectic form* $\langle\cdot,\cdot,\rangle$*, where* $\|\cdot\|_a$ *is defined with respect to the same form. Then, for any* $x \in \mathbb{X}$*, we have* $\|x\|_{a,a} = \|x\|$*. Furthermore* $x \perp_{\|.\|} Gy$ *if, and only if,* $y \perp_{\|.\|_a} Gx$*.*

Proof By definition,

$$\|x\|_{a,a} = \sup\{\langle y, x\rangle : \|y\|_a = 1\}.$$

Observe that (6) yields that $|\langle x, y\rangle| \leq \|x\| \cdot \|y\|_a$ for any $x, y \in \mathbb{X}$. Thus, it follows that $\|x\|_{a,a} \leq \|x\|$. On the other hand, let $y = G^{-1}Fx$. By definition, $Fx = f_x$ is the linear functional with the property that $f_x(x) = \|x\|$ and for any $z \in \mathbf{B}$ we have $|f_x(z)| \leq 1$, which yield that $\|f_x\|^* = 1$. Thus, if we set $g_y = Gy$, then $f_x(z) = g_y(z) = \langle z, y\rangle \leq 1$ for any $z \in \mathbf{B}$, and $g_y\left(\frac{x}{\|x\|}\right) = 1$, implying that $\|y\|_a = 1$ and $\langle x, y\rangle = \|x\|$. Hence, by definition, $\|x\|_{a,a} \geq \|x\|$, and the first statement follows.

Now, consider some $x, y \in \mathbb{X}$, and assume that $|\langle x, y\rangle| = \|x\| \cdot \|y\|_a$. By the definition of antinorm, this is equivalent to saying that the function $|\langle ., y\rangle|$ is maximized on $\|x\|\mathbf{B}$ at x. In other words, the supporting hyperplane of $\|x\|\mathbf{B}$ at x is a level surface of the linear functional $Gy = \langle\cdot, y\rangle$. On the other hand, since $\|x\|_{a,a} = \|x\|$ and $|\langle ., .\rangle|$ is symmetric, we have that $Gx = |\langle ., x\rangle|$ is maximized on $\|y\|_a\mathbf{B}_a$ at y. Thus, we have the following.

$$x \perp_{\|.\|} Gy \quad \Longleftrightarrow \quad |\langle x, y\rangle| = \|x\| \cdot \|y\|_a \quad \Longleftrightarrow \quad y \perp_{\|.\|_a} Gx.$$

The normality relation defined at the beginning of Section 2 is not symmetric. Nevertheless, it was shown in [19, § 3] that for any normed plane $(\mathbb{X}, \|\cdot\|)$, for any $x, y \in \mathbb{X}$, x is normal to y with respect to $\|\cdot\|$ if, and only if, y is normal to x with respect to $\|\cdot\|_a$, which we denote $y \dashv_a x$. We show that this property cannot be generalized for higher dimensions, in a strong sense.

Theorem 32 *Let* $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *be two norms defined on the real linear space* $\mathbb{X}$*, where* $\dim \mathbb{X} > 2$*. For* $i = 1, 2$*, let* $\mathbf{B}_i$ *and* $\dashv_i$ *denote the unit ball and the normality relation of the norm* $\|\cdot\|_i$*. Then the relations* $x \dashv_1 y$ *and* $y \dashv_2 x$ *are equivalent for all* $x, y \in \mathbb{X}$ *if, and only if* $\mathbf{B}_1$ *and* $\mathbf{B}_2$ *are homothetic ellipsoids.*

Proof Clearly, if $\mathbf{B}_1$ and $\mathbf{B}_2$ are homothetic ellipsoids, then $x \dashv_1 y$ and $y \dashv_2 x$ are the same relation.

Observe that the condition $x \dashv_1 y$ geometrically means that y is parallel to the supporting hyperplane of $\|x\|_1 \mathbf{B}_1$ at x. In particular, it follows that the set $\{y \in \mathbb{X} : x \dashv_1 y\}$ is a hyperplane for every $x \neq o$. On the other hand, the set $\{y \in \mathbb{X} : y \dashv_2 x\}$ is the union of the *shadow boundaries* of $\lambda \mathbf{B}_2$, $\lambda > 0$, in the direction of x (for the definition of shadow boundary, cf., e.g. [12]). Thus, if $x \dashv_1 y$ and $y \dashv_2 x$ are equivalent, then for any direction, the shadow boundary of $\mathbf{B}_2$ lies in a hyperplane. By a result of Blaschke (cf. Theorem 10.2.3 of [21]), this implies that $\mathbf{B}_2$ is an ellipsoid, and hence, $x \dashv_2 y$ and $y \dashv_2 x$ are the same relation. We obtain similarly that $\mathbf{B}_1$ is an ellipsoid, which yields that $x \dashv_1 y$ and $y \dashv_1 x$ are the same relation. Since it follows that $x \dashv_1 y$ and $x \dashv_2 y$ are the same relation as well, it is easy to see that $\mathbf{B}_1$ and $\mathbf{B}_2$ are homothetic.

In light of Theorem 31, it is reasonable to ask if $\|\cdot\|$ and $\|\cdot\|_a$ can be proportional, or equivalently, equal for some non-Euclidean norm. For normed planes this question was answered by Busemann [4] (cf. also [19]), who proved that this happens exactly for Radon norms; i.e. for 2-dimensional norms in which the normality relation is symmetric. Whereas for dimensions $n > 2$, normality is symmetric only in Euclidean spaces, Theorem 33 shows that the answer to our question is not so straightforward.

Before stating it, let us recall that a *polar decomposition* of a symplectic form $\langle \cdot, \cdot \rangle$ on $\mathbb{X}$, where $\dim \mathbb{X} = 2n$ is a basis $\{e_1, e_2, \ldots, e_{2n}\}$ such that $\langle e_i, e_j \rangle = 0$ if $|i - j| \neq n$, and for $i = 1, 2, \ldots, n$, $1 = \langle e_i, e_{i+n} \rangle = -\langle e_{i+n}, e_i \rangle$. Clearly, in this case for any $u = \sum_{i=1}^{2n} x_i e_i$ and $v = \sum_{i=1}^{2n} y_i e_i$, their product can be written as

$$\langle u, v \rangle = \sum_{i=1}^{n} x_i y_{i+n} - \sum_{i=1}^{n} y_i x_{i+n}.$$

Set $U = \mathrm{lin}\{\{e_1, \ldots, e_n\}$ and $V = \mathrm{lin}\{e_{n+1}, \ldots, e_{2n}\}$. It is known [2] that U and V are *Lagrangian subspaces* of $\mathbb{X}$ (i.e. they are their own orthogonal complements), and, furthermore, the polar decompositions of $\mathbb{X}$ can be identified with pairs of transversal Lagrangian subspaces U and V of $\mathbb{X}$. Thus, for brevity, we may call $\{U, V\}$ a polar decomposition of $\langle \cdot, \cdot \rangle$ on $\mathbb{X}$.

Now, we say that $\{U, V\}$ is a *Euclidean decomposition* of the norm $\|\cdot\|$, if $[\cdot, \cdot]$ is the direct sum of its restrictions to U and V; or in other words, if for any $u \in U$ and $v \in V$, we have $\|u + v\| = \sqrt{\|u\|_U^2 + \|v\|_V^2}$. Geometrically, this condition is equivalent to the requirement that for any $u \in U$ and $v \in V$, the intersection of $\mathbf{B}$ with $\mathrm{lin}\{u, v\}$ is an ellipse, where u and v belong to a pair of conjugate diameters. We note that the semi-inner product defined in this way is also a semi-inner product [13], and that this property (and its geometric variant) appeared also in [15].

Theorem 33 *Assume that* $\{U, V\}$ *is a Euclidean decomposition of* $(\mathbb{X}, \|\cdot\|)$, *where* $\dim \mathbb{X} = 2n$. *Let* $\langle \cdot, \cdot \rangle$ *be a symplectic form on* $\mathbb{X}$ *with a polar decomposition* $\{U, V\}$. *Then the following are equivalent.*

(i) The antinorm $\|\cdot\|_a$, *with respect to* $\langle \cdot, \cdot \rangle$, *is equal to* $\|\cdot\|$.

(ii) We have

$$\mathbf{B} \cap U = \{x \in U : |\langle x, y\rangle| \leq 1 \textit{ for every } y \in \mathbf{B} \cap V\}, \tag{7}$$

$$\mathbf{B} \cap V = \{y \in V : |\langle x, y\rangle| \leq 1 \textit{ for every } x \in \mathbf{B} \cap U\}.$$

Note that if we imagine U and V as orthogonal subspaces, then (ii) states that, identifying U and V via a symplectic basis, $U \cap \mathbf{B}$ and $V \cap \mathbf{B}$ are polars of each other.

Proof We set $U = \text{lin}\{e_1, e_2, \dots, e_n\}$, $V = \text{lin}\{e_{n+1}, \dots, e_{2n}\}$, where $\langle e_i, e_j\rangle = 0$ if $|i-j| \neq n$, and for $i = 1, 2, \dots, n$, $\langle e_i, e_{i+n}\rangle = 1$. For simplicity, we imagine this basis as the standard orthonormal basis of an underlying Euclidean space. Let $K = U \cap \mathbb{S}$ and $L = V \cap \mathbb{S}$. By straightforward computation, from the definition of Euclidean decomposition of $(\mathbb{X}, \|\cdot\|)$, we obtain that

$$\mathbb{S} = \{u \cos\phi + v \sin\phi : u \in K, v \in L, \text{ and } \phi \in [0, 2\pi]\}.$$

(We note that the opposite direction also holds, for the idea of the proof see [15, Lemma 2].)

First we prove (ii) $\Rightarrow$ (i).

To prove (i) observe that, by the definition of antinorm, we have $\|x\|_a = \sup\{\langle x, y\rangle : \|y\| = 1\}$. By homogeneity, it suffices to show that for our norm, $\|x\| = 1$ yields $\sup\{\langle x, y\rangle : \|y\| = 1\} = 1$. In other words, we need to show that $\langle \mathbb{S}, \mathbb{S}\rangle = [-1, 1]$, and for every $x \in \mathbb{S}$, there is some $y \in \mathbb{S}$ satisfying $\langle x, y\rangle = 1$.

Consider some $x, y \in \mathbb{S}$. Then $x = (x_1 \cos\alpha, \dots, x_n \cos\alpha, x_{n+1} \sin\alpha, \dots, x_{2n} \sin\alpha)$ and $y = (y_1 \cos\beta, \dots, y_n \cos\beta, y_{n+1} \sin\beta, \dots, y_{2n} \sin\beta)$, where $(x_1, \dots, x_n), (y_1, \dots y_n) \in K$, $(x_{n+1}, \dots x_{2n}), (y_{n+1}, \dots, y_{2n}) \in L$, and, without loss of generality, $0 \leq \alpha, \beta \leq \frac{\pi}{2}$. An elementary computation yields that

$$\langle x, y\rangle = (x_1 y_{n+1} + \dots + x_n y_{2n}) \cos\alpha \sin\beta - (x_{n+1} y_1 + \dots + x_{2n} y_n) \sin\alpha \cos\beta.$$

By the definitions of K and L, we have $|x_1 y_{n+1} + \dots + x_n y_{2n}| \leq 1$ and $|x_{n+1} y_1 + \dots + x_{2n} y_n| \leq 1$. Thus,

$$|\langle x, y\rangle| \leq \cos\alpha \sin\beta + \sin\alpha \cos\beta = \sin(\alpha + \beta) \leq 1.$$

On the other hand, consider any $x \in \mathbb{S}$. Then, using the notations of the previous paragraph, we have $(x_1, \dots, x_n, 0, \dots, 0) \in K$ and $(0, \dots, 0, x_{n+1}, \dots, x_{2n}) \in L$. Thus, by the condition in (ii), there are some $(y_1, \dots, y_n, 0, \dots, 0) \in K$ and $(0, \dots, 0, y_{n+1}, \dots, y_{2n}) \in L$ satisfying $\sum_{i=1}^n x_i y_{n+i} = -\sum_{i=1}^n y_i x_{n+i} = 1$. Now, setting

$$y = \left(y_1 \cos\left(\frac{\pi}{2} - \alpha\right), \dots, y_n \cos\left(\frac{\pi}{2} - \alpha\right), y_1 \sin\left(\frac{\pi}{2} - \alpha\right), \dots, y_n \sin\left(\frac{\pi}{2} - \alpha\right)\right),$$

we have $\langle x, y\rangle = \sin\frac{\pi}{2} = 1$.

Finally, we prove (i) $\Rightarrow$ (ii).

Assume that $\|\cdot\| = \|\cdot\|_a$ holds, and let $x = (x_1, \ldots, x_{2n}) \in \mathbb{S}$. Then, we have

$$1 = \sup\{\langle x, y\rangle : y \in \mathbb{S}\} = \sup\left\{\sum_{i=1}^{n} x_i y_{i+n} : (y_1, \ldots, y_{2n}) \in \mathbb{S}\right\}.$$

Observe that, by the definition of Euclidean decomposition, the orthogonal projection of $\mathbb{S}$ onto V is $V \cap \mathbf{B} = \operatorname{conv} L$. Thus, we have

$$1 = \sup\left\{\sum_{i=1}^{n} x_i y_{i+n} : (0, \ldots, 0, y_{n+1}, \ldots, y_{2n}) \in L\right\} = \sup\left\{\langle x, y\rangle : y \in L\right\}.$$

This yields the first equality in (ii), which readily implies the second inequality as well.

Corollary 31 *There are infinitely many non-Euclidean norms coinciding with their antinorms with respect to some symplectic form.*

Corollary 32 *Assume that in Theorem 33, $\mathbf{B} \cap U$ and $\mathbf{B} \cap V$ are ellipsoids, and that the polar decomposition defined by U, V is an orthogonal basis of an underlying Euclidean space. Let $\mathbf{B}^*$ denote the (Euclidean) polar of $\mathbf{B}$ in this space (cf. (9)), defining the norm $\|\cdot\|_*$ and antinorm $\|\cdot\|_{*,a}$. Then we have*

$$(7) \Leftrightarrow \|\cdot\| = \|\cdot\|_a \Leftrightarrow \|\cdot\|_* = \|\cdot\|_{*,a}.$$

Proof Clearly, it suffices to prove the second equivalence. Let the polar basis of $\langle\cdot,\cdot\rangle$ be $\{e_1, \ldots, e_{2n}\}$, and assume that $\|\cdot\| = \|\cdot\|_a$. This, by the definition of polar decomposition, yields that the semi-axes of $\mathbf{B}$ are $a_1, \ldots, a_n, \frac{1}{a_1}, \ldots, \frac{1}{a_n}$, in the directions of the corresponding basis vectors, respectively. On the other hand, $\mathbf{B}^*$ is also an ellipsoid, with semi-axes $\frac{1}{a_1}, \ldots, \frac{1}{a_n}, a_1, \ldots, a_n$ in the same directions, respectively, and thus, it satisfies the conditions in (7). The opposite direction follows from $(\mathbf{B}^*)^* = \mathbf{B}$.

Note that if $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on $\mathbb{X}$, then for any $1 \le p \le \infty$, $\|\cdot\| = (\|\cdot\|_1^p + \|\cdot\|_2^p)^{\frac{1}{p}}$ is a norm as well. In the following, we examine the relation between antinorm and this operation. We remark that for $p = 1$, using the identities between the support and the gauge/radial functions (cf. [8]), we have that the unit ball $\mathbf{B}$ of $\|\cdot\|$ is the convex body $(\mathbf{B}_1^* + \mathbf{B}_2^*)^*$, where $\mathbf{B}_1$ and $\mathbf{B}_2$ is the unit ball of $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively.

Proposition 31 *Let $(\mathbb{X}, \langle\cdot,\cdot\rangle)$ be a symplectic vector space. For $i = 1, 2$, let $\|\cdot\|_i$ be a norm on $\mathbb{X}$ with unit ball $\mathbf{B}_i$, and with antinorm $\|\cdot\|_{i,a}$. Let $1 \le p \le \infty$, and let $\|\cdot\| = (\|\cdot\|_1^p + \|\cdot\|_2^p)^{\frac{1}{p}}$. Then, for every $x \in \mathbb{X} \setminus \{o\}$, we have*

$$\|x\|_a \le \min\{\|x\|_{1,a}, \|x\|_{2,a}\} \le \left(\|x\|_{1,a}^p + \|x\|_{2,a}^p\right)^{\frac{1}{p}},$$

with equality in the first inequality if, and only if, $p = \infty$, and in the second one if, and only if $p = \infty$ and $\|x\|_{1,a} = \|x\|_{2,a}$.

Proof By the definition of antinorm, for $i = 1, 2$, we have

$$\|x\|_a = \sup\left\{\langle x, y\rangle : (\|y\|_1^p + \|y\|_2^p)^{\frac{1}{p}} \le 1\right\} \le \sup\left\{\langle x, y\rangle : \|y\|_i \le 1\right\} \le \|x\|_{i,a}.$$

From this, the assertion readily follows.

4 The normality map

Let $\mathbb{X}$ be even dimensional, $[\cdot,\cdot]$ induced by a strictly convex, smooth norm, and $\langle\cdot,\cdot\rangle$ a symplectic form on $\mathbb{X}$. Recall the maps F from Remark 22 defined on $(\mathbb{X}, \|\cdot\|)$, and G, defined in (5) for $(\mathbb{X}, \langle\cdot,\cdot\rangle)$. The main concept of this section is the following.

Definition 41 *The product* $J = G^{-1}F$, $J : \mathbb{X} \to \mathbb{X}$ *is called the* normality map *of* $(\mathbb{X}, \|\cdot\|)$, *with respect to* $\langle\cdot,\cdot\rangle$.

We remark that the normality map J also appears in [11, p. 308] as T.

Remark 41 *As G is linear, the linearity of J implies that F is linear, and thus, J is linear if, and only if $(\mathbb{X}, \|\cdot\|)$ is a Hilbert space.*

In light of Remark 41, we would like to emphasize that by *isometry*, we mean a (not necessarily linear) transformation $Z : \mathbb{X} \to \mathbb{X}$ with the property that for every $x \in \mathbb{X}$, $\|Z(x)\| = \|x\|$.

Before stating our first result in this section, we set $J_a = G^{-1}F_a$, and recall that $\mathbf{B}_a$ and $\mathbb{S}_a$ denote, respectively, the unit ball and the unit sphere of the antinorm.

Theorem 41 *For any $x, y \in \mathbb{X}$ and any $\lambda \in \mathbb{R}$, we have*

(i) $\|x\| = \|Jx\|_a$ *and* $\|x\|_a = \|J_a x\|$;
(ii) $J\mathbb{S} = \mathbb{S}_a$ *and* $J\mathbf{B} = \mathbf{B}_a$;
(iii) $[x, y] = \langle x, Jy\rangle$ *and* $[x, y]_a = \langle x, J_a y\rangle$, *where* $[\cdot,\cdot]_a$ *is the semi-inner product induced by* $\|\cdot\|_a$;
(iv) $x \dashv Jx$ *and* $x \dashv_a J_a x$;
(v) $[Jx, y] = -[Jy, x]$;
(vi) $[Jx, y]_a = -[J_a y, x]$;
(vii) $J(\lambda x) = \lambda Jx$;
(viii) $J_a J = J J_a = -I$, *where I denotes the identity map of* $\mathbb{X}$;
(ix) $[x, y] = [Jy, J_a x]$ *and* $[x, J_a y] = -[y, J_a x]$.

Proof Let $x \overset{F}{\longmapsto} f_x \overset{G^{-1}}{\longmapsto} Jx$ and $x \overset{F_a}{\longmapsto} f_x^a \overset{G^{-1}}{\longmapsto} J_a x$. Since $GJx = f_x$ and $\|x\|_{a,a} = \|x\|$, we have

$$\|Jx\|_a = \|GJx\|^* = \|f_x\|^* = \|x\|, \tag{8}$$

by (3) and (6). The second equality in (i) is implied by $GJ_a x = f_x^a$.

The equality (8) yields (ii). According to the definition of G in (5), we have $GJx = \langle\cdot, Jx\rangle$. On the other hand, $GJx = GG^{-1}Fx = Fx = [\cdot, x]$. The same holds also for J_a, and thus we obtain (iii). Setting $x = Jy$ in the first equality in (iii) and $x = J_a y$ in the second one yields (iv).

By (iii) and the skew-symmetry of $\langle \cdot, \cdot \rangle$ it follows that

$$[Jx, y] = \langle Jx, Jy \rangle = -\langle Jy, Jx \rangle = -[Jy, x],$$

which proves (v). By a similar argument, we may obtain (vi). The homogeneity of $[\cdot, \cdot]$ and $\langle \cdot, \cdot \rangle$ yields (vii).

By (i) and $\|x\|_{a,a} = \|x\|$, we have $\|J_a J x\| = \|x\|$ for every $x \in \mathbb{X}$. Thus, $J_a J \mathbf{B} = \mathbf{B}$, which yields that $J_a J$ is contained in the symmetry group of $\mathbf{B}$. Since $\mathbf{B}$ is o-symmetric, this group contains I and $-I$, and possibly some other transformations. First, consider the case that the only symmetries of $\mathbf{B}$ are I and $-I$, which implies that $J_a J = I$ or $J_a J = -I$. Furthermore, if $J_a J = I$, then, applying (vi) with $y = Jx$ yields $[Jx, Jx]_a = -[x, x]$. Since for every $x \in \mathbb{X}$, we have $[Jx, Jx]_a \geq 0$, and $[x, x] \geq 0$ with equality only for $x = o$, we have reached a contradiction, implying that $J_a J = -I$ in this case. If $\mathbf{B}$ has symmetries different from I and $-I$, then we may approach $\mathbf{B}$ with a sequence of o-symmetric convex bodies which have no other symmetries, and apply a continuity argument. This proves the first part of (viii), whereas the second part follows from the same argument.

Finally, from (v) it follows that $[J J_a x, y] = -[Jy, J_a x]$, which, together with (viii), yields the first relation of (ix), implying $[x, J_a y] = [J J_a y, J_a x] = -[y, J_a x]$ as well.

Remark 42 *If $(\mathbb{X}, \|\cdot\|)$ is the Euclidean plane, then $J : \mathbb{X} \longrightarrow \mathbb{X}$ is simply the rotation about the origin by $\frac{\pi}{2}$. Furthermore, if $(\mathbb{X}, \|\cdot\|)$ is two-dimensional, then $J\mathbb{S} = \mathbb{S}_a$ is the* isoperimetrix *of $(\mathbb{X}, \|\cdot\|)$ (cf. [22]).*

For our next theorem, we need some preparation. First, recall the so-called 'linear Darboux Theorem' (cf. [2]) that states that any two symplectic spaces $(\mathbb{X}_1, \langle \cdot, \cdot \rangle_1)$ and $(\mathbb{X}_2, \langle \cdot, \cdot \rangle_2)$ of the same dimension are *symplectically isomorphic*; that is, there is a linear isomorphism $L : \mathbb{X}_1 \to \mathbb{X}_2$ satisfying $\langle x, y \rangle_1 = \langle Lx, Ly \rangle_2$ for all $x, y \in \mathbb{X}_1$. Such a map is called a *symplectic isomorphism* between the two spaces. Furthermore, observe that if $(\mathbb{X}, \langle \cdot, \cdot \rangle)$ is a symplectic space and $L : \mathbb{X} \longrightarrow \mathbb{X}$ is a linear transformation, then there is a unique linear transformation L' satisfying $\langle Lx, y \rangle = \langle x, L'y \rangle$, or equivalently, $\langle x, Ly \rangle = \langle L'x, y \rangle$, which we call the *left adjoint* of L.

Remark 43 *If, in a polar decomposition of $\mathbb{X}$, where $\dim \mathbb{X} = 2n$, the matrix of L is $\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$, where each block is an $n \times n$ matrix, then the matrix of its left adjoint is $\begin{bmatrix} A_4^T & -A_2^T \\ -A_3^T & A_1^T \end{bmatrix}$.*

Theorem 42 *Let $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ be two symplectic forms on $(\mathbb{X}, \|\cdot\|)$, and let $L : (\mathbb{X}, \langle \cdot, \cdot \rangle_1) \longrightarrow (\mathbb{X}, \langle \cdot, \cdot \rangle_2)$ be a symplectic isomorphism. For $i = 1, 2$ and any linear transformation A on $\mathbb{X}$, let A_i^*, J_i and $\|\cdot\|_{i,a}$ denote the (left or right) adjoint of A, the normality map and the antinorm, respectively, with respect to $\langle \cdot, \cdot \rangle_i$. Then the following are equivalent.*

(i) $J_1^{-1}L_2^*LJ_1$ *is an isometry of* $(\mathbb{X}, \|\cdot\|)$*.*
(ii) $J_2^{-1}(L^{-1})_1^*L^{-1}J_2$ *is an isometry of* $(\mathbb{X}, \|\cdot\|)$*.*
(iii) For any $x \in \mathbb{X}$*, we have* $\|x\|_{1,a} = \|x\|_{2,a}$*.*

We note that since the unit ball $\mathbf{B}$ of $(\mathbb{X}, \|\cdot\|)$ is o-symmetric, it makes no difference if, in Theorem 42 we mean right or left adjoint.

Proof By symmetry, it suffices to show that (i) and (iii) are equivalent. First, assume that (i) holds. Then, for any $x \in \mathbb{X}$,

$$\|x\|_{1,a} = \sup\{\langle y, x\rangle_1 : y \in \mathbb{S}\} = \sup\{\langle Ly, Lx\rangle_2 : y \in \mathbb{S}\} =$$

$$= \sup\{\langle y, L_2^*Lx\rangle_2 : y \in \mathbb{S}\} = \|L_2^*Lx\|_{2,a}.$$

Hence $L_2^*L\mathbb{S}_{1,a} = \mathbb{S}_{2,a}$, where, for $i = 1, 2$, $\mathbb{S}_{i,a}$ is the unit sphere of the norm $\|\cdot\|_{i,a}$. Now, from (ii) of Theorem 41 it follows that

$$\mathbb{S}_{2,a} = L_2^*L\mathbb{S}_{1,a} = J_1J_1^{-1}L_2^*LJ_1\mathbb{S} = J_1\mathbb{S} = \mathbb{S}_{1,a},$$

implying (iii).

Conversely, (iii) of Theorem 41 yields

$$\langle y, J_2x\rangle_2 = [y, x] = \langle y, J_1x\rangle_1 = \langle Ly, LJ_1x\rangle_2 = \langle y, L_2^*LJ_1x\rangle_2$$

holds for all $x, y \in \mathbb{X}$. Since $\langle\cdot,\cdot\rangle_2$ is nondegenerate and bilinear, from this $J_2x = L_2^*LJ_1x$ follows for all $x \in \mathbb{X}$. Now, using (ii) of Theorem 41 and (iii), we obtain

$$J_1\mathbb{S} = \mathbb{S}_{1,a} = \mathbb{S}_{2,a} = J_2\mathbb{S} = L_2^*LJ_1\mathbb{S},$$

which readily yields (i).

In light of Theorem 42, we may introduce a more refined classification system on symplectic forms than standard symplectic isomorphism. Theorem 42 shows also that antinorms and the isometries of a normed space are related.

Definition 42 *Let* $\langle\cdot,\cdot\rangle_1$ *and* $\langle\cdot,\cdot\rangle_2$ *be two symplectic forms on* $(\mathbb{X}, \|\cdot\|)$*. For* $i = 1, 2$*, let* $\|\cdot\|_{i,a}$ *denote the antinorm with respect to* $\langle\cdot,\cdot\rangle_i$*. We say that the two symplectic forms are* equivalent with respect to the norm $\|\cdot\|$*, if for all* $x \in \mathbb{X}$*, we have* $\|x\|_{1,a} = \|x\|_{2,a}$*.*

5 The concept of semi-polarity

The concept of *polarity* (or *polar duality*) is a very important tool in several areas of convexity. From a functional analytic point of view, for a real vector space $\mathbb{X}$, the *polar* of a set $X \subset \mathbb{X}$ is defined as the subset $\{f \in \mathbb{X}^* : f(x) \leq 1 \text{ for all } x \in X\}$ of the dual space. Nevertheless, in geometry, this subset is usually identified with the subset

$$X^* = \{y \in \mathbb{X} : [x, y]_{\mathbb{E}} \leq 1 \text{ for all } x \in X\} \tag{9}$$

of $\mathbb{X}$, induced by an inner product $[\cdot,\cdot]_{\mathbb{E}}$ of $\mathbb{X}$. This subset X^* is also called the polar, and we use this definition in our paper. We note that the identification in (9) assumes an inner product structure on the space, as there is no canonical isomorphism between $\mathbb{X}$ and $\mathbb{X}^*$. The following theorem summarizes some of the important properties of polar sets; see, e.g., [21, § 1.6 and Remark 1.7.7], [23, § 2.8], [3, § 3], and [10, § 4.1, p. 56]. Recall from Section 2 that $\mathfrak{X}_o$ denotes the family of convex bodies in $\mathbb{X}$, with the origin o as an interior point, and in a normed space with unit ball $\mathbf{B}$, the support function of $K \in \mathfrak{X}_o$ in the direction $u \neq o$ is $h_B(K,u) = \sup\{[x,u] : x \in K\}$, and the gauge function of K is $g(K,x) = \min\{\lambda \geq 0 : x \in \lambda K\}$, $x \in \mathbb{X}$. Furthermore, $\mathbf{B}_{\mathbb{E}}$ is the Euclidean unit ball of $\mathbb{X}$.

Theorem 51 *Let $M, N \subset \mathbb{X}$, and $\lambda \neq 0$. Then*

(i) $M \subseteq N$ *implies* $N^* \subseteq M^*$;
(ii) $(M \cup N)^* = M^* \cap N^*$;
(iii) $(\lambda M)^* = (1/\lambda) M^*$;
(iv) $\mathbf{B}_{\mathbb{E}}^* = \mathbf{B}_{\mathbb{E}}$.

If $M \in \mathfrak{X}_o$, then

(v) $M^{**} = M$,
(vi) $g(M^*,x) = h(M,x)$ *and* $h(M^*,x) = g(M,x)$.

If, in addition, M is centered at o, then

(vii) $h(M,x) = \|x\|_{M^*}$ *and* $h(M^*,x) = \|x\|_M$ *for* $x \in X$, *where* $\|\cdot\|_N$ *is the norm induced by the o-symmetric convex body N.*

To generalize this notion, instead of an inner product, we use the semi-inner product of $(\mathbb{X}, \|\cdot\|)$ to identify elements of $\mathbb{X}$ and $\mathbb{X}^*$. Unless we specifically state, in this section we do not restrict our investigation to even dimensional spaces but consider only strictly convex, smooth norms.

Definition 51 *Let $(\mathbb{X}, \|\cdot\|)$ be a normed space with unit ball $\mathbf{B}$ and semi-inner product $[\cdot,\cdot]$ and let $m \in \mathbb{X}$. Then the* left/right semi-polar of *m is*

$$m_\circ = \{x \in \mathbb{X} : [m,x] \leq 1\} \quad \text{and} \quad m^\circ = \{x \in \mathbb{X} : [x,m] \leq 1\}, \tag{10}$$

respectively. If $M \subset \mathbb{X}$, then the left/right semi-polar of *M is*

$$M_\circ = \bigcap_{m \in M} m_\circ = \{x \in \mathbb{X} : [m,x] \leq 1 \textit{ for all } m \in M\} \tag{11}$$

$$M^\circ = \bigcap_{m \in M} m^\circ = \{x \in \mathbb{X} : [x,m] \leq 1 \textit{ for all } m \in M\}, \tag{12}$$

respectively.

We note that, by this definition $o_\circ = o^\circ = \mathbb{X}$. Observe that $f_x = [\cdot, x]$ is a linear functional on $\mathbb{X}$, but $[x,\cdot]$ is not necessarily so. Thus, if $M \in \mathfrak{X}_o$, then $M^\circ \in \mathfrak{X}_o$ as well, but $M_\circ$ is not necessarily convex. On the other hand, if $[\cdot,\cdot]$ is symmetric (e.g. $(\mathbb{X}, \|\cdot\|)$ is an inner product space), then both $M_\circ$ and M° coincide with the usual polar of M in this space.

Theorem 52 *Let $M, N \subset \mathbb{X}$ and $\lambda \neq 0$. Then*

(i) $M \subseteq N$ *implies* $N^\circ \subseteq M^\circ$ *and* $N_\circ \subseteq M_\circ$*;*
(ii) $(M \cup N)^\circ = M^\circ \cap N^\circ$ *and* $(M \cup N)_\circ = M_\circ \cap N_\circ$*;*
(iii) $(\lambda M)^\circ = (1/\lambda)M^\circ$ *and* $(\lambda M)_\circ = (1/\lambda)M_\circ$*;*
(iv) $\mathbf{B}^\circ = \mathbf{B}_\circ = \mathbf{B}$.
(v) If $M \in \mathfrak{X}_o$*, then* $(M_\circ)^\circ = M$.

Proof Note that (i)-(iii), and the equality $\mathbf{B}^\circ = \mathbf{B}$ are a straightforward consequence of Definition 51.

We prove that $\mathbf{B}_\circ = \mathbf{B}$. By definition, we have $\mathbf{B}_\circ = \{y \in \mathbb{X} : [x, y] \leq 1 \text{ for any } x \in \mathbf{B}\}$. Since $[x, y] \leq 1$ for any $x, y \in \mathbf{B}$, we clearly have $\mathbf{B} \subseteq \mathbf{B}_\circ$. On the other hand, let $y \in \mathbb{X} \setminus \mathbf{B}$. Then $\|y\| > 1$, and we have $\left[\frac{y}{\|y\|}, y\right] = \frac{1}{\|y\|}[y, y] = \|y\| > 1$. As $\frac{y}{\|y\|} \in \mathbf{B}$, it follows that $y \notin \mathbf{B}_\circ$.

Finally, we show (v). By definition, for any $M \subset \mathbb{X}$, we have $M \subseteq (M_\circ)^\circ$. Let $x \notin M \in \mathfrak{X}_o$. Then there is a hyperplane H strictly separating M from x. Since H cannot pass through o, using the identification F in Remark 22, there is some $y \in \mathbb{X}$ such that $H = \{z \in \mathbb{X} : [z, y] = 1\}$. Now, for any $z \in M$, we have $[z, y] < 1$, implying that $y \in M_\circ$. On the other hand, $[x, y] > 1$, which yields that $x \notin (M_\circ)^\circ$ and $(M_\circ)^\circ = M$.

Theorem 53 *For* $(\mathbb{X}, \|\cdot\|)$*, the following are equivalent.*

(i) $(\mathbb{X}, \|\cdot\|)$ *is an inner product space;*
(ii) for any $m \in \mathbb{X}$*,* $m_\circ$ *is convex;*
(iii) for any $m \in \mathbb{X}$*,* $m \in (m^\circ)^\circ$*;*
(iv) for any $m \in \mathbb{X}$*,* $m \in (m_\circ)_\circ$.

Proof Clearly, (i) implies (ii)-(iv).

First, we show that (ii) implies (i). Assume that (ii) holds. Then, since the intersection of convex sets is convex, we have that $M_\circ$ is convex for any $M \subset \mathbb{X}$. Let $m \neq o$ arbitrary, and $M = \{\lambda m : \lambda \in \mathbb{R}\}$. Then $x \in M_\circ$ if, and only if $[m, x] = 0$, or in other words, if $x \dashv m$. Observe that the set of these points is exactly the conic hull of the shadow boundary of $\mathbf{B}$, in the direction of m. Since $\mathbf{B}$ is strictly convex, or in other words, $\mathbb{S}$ does not contain a nondegenerate segment, from the convexity of $M_\circ$ it follows that $M_\circ$ is a hyperplane, passing through the origin. Thus, similarly as in the proof of Theorem 32, to finish the proof it suffices to apply the result of Blaschke (cf. Theorem 10.2.3 of [21]), stating that in this case $\mathbf{B}$ is an ellipsoid.

Now we prove that (iii) yields (i). Assume that for any $m \in \mathbb{X}$, $m \in (m^\circ)^\circ$. Then we have $[m, x] \leq 1$ for any $x \in m^\circ$; or in other words, $[x, m] \leq 1$ implies $[m, x] \leq 1$, for any $x, m \in \mathbb{X}$. We show that from this, it follows that $[x, m] = 1$ and $[m, x] = 1$ are equivalent. Indeed, assume that $[x, m] = 1$ and $[m, x] < 1$ for some $x, m \in \mathbb{X}$. Then, by the homogeneity of the second variable, there is some $\lambda > 1$ such that $[m, \lambda x] \leq 1$, which implies $1 \geq [\lambda x, m] = \lambda[x, m] = \lambda > 1$; a contradiction. Hence, we have that $[x, m] = 1$ and $[m, x] = 1$ are equivalent, which yields, by homogeneity, that $[x, m] = [m, x]$ for any $m, x \in \mathbb{X}$. Thus $[\cdot, \cdot]$ is an inner product.

To show that (iv) yields (i), we may apply a similar argument.

For even dimensional spaces, Theorem 54 seems to be the analogue of (v) of Theorem 51.

Theorem 54 *Let* $(\mathbb{X}, \|\cdot\|)$ *be even dimensional, and let* J *be the normality map with respect to a symplectic form* $\langle\cdot,\cdot\rangle$ *on* $\mathbb{X}$. *If* $M \in \mathfrak{X}_o$, *then* $\operatorname{conv}(JM) = (J_a M^\circ)^\circ$.

We note that JM is not necessarily convex, even in the plane. As an example, we can take $\|.\|$ as the ℓ_p-norm with $p \approx \infty$, and M as the unit disk of the ℓ_1 norm.

Proof (Proof of Theorem 54) By definition, $M^\circ = \{x \in \mathbb{X} : [x,m] \leq 1 \quad \text{for all} \quad m \in M\}$. Hence, by (ix) of Theorem 41, it follows that $[Jm, J_a x] \leq 1$ holds for every $x \in M^\circ$ and every $m \in M$. Therefore $Jm \in (J_a M^\circ)^\circ$, implying $\operatorname{conv}(JM) \subseteq (J_a M^\circ)^\circ$.

To prove that $(J_a M^\circ)^\circ \subseteq \operatorname{conv}(JM)$, consider some $z \notin \operatorname{conv}(JM)$. Then there is a hyperplane H strictly separating z and $\operatorname{conv}(JM)$. Since $\operatorname{conv}(JM) \in \mathfrak{X}_o$, this hyperplane cannot pass through the origin, and, using the identification F of the elements of $\mathbb{X}$ and $\mathbb{X}^*$ in Remark 22, $H = \{x \in \mathbb{X} : [x,u] = 1\}$ for some $u \in \mathbb{X}$. Then

$$[z,u] > 1, \quad \text{and} \quad [Jm,u] < 1 \quad \text{for any} \quad m \in M. \tag{13}$$

Then (ix) of Theorem 41 implies that

$$[J_a^{-1}u, m] = [Jm, J_a J_a^{-1} u)] = [Jm, u] < 1,$$

for every $m \in M$, from which $J_a^{-1}u \in M^\circ$ and $u \in J_a M^\circ$ follows. Thus, by (13) we have that $z \notin (J_a M^\circ)^\circ$, which completes the proof.

The next theorem shows how the gauge function of the semi-polar of a convex body is related to the normed support function of this body.

Theorem 55 *Let* $(\mathbb{X}, \|\cdot\|)$ *be even dimensional, and let* J *be the normality map with respect to a symplectic form* $\langle\cdot,\cdot\rangle$ *on* $\mathbb{X}$. *Assume that* $M, JM, J_a M^\circ \in \mathfrak{X}_o$. *Then*

$$h_B(M^\circ, x) = g(M,x) \quad \text{and} \quad h_B(JM, x) = g(J_a M^\circ, x) \tag{14}$$

for every $x \in \mathbb{X} \setminus \{o\}$.

Proof First we show that the second equation in (14) implies the first one. Applying the second equation for $J_a M^\circ$ and using Theorem 54, we obtain $h_B(JJ_a M^\circ, x) = g(J_a JM, x)$, which, by (viii) of Theorem 41, is equivalent to $h_B(-M^\circ, x) = g(-M, x)$, and $h_B(M^\circ, x) = g(M,x)$.

Now, we prove the second equation. Note that by our assumptions, o is an interior point of $J_a M^\circ$. Hence, for any $x \neq 0$, we may denote by x_0 the intersection point of bd $J_a M^\circ$ with the conic hull of x. Let $s = J_a^{-1} x_0 \in M^\circ$. Then, by (ix) of Theorem 41, for every $m \in M$ we have

$$1 \geq [s, m] = [J_a^{-1} x_0, m] \Longrightarrow 1 \geq [Jm, J_a J_a^{-1} x_0] = [Jm, x_0].$$

Thus, we obtain that $h_B(JM, x_0) = \sup\{[Jm, x_0] : m \in M\} \leq 1$, which yields that

$$h_B(J(M), x) = h_B\left(JM, \frac{\|x\|}{\|x_0\|} x_0\right) \leq \frac{\|x\|}{\|x_0\|} = g(J_a M^\circ, x). \tag{15}$$

On the other hand, let $0 < \lambda < g(J_a M^\circ, x)$ be arbitrary. Then, by (vii) of Theorem 41 and (iii) of Theorem 52, we have

$$x \notin \lambda J_a M^\circ = J_a(\lambda M^\circ) = J_a\left(\left(\frac{1}{\lambda} M\right)^\circ\right). \tag{16}$$

Applying this for $y = J_a^{-1} x$, we obtain that $y \notin \left(\frac{1}{\lambda} M\right)^\circ$, which yields $\left[y, \frac{1}{\lambda} m_0\right] > 1$ for some $m_0 \in M$. Hence, by (ix) of Theorem 41 and the homogeneity of $[\cdot, \cdot]$,

$$\lambda < [J_a^{-1} x, m_0] = [Jm_0, J_a J_a^{-1} x] = [Jm_0, x],$$

and therefore $h_B(JM, x) = \sup\{[Jm, x] : m \in M\} > \lambda$. Since $0 < \lambda < g(J_a M^\circ, x)$ is arbitrary, it follows that $h_B(JM, x) \geq g(J_a M^\circ, x)$, which, combined with (15) proves the assertion.

Corollary 51 *If $M \in \mathfrak{X}_o$, then $h(M^*, x) = h_B(M^\circ, x)$ and $h(M^*, x) h(\mathbf{B}, x) = h(M^\circ, x)$.*

Proof It follows from (vi) of Theorem 51, and (14), that $h(M^*, x) = g(M, x) = h_B(M^\circ, x)$.

The next corollary is an analogue of (vii) of Theorem 51. We note that if M is o-symmetric, then JM, $J_a M$ and M° are o-symmetric as well.

Corollary 52 *If $M, JM, J_a M^\circ \in \mathfrak{X}_o$ and M is o-symmetric, then*

$$h_B(M^\circ, x) = \|x\|_M \quad \text{and} \quad h_B(JM, x) = \|x\|_{J_a M^\circ}.$$

6 Remarks and questions

Remark 61 *One can attribute a geometric meaning to a symplectic form in any dimensions. More specifically, if $\{e_1, \ldots, e_{2n}\}$ is a polar decomposition of the symplectic product $\langle \cdot, \cdot \rangle$, then $\langle x, y \rangle$ is the sum of the areas of the projections onto the n coordinate planes $\{e_i, e_{n+i}\}$ of the oriented parallelogram which x and y span.*

It is clear from Theorem 33 that if $\mathbb{X}$ is a Euclidean space, and $\langle \cdot, \cdot \rangle$ has a polar decomposition into an orthonormal basis of $\mathbb{X}$, then, with respect to this form, $\| \cdot \|_a = \| \cdot \|$. On the other hand, it is easy to see that these two norms are not even proportional for *each* symplectic form for any norm. Note that, for normed spaces, the counterpart of an orthogonal basis is a so-called *Auerbach basis*, which is a basis containing pairwise normal unit vectors with respect to the norm. This leads to the following question.

Problem 1 Prove or disprove that if $\|\cdot\|_a = \|\cdot\|$ with respect to any symplectic form with a polar decomposition into an Auerbach basis of $(\mathbb{X}, \| \cdot \|)$, then $(\mathbb{X}, \| \cdot \|)$ is Euclidean.

Problem 2 Characterize the norms $\| \cdot \|$ satisfying $\| \cdot \|_a = \| \cdot \|$ with respect to some symplectic form.

Note that the normality map J depends on the choice of the symplectic form $\langle \cdot, \cdot \rangle$ on $\mathbb{X}$.

Question 1 *Let $(\mathbb{X}, \| \cdot \|)$ be of dimension $2n > 2$. Do there exist symplectic forms with respect to which $J\mathbb{S}$ is the isoperimetrix of $(\mathbb{X}, \| \cdot \|)$ in the sense of Busemann or Holmes-Thompson?*

For both these concepts of isoperimetrices see, e.g., Chapter 5 of [22], or [17].

Question 2 *We have shown in Theorem 52 that for any $M \in \mathfrak{X}_o$, we have $(M_\circ)^\circ = M$. Clearly, $M \subseteq (M^\circ)_\circ$ also holds. Is it true that $M = (M^\circ)_\circ$?*

The requirements that the underlying normed space $(\mathbb{X}, \| \cdot \|)$ is smooth, strictly convex and of even dimension are not necessary for the definition of semi-polarities; these requirements are only needed for the purpose that the normality map J is well defined.

Question 3 *Is there a counterpart of Theorem 54 for odd dimensional spaces?*

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