

An abstract approach to polychromatic coloring: shallow hitting sets in ABA-free hypergraphs and pseudohalfplanes

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Abstract. The goal of this paper is to give a new, abstract approach to cover-decomposition and polychromatic colorings using hypergraphs on ordered vertex sets. We introduce an abstract version of a framework by Smorodinsky and Yuditsky, used for polychromatic coloring halfplanes, and apply it to so-called *ABA-free hypergraphs*, which are a special case of *shift-chains*. Using our methods, we prove that $(2k - 1)$ -uniform ABA-free hypergraphs have a polychromatic k -coloring, a problem posed by the second author. We also prove the same for hypergraphs defined on a point set by pseudohalfplanes. These results are best possible.

We also introduce several new notions that seem to be important for investigating polychromatic colorings and ϵ -nets, such as *shallow hitting sets* and *balanced polychromatic colorings*. We pose several open problems related to them. For example, is it true that given a finite point set S on a sphere and a set of halfspheres \mathcal{F} , such that $\{S \cap F \mid F \in \mathcal{F}\}$ is a Sperner family, we can select an $R \subset S$ such that $1 \leq |F \cap R| \leq 2$ holds for every $F \in \mathcal{F}$?

1 Introduction

The study of proper and, more generally, polychromatic colorings of geometric hypergraphs has attracted much attention, not only because this is a very basic and natural theoretical problem but also because such problems often have important applications. One such application area is resource allocation, e.g., battery consumption in sensor networks. Moreover, the coloring of geometric shapes in the plane is related to the problems of cover-decomposability, conflict-free colorings and ϵ -nets; these problems have applications in sensor networks and frequency assignment as well as other areas. For surveys on these and related problems see [18,25].

In a (primal) *geometric hypergraph polychromatic coloring* problem we are given a set of points and a collection of regions in \mathbb{R}^d , and our goal is to k -color the points such that every region that contains at least $m(k)$ points contains

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a point of every color. We call such a coloring a *polychromatic k -coloring*. In a *dual* geometric hypergraph polychromatic coloring problem, our goal is to k -color the regions such that every point which is contained in at least $m(k)$ regions is contained in a region of every color. In other words, in the dual version our goal is to *decompose* an $m(k)$ -fold covering of some point set into k coverings. The primal and the dual versions are equivalent if the underlying regions are the translates of some fixed set. For the proof of this statement and an extensive survey of results related to *cover-decomposition*, see e.g., [18]. Below we mention some of these results, stated in the equivalent primal form.

The most general result about polygons is that given a fixed convex polygon, there exists a c (that depends only on the polygon) such that any *finite* point set has a polychromatic k -coloring such that any translate of the fixed convex polygon that contains at least $m(k) = c \cdot k$ points contains a point of every color [9]. Polygons for which such a finite $m(k)$ (for any $k \geq 2$) exists have been classified [21,24].

As it was shown recently [23], there is no such finite $m(2)$ for convex sets with a smooth boundary, e.g., for the translates of a disc. However, it was also shown in the same paper that for the translates of any *unbounded* convex set $m(2) = 3$ is sufficient. In this paper we extend this result to every k , showing that $m(k) = 2k - 1$ is an optimal function for unbounded convex sets.

For homothets of a given shape the primal and dual problems are not equivalent. For homothets of a triangle (a case closely related to the case of translates of octants [13,14]), there are several results, the current best are $m(k) = O(k^{4.53})$ in the primal version [3,15] and $m(k) = O(k^{5.53})$ in the dual version [4]. For the homothets of other convex polygons, in the dual case there is no finite $m(2)$ [17], and in the primal case only conditional results are known [15], namely, that the existence of a finite $m(2)$ implies the existence of an $m(k)$ that grows at most polynomially in k . In fact, it is even possible that for *any* polychromatic coloring problem $m(k) = O(m(2))$.

For other shapes, cover-decomposability has been less studied, in these cases the investigation of polychromatic-colorings is motivated rather by conflict-free colorings or ϵ -nets. Most closely related to our paper, coloring halfplanes for small values were investigated in [11],[12] and [8], and polychromatic k -colorings in [26]. We generalize all the (primal and dual) results of the latter paper to pseudohalfplanes. Note that translates of an unbounded convex set form a set of pseudohalfplanes, thus the above mentioned result about unbounded convex sets is a special case of this generalization to pseudohalfplanes.

Axis-parallel rectangles are usually investigated from the ϵ -net point of view (e.g., [5,19]), for which the coloring function f is not independent of the number of points/regions. Motivated by these, bottomless rectangles are regarded for small values in [11,12] and polychromatic k -colorings in [1]. In this paper we place bottomless rectangles in our abstract context and pose some further problems about them.

Besides improving earlier results, our contribution is a more abstract approach to the above problems. Namely, we introduce the notion of *t -as-free*

families (see Definition 1), *shallow hitting sets* (see Definition 4) and *balanced polychromatic colorings* (see Definition 38), and discuss their relevance.

1.1 Definitions and statements of main results

We start with a definition, resembling the notion of a Davenport-Schinzel sequence [6]. For some set, we call a *(total) ordering of the set* an injection of its elements to the real numbers.

Definition 1 *Given two sets of real numbers, A and B , we say that they contain an alternating sequence of length s , if there is a sequence $x_1 < \dots < x_s$ such that either of the below conditions holds.*

- *If i is odd, then $x_i \in A \setminus B$ and if i is even, then $x_i \in B \setminus A$.*
- *If i is even, then $x_i \in A \setminus B$ and if i is odd, then $x_i \in B \setminus A$.*

*A hypergraph whose vertices are ordered (i.e., are real numbers) contains an alternating sequence of length s if it contains two hyperedges (sets), A and B , which contain an alternating sequence of length s . The hypergraph is t -as-free if it does not contain an alternating sequence of length t . For a hypergraph with an unordered vertex set we say that it is t -as-free if its vertices have an ordering with which the hypergraph is t -as-free.³ 3- and 4-as-free hypergraphs are also called, respectively, *ABA-free* and *ABAB-free*.*

Remark 1. ABA-free hypergraphs were first defined in [23] under the name *special shift-chains*, as they are a special case of *shift-chains* introduced in [22].

Example 2 ([23]) *Let S be a set of points in the plane with different x -coordinates and let C be a convex set that contains a vertical halfline. Define a hypergraph \mathcal{H} whose vertex set is the x -coordinates of the points of S . A set of numbers H is a hyperedge of \mathcal{H} if there is a translate of C such that the x -coordinates of the points of S contained in the translate is exactly H . The hypergraph \mathcal{H} defined this way is ABA-free.*

Example 3 *Let S be a set of points in the plane in general position. Define a hypergraph \mathcal{H} whose vertex set is the x -coordinates of the points of S . A set of numbers H is a hyperedge of \mathcal{H} if there is a positive halfplane H (i.e., that contains a vertical positive halfline) such that the x -coordinates of the points of S contained in the translate is exactly H . The hypergraph \mathcal{H} defined this way is ABA-free.*

The above examples show how to reduce geometric problems to abstract problems about t -as-free hypergraphs. Observe that given an S , by choosing an

³ While it might seem that using the same notion for ordered and unordered hypergraphs might lead to confusion, as by forgetting the ordering of an ordered hypergraph containing an alternating sequence of length t , it might become t -as-free, from the context it will always be perfectly clear what we mean.

appropriately big parabola, all hyperedges defined by positive halfplanes is also defined by some translate of this parabola, thus the first example is more general than the second. Even more, as we will see later in Section 3, ABA-free hypergraphs have an equivalent geometric representation with graphic pseudoline arrangements (sets are defined by the regions above the pseudolines, for the definitions and details see Section 3) and both translates of the boundary of an unbounded convex set and lines in the plane form graphic pseudoline arrangements, showing again that the above examples are special cases of ABA-free hypergraphs.

To study polychromatic coloring problems, we also introduce the following definition, which is implicitly used in [26], but deserves to be defined explicitly as it seems to be important in the study of polychromatic colorings.

Definition 4 *A set R is a c -shallow hitting set of the hypergraph \mathcal{H} if for every $H \in \mathcal{H}$ we have $1 \leq |R \cap H| \leq c$.*

We denote the smallest (resp. largest) element of an ordered set H (if exists) by $\min(H)$ (resp. $\max(H)$).

Observation 5 *An induced subhypergraph of a t -as-free hypergraph is also t -as-free.*

Our main results and the organization of the rest of this paper is as follows.

In Section 2 we prove (following closely the ideas of Smorodinsky and Yuditsky [26]) that every $(2k - 1)$ -uniform ABA-free hypergraph has a polychromatic coloring with k colors. We then observe that the dual of this problem is equivalent to the primal, which implies that the edges of every $(2k - 1)$ -uniform ABA-free hypergraph can be colored with k colors, such that if a vertex v is in a subfamily \mathcal{H}_v of at least $m(k) = 2k - 1$ of the edges of \mathcal{H} , then \mathcal{H}_v contains a hyperedge from each of the k color classes.

In Section 3 we give an abstract equivalent definition (using ABA-free hypergraphs) of hypergraphs defined by pseudohalfplanes, and we prove that given a finite set of points S and a pseudohalfplane arrangement \mathcal{H} , we can k -color S such that any pseudohalfplane in \mathcal{H} that contains at least $m(k) = 2k - 1$ points of S contains all k colors. Both results are sharp. Note that these results imply the same for hypergraphs defined by unbounded convex sets.

In Section 4 we discuss dual and other versions of the problem. For example we prove that given a pseudohalfplane arrangement \mathcal{H} , we can k -color \mathcal{H} such that if a point p belongs to a subfamily \mathcal{H}_p of at least $m(k) = 3k - 2$ of the pseudohalfplanes of \mathcal{H} , then \mathcal{H}_p contains a pseudohalfplane from each of the k color classes. This result might not be sharp, the best known lower bound for $m(k)$ is $2k - 1$ [26]. We also discuss consequences about ϵ -nets on pseudohalfplanes. Please note that in the current version some parts of this section have been moved to the Appendix.

In Appendix F, we discuss ABAB-free hypergraphs and related problems.

We denote the symmetric difference of two sets, A and B , by $A\Delta B$, the complement of a hyperedge F by \bar{F} and for a family \mathcal{F} we use $\bar{\mathcal{F}} = \{\bar{F} \mid F \in \mathcal{F}\}$.

2 ABA-free hypergraphs

Suppose we are given an ABA-free hypergraph \mathcal{H} on n vertices. As the hypergraph is ABA-free, for any pair of sets $A, B \in \mathcal{H}$ either there are $a < b$ such that $a \in A \setminus B$ and $b \in B \setminus A$, or there are $b < a$ such that $a \in A \setminus B$ and $b \in B \setminus A$, or none of them, but not both as that would contradict ABA-freeness.

Define $A < B$ if and only if there are $a < b$ such that $a \in A \setminus B$ and $b \in B \setminus A$. Define $A \leq B$ if and only if either $A = B$ (as sets) or $A < B$. By the above observation, this is well-defined, and it gives a partial ordering of the sets. First, we show that reordering the vertices in an appropriate way keeps the ordered hypergraph ABA-free.

Lemma 6 *Suppose \mathcal{F} is an ordered ABA-free hypergraph on vertex set S . Let $F \in \mathcal{F}$ be a smallest hyperedge in the above ordering. If we reorder S as (F, \bar{F}) , i.e., the vertices of F go to the front but otherwise the order inside F and \bar{F} unchanged, then the ordered hypergraph remains ABA-free.*

Proof. Let us denote the original order by $<$ and the new one by \prec . Suppose on the contrary, that for some $A, B \in \mathcal{F}$ we have some $a, c \in A \setminus B$ and $b \in B \setminus A$ that satisfy $a \prec b \prec c$. The proof is a simple case analysis of how this could happen. Notice that $c \in F$ implies $b \in F$ and $b \in F$ implies $a \in F$, so there are four cases. If $a, b, c \in F$ or $a, b, c \notin F$, then $a < b < c$. In this case A and B contradict that \mathcal{F} is ABA-free. If $a \in F$ and $b, c \notin F$, then also $b < c < a$. In this case $B < F$, contradicting that F is smallest. If $a, b \in F$ and $c \notin F$, then also $c < a < b$. In this case $A < F$, contradicting that F is smallest.

Next we show a lemma that if we modify the order of the vertices leaving their circular order unchanged, then we get another ABA-free hypergraph as follows.

Lemma 7 *Suppose \mathcal{F} is an ordered ABA-free hypergraph on vertex set S . Take a partition of its vertices $S = (Y, Z)$ such that the vertices in Y precede the ones in Z . Then $\mathcal{F}' = \mathcal{F} \Delta Y = \{F \Delta Y \mid F \in \mathcal{F}\}$ is an ordered ABA-free hypergraph on the vertices ordered as (Z, Y) , i.e., Z precedes Y but otherwise the order inside Y and Z is unchanged.*

Proof. It is enough to show the statement if $|Y| = 1$, as then by induction we can proceed with the vertices of $|Y| > 1$ one by one. Let us denote the original order by $<$ and the new one by \prec . It is enough to show that if for some $A, B \in \mathcal{F}$ we have $A < B$, then for $A' = A \Delta Y, B' = B \Delta Y \in \mathcal{F}'$ we also have $A' \prec B'$. Denote $\{y\} = Y$. If $y \notin A \cup B$, then $A \Delta B$ is unchanged by the transformation, thus $A' \prec B'$. If $y \in A \setminus B$, then $y \in B' \setminus A'$, so in A', B' an ABA-sequence in the new order would imply that $B < A$, which is not possible. Thus $A' \prec B'$ (with allowing $A' = \emptyset$) or they are incomparable. Finally, $y \in B \setminus A$ is not possible if $A < B$.

Lemma 7 shows that instead of our ordering we could consider some sort of circular order but we have found that it would only make the arguments more complicated. We proceed with another definition.

Definition 8 A vertex a is *skippable* if there exists an A such that $\min(A) < a < \max(A)$ and $a \notin A$. In this case we say that A skips a . A vertex a is *unskippable* if there is no such A .

Observation 9 If a vertex a is unskippable in some ABA-free hypergraph \mathcal{H} , then after adding the one-element edge $\{a\}$ to \mathcal{H} , it remains ABA-free.

Note that the following two lemmas show that the unskippable vertices of an ABA-free hypergraph behave similarly to vertices on the convex hull of a hypergraph on a point set defined by halfplanes. These two lemmas make possible to use the framework of [26] on ABA-free hypergraphs.

Lemma 10 If \mathcal{H} is finite ABA-free, then every $A \in \mathcal{H}$ contains an unskippable vertex.

Remark 2. Note that finiteness is needed, as the hypergraph whose vertex set is \mathbb{Z} and edge set is $\{\mathbb{Z} \setminus \{n\} \mid n \in \mathbb{Z}\}$ is ABA-free without unskippable vertices.

Proof. Take an arbitrary set $A \in \mathcal{H}$, suppose that it does not contain an unskippable vertex, we will reach a contradiction. Call $a \in A$ *rightskippable* if there is a $B \in \mathcal{H}$ rightskipping a , that is for which $a \in A \setminus B$ and there are $b_1, b_2 \in B$ such that $b_1 < a < b_2$ where $b_2 \in B \setminus A$.

If A contains no unskippable vertex, $\max(A)$ must be rightskippable (the set skipping $\max(A)$ must also rightskip $\max(A)$). Also, $\min(A)$ cannot be rightskippable, as otherwise A and the set B rightskipping $\min(A)$ would violate ABA-freeness (we would get $b_1 < \min(A) < b_2$ where $b_1, b_2 \in B \setminus A, \min(A) \in A \setminus B$). Therefore we can take the largest $a \in A$ that is not rightskippable. By the assumption, it is skipped by a set, call it B , i.e., $b_1 < a < b_2$ where $b_1, b_2 \in B \not\subset A$. Moreover, wlog., suppose that b_2 is the smallest element of B which is bigger than a . Since a is not rightskippable, $b_2 \in A$ must also hold. As $b_2 \in A$ is rightskippable, there is a C such that $c_1 < b_2 < c_2$ where $c_1, c_2 \in C$ and $b_2 \notin C, c_2 \notin A$. Wlog., suppose c_1 is the largest element of C which is smaller than b_2 . If $c_1 < a$, then C would rightskip a , a contradiction. Thus, $b_1 < a \leq c_1$. As $c_2 \notin A$, also $c_2 \notin B$, otherwise B would rightskip a . Putting all together, we get $c_1 < b_2 < c_2$, thus B and C contradict ABA-freeness.

Recall that a hypergraph is called *Sperner* if no two of its sets (i.e., hyperedges) contain each other, we further assume in it that the hypergraph is nonempty, i.e., it contains at least one edge.

Lemma 11 If \mathcal{H} is finite, ABA-free and Sperner, then any minimal hitting set of \mathcal{H} that contains only unskippable vertices is 2-shallow.

Proof. Let R be a minimal hitting set of unskippable vertices. Assume to the contrary that there exists a set A such that $|A \cap R| \geq 3$. Let $l = \min(A \cap R)$ and $r = \max(A \cap R)$. There exists a third vertex $a \in A \cap R$ with $l < a < r$. We claim that $R' = R \setminus \{a\}$ hits all sets of \mathcal{H} , contradicting its minimality. Assume on the contrary that R' is disjoint from some $B \in \mathcal{H}$. As R must hit B , we have

$R \cap B = \{a\}$. If there is a $b \in B \setminus A$ such that $l < b < r$, that would contradict the ABA-free property. If there is a $b \in B$ such that $b < l < a$ or $a < r < b$, that would contradict that l and r are unskippable. Thus $B \subset A$, contradicting that \mathcal{H} is Sperner.

Now we present the framework of [26] modified for ABA-free hypergraphs. Our algorithm to give a polychromatic k -coloring of the vertices of an ABA-free hypergraph with edges of size at least $2k - 1$ is as follows.

Algorithm 12 *At the beginning we are given an ABA-free hypergraph \mathcal{H} with edges of size at least $2k - 1$. Repeat k times ($i = 1, \dots, k$) the **general step i of the algorithm**:*

At the beginning of step i we have an ABA-free hypergraph \mathcal{H} with edges of size at least $2k - 2i + 1$. If any edge contains another, then we delete the bigger edge, thus making our hypergraph Sperner. Repeat this until no edge contains another. Next, we take the set of all unskippable vertices, which is a hitting set by Lemma 10 and delete vertices from this set until it becomes a minimal hitting set R . By Lemma 11 R is a 2-shallow hitting set, color its vertices with the i -th color. Delete these vertices from \mathcal{H} (the edges of the new hypergraph are the ones induced by the remaining vertices). Using Observation 5, at the end of this step i , we have an ABA-free hypergraph with edges of size at least $2k - 2i - 1$.

Algorithm 12 implies the following theorem.

Theorem 13 *Given a finite ABA-free \mathcal{H} we can color its vertices with k colors such that every $A \in \mathcal{H}$ whose size is at least $2k - 1$ contains all k colors.*

Notice that the above theorem is sharp, as taking \mathcal{H} to be all subsets of size $2k - 2$ from $2k - 1$ vertices, in any coloring of the vertices, one color must occur at most once and is thus missed by some edge.

We state another corollary of Lemma 10 that we need later. Before that, we need another simple claim.

Proposition 14 *Suppose we insert a new vertex, v , somewhere into the (ordered) vertex set of an ABA-free hypergraph, \mathcal{H} , and add v to every edge that contains a vertex before and another vertex after v , then we get an ABA-free hypergraph.*

Proof. We show that if in the new hypergraph, \mathcal{H}' , two hyperedges A' and B' violate ABA-freeness, then we can find two hyperedges A and B in the original hypergraph, \mathcal{H} that also violate ABA-freeness, which would be a contradiction. We define $A = A' \setminus \{v\}$ and $B = B' \setminus \{v\}$. If both A' and B' contain or do not contain v , then from the definition A and B also violate the condition. If, say, $v \notin A'$ and $v \in B'$, then without loss of generality we can suppose that all the vertices of $A = A'$ are before v . This means that if there is a 3 element alternating sequence in A' and B' , then v can only be its third vertex. But as B' has an element that is bigger than v , we can replace v in this case by a bigger element of B' . Therefore, we can also find a 3 element alternating sequence in A and B , a contradiction.

Lemma 15 *If \mathcal{H} is ABA-free, $A \in \mathcal{H}$, then there is a vertex $a \in A$ such that we can extend \mathcal{H} with $A' = A \setminus \{a\}$, i.e., $\mathcal{H} \cup \{A'\}$ is also ABA-free.*

Proof. If $|A| = 1$, then trivially \mathcal{H} can be extended with \emptyset . If $|A| > 1$, then we proceed by induction on the size of A . Using Lemma 10, there is an unskippable vertex $v \in A$. Delete this vertex from \mathcal{H} to obtain some ABA-free \mathcal{H}_v and let $A_v = A \setminus \{v\}$. Using induction on A_v , there is an $A'_v = A_v \setminus \{a\}$ such that $\mathcal{H}_v \cup \{A'_v\}$ is also ABA-free. We claim that adding to \mathcal{H} the set $A' = A'_v \cup \{v\} = A \setminus \{a\}$, $\mathcal{H} \cup \{A'\}$ is also ABA-free.

Notice that adding back v to \mathcal{H}_v is very similar to the operation of Proposition 14, as v is unskippable in \mathcal{H} . The only difference is, that we might also have to add it to some further hyperedges, ending in or starting at v . But a hyperedge that contains v cannot form a 3 element alternating sequence with A' , since it also contains v , so the corresponding hyperedges in \mathcal{H}_v would also contain a 3 element alternating sequence.

Notice that with the repeated application of Lemma 15 we can extend any ABA-free hypergraph, such that in any set A there is a vertex a for which $\{a\}$ is a singleton edge, implying that a is unskippable in A . Thus in fact Lemma 15 is equivalent to Lemma 10. Moreover, in Section 3, in the more general context of pseudohalfplanes, it will be the abstract equivalent of a known and important property of pseudohalfplanes.

We prove another interesting property of ABA-free hypergraphs before which we need a definition.

Definition 16 *The dual of a hypergraph \mathcal{H} , denoted by \mathcal{H}^* , is such that its vertices are the hyperedges of \mathcal{H} and its hyperedges are the vertices of \mathcal{H} with the same incidences as in \mathcal{H} .*

Proposition 17 *If \mathcal{H} is ABA-free, then its dual \mathcal{H}^* is also ABA-free with respect to some ordering of its vertices.*

Proof. Take the partial order “ $<$ ” of the hyperedges of \mathcal{H} and extend this arbitrarily into a total order, it is easy to check that \mathcal{H}^* is ABA-free with this order.

Corollary 18 *The edges of every $(2k-1)$ -uniform ABA-free hypergraph can be colored with k colors, such that if a vertex v is in a subfamily \mathcal{H}_v of at least $m(k) = 2k-1$ of the edges of \mathcal{H} , then \mathcal{H}_v contains a hyperedge from each of the k color classes.*

Corollary 19 *Any $(2k-1)$ -fold covering of a finite point set with the translates of an unbounded convex planar set is decomposable into k coverings.*

3 Pseudohalfplanes

Here we extend a result of Smorodinsky and Yuditsky [26]. A *pseudoline arrangement* is a finite collection of simple curves in the plane such that any two

are either disjoint or intersect once and in the intersection point they cross. We suppose that they are in general position, i.e. no three curves have a common point. Some well-known results about pseudoline arrangements are collected in Appendix A, which can be found in [2]. We also recommend [7] where generalizations of classical theorems are proved for *topological affine planes*. From these, it follows that the hypergraphs defined by points contained in pseudohalfplanes have the following structure.

Definition 20 *A hypergraph \mathcal{H} on an ordered set of points S is called a pseudohalfplane-hypergraph if there exists an ABA-free hypergraph \mathcal{F} on S such that $\mathcal{H} \subset \mathcal{F} \cup \bar{\mathcal{F}}$.*

Note that $\bar{\mathcal{F}}$ is also ABA-free with the same ordering of the points. We refer to the edges of a pseudohalfplane-hypergraph also as pseudohalfplanes.

Using Lemma 15 on a hyperedge of a pseudohalfplane-hypergraph, we get the following.

Proposition 21 *Given a pseudohalfplane-hypergraph \mathcal{H} , and an edge A of \mathcal{H} , we can add a new hyperedge A' contained completely in A that contains all but one of the points of A , such that \mathcal{H} remains a pseudohalfplane-hypergraph.*

In the geometric setting this corresponds to the known and useful fact that given a pseudohalfplane arrangement and a finite set of points A contained in the pseudohalfplane H , we can add a new pseudohalfplane H' contained completely in H that contains all but one of the points of A .

Now we show how to extend Theorem 13 to pseudohalfplane arrangements, i.e., to the case when the points of S below a line also define a hyperedge. The proof is based on the proof from Section 2 and can be found in Appendix B.

Theorem 22 *Given a finite set of points S and a pseudohalfplane arrangement \mathcal{H} , we can color S with k colors such that any pseudohalfplane in \mathcal{H} that contains at least $2k - 1$ points of S contains all k colors. Equivalently, the vertices S of a finite pseudohalfplane-hypergraph can be colored with k colors such that any hyperedge containing at least $2k - 1$ points contains all k colors.*

We remark that a similar statement is not true for the union of two arbitrary ABA-free hypergraphs (instead of an ABA-free hypergraph and its complement), in fact the union of two arbitrary ABA-free hypergraphs might not be 2-colorable, see [23] for such a construction.

4 The dual problem and signed ABA-free hypergraphs

We are also interested in coloring pseudohalfplanes with k colors such that all points that are covered many times will be contained in each k colors. For example, we can also generalize the dual result about coloring halfplanes of [26] to pseudohalfplanes.

Theorem 23 *Given a pseudohalfplane arrangement \mathcal{H} , we can color \mathcal{H} with k colors such that if a point p belongs to a subset \mathcal{H}_p of at least $3k - 2$ of the pseudohalfplanes of \mathcal{H} then \mathcal{H}_p contains a pseudohalfplane of every color.*

Theorem 23 follows from Theorem 27, that we will state and prove later.

However, instead of coloring pseudohalfplanes, we stick to coloring points with respect to pseudohalfplanes and work with *dual hypergraphs*, where the vertex-hyperedge incidences are preserved, but vertices become hyperedges and hyperedges become vertices. Since we have already seen in Section 3 the equivalence of our abstract definition and the standard definition of pseudohalfplane arrangement, we can use the well-known properties of the dual arrangement (see, e.g., [2]) to obtain the following.

Proposition 24 *A dual pseudohalfplane-hypergraph is a hypergraph \mathcal{H} on an ordered set of vertices S such that there exists a set $X \subset S$ and an ABA-free hypergraph \mathcal{F} on S such that the edges of \mathcal{H} are the edges $F \Delta X$ for every $F \in \mathcal{F}$.*

The proof of Proposition 24 is in Appendix C. Now we define a common generalization of the primal and dual definitions.

Definition 25 *A signed pseudohalfplane-hypergraph is a hypergraph \mathcal{H} on an ordered set of vertices S such that there exists a set $X \subset S$ and an ABA-free hypergraph \mathcal{F} on S such that the edges of \mathcal{H} are some subset of $\{F \Delta X, \bar{F} \Delta X \mid F \in \mathcal{F}\}$.*

It is easy to see that the dual of such a signed pseudohalfplane-hypergraph is also a signed pseudohalfplane-hypergraph, just like in Proposition 17. Furthermore, there is a nice geometric representation of such hypergraphs; \mathcal{H} is a signed pseudohalfplane-hypergraph if and only if there is a set of points, S , on the surface of a sphere and a *pseudohalfsphere arrangement* \mathcal{F} on the sphere such that the incidences among S and \mathcal{F} give \mathcal{H} . (Here we omit the exact definition of pseudohalfsphere arrangements, which are a generalization of a collection of some halfspheres of a sphere. The interested reader can find it in [2].)

Another popular geometric representation on the plane, adding *signs* to lines and points, is the following. The vertices correspond to a set of points in the plane together with a direction (up or down), and the edges correspond to a set of (x-monotone) pseudolines with a sign (+ or -). The hyperedge corresponding to a positive pseudoline is the set of points that point *towards* the pseudoline, while the hyperedge corresponding to a negative pseudoline is the set of points that point *away* the pseudoline. Positive pseudolines correspond to \mathcal{F} , negative pseudolines to $\bar{\mathcal{F}}$, up points correspond to X and down points correspond to \bar{X} . With this interpretation, ABA-free hypergraphs have only + and up signs, pseudohalfplane-hypergraphs have \pm and up signs, dual pseudohalfplane-hypergraphs have + and up/down signs.

In the next table we summarize the best known results about these hypergraphs, with respect to how many points each edge has to contain to have a

| | Polychromatic k -coloring | Shallow hitting set |
|------------------------------------|------------------------------|-----------------------|
| ABA-free hypergraphs | $2k - 1$ (Theorem 13) | 2 (Lemma 11) |
| Pseudohalfplane-hypergraphs | $2k - 1$ (Theorem 22) | 2 (Lemma 30) |
| Dual pseudohalfplane-hypergraphs | $\leq 3k - 2$ (Theorem 23) | ≤ 3 (Theorem 27) |
| Signed pseudohalfplane-hypergraphs | $\leq 4k - 3$ (Corollary 26) | ? |

polychromatic k -coloring and the values of the smallest c for which there exists a c -shallow hitting set for Sperner families.

We conjecture that even Sperner pseudohalfsphere arrangements have a 2-shallow hitting set, which would also imply (using the framework described above Theorem 13) that any family whose sets have size at least $2k - 1$ admits a polychromatic k -coloring, but we could not even prove for any constant c that a c -shallow hitting set exists.

As we can find a polychromatic k -coloring of the points of X and \bar{X} independently with respect to the sets of \mathcal{F} and $\bar{\mathcal{F}}$, respectively, of size at least $2k - 1$ using Theorem 22, the following is true.

Corollary 26 *Given a finite set of points S on the sphere and a pseudohalf-sphere arrangement \mathcal{H} , we can color S with k colors such that any pseudohalf-sphere in \mathcal{H} that contains at least $4k - 3$ points of S contains all k colors. Equivalently, the vertices S of a finite signed pseudohalfplane-hypergraph can be colored with k colors such that any hyperedge containing at least $4k - 3$ points contains all k colors.*

To finish, we prove the following theorem in Appendix D, which, using the usual framework, implies Theorem 23.

Theorem 27 *Every Sperner dual pseudohalfplane hypergraph has a 3-shallow hitting set.*

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A Simple facts about pseudolines

An *infinite* pseudoline arrangement is such that cutting a pseudoline in two, both parts are unbounded. A curve is *graphic* if it is the graph of a function, i.e., an x -monotone infinite curve that intersects every vertical line of the plane. A *graphic* pseudoline arrangement is such that every curve is graphic. We say that two pseudoline arrangements are *equivalent* if there is a bijection between their pseudolines such that the order in which a pseudoline intersects the other pseudolines remains the same. A *pseudohalfplane arrangement* is an infinite pseudoline arrangement, with a side of each pseudoline selected (note that the two sides of a pseudoline are well-defined regions in this case).

Facts about pseudoline arrangements

- I. (Levi Enlargement Lemma) Given a pseudoline arrangement, any two points of the plane can be connected by a new pseudoline (if they are not connected already).
- II. Given a pseudoline arrangement, we can find a(n infinite) pseudoline arrangement in which every pair of pseudolines intersects exactly once, and the order in which a pseudoline intersects the other pseudolines remains the same (ignoring the new intersections).
- III. Given an infinite pseudoline arrangement, we can find an equivalent graphic pseudoline arrangement.

From these facts it follows that in the definition of a pseudohalfplane we can (and will) suppose that the underlying pseudoline arrangement is a graphic pseudoline arrangement.

Notice that ABA-free hypergraphs are in a natural bijection with (graphic) pseudoline arrangements and sets of points, such that each hyperedge corresponds to the subset of points *above* a pseudoline.

Proposition 28 *Given in the plane a set of points S (with all different x -coordinates) and a graphic pseudoline arrangement L , define the hypergraph $\mathcal{H}_{S,L}$ with vertex set S such that for each pseudoline $l \in L$ the set of points above l is a hyperedge of $\mathcal{H}_{S,L}$. Then $\mathcal{H}_{S,L}$ is ABA-free with the order on the vertices defined by the x -coordinates.*

Conversely, given an ABA-free hypergraph \mathcal{H} , there exists a set of points S and a graphic pseudoline arrangement L such that $\mathcal{H} = \mathcal{H}_{S,L}$.

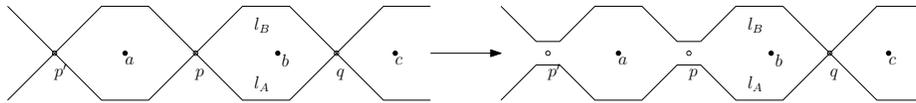


Fig. 1: Redrawing a lens to decrease the number of intersections

Proof. The first part is almost trivial, suppose that there are two hyperedges A, B in $\mathcal{H}_{S,L}$ having an ABA-sequence on the vertices corresponding to the points $a, b, c \in S$. The pseudolines corresponding to the hyperedges A and B are denoted by ℓ_A and ℓ_B . The pseudoline ℓ_A intersects the vertical line through a below a , the vertical line through b above b and the vertical line through c above c , while ℓ_B intersects these in the opposite way (above/below/above). Thus these lines must intersect in the vertical strip between a and b and also in the strip between b and c , thus having two intersections, a contradiction.

The second part of the proof is also quite natural. Given an ABA-free hypergraph $\mathcal{H}(V, E)$ with an ordering on V , we want to realize it with a planar point set S and a graphic pseudoline arrangement L . Let S be $|V|$ points on the x axis corresponding to the vertices in V such that the order on V is the same as the order given by the x -coordinates on S . From now on we identify the vertices of V with the corresponding points of V .

For a given $A \in \mathcal{H}$ it is easy to draw an ℓ_A graphic curve for which the points of S above ℓ_A are exactly in A . Draw a pseudoline ℓ_A for every $A \in \mathcal{H}$, such that there are finitely many intersections among these pseudolines, all of them crossings. What we get is an arrangement of graphic curves, but it can happen that they intersect more than twice. Now among such drawings take one which has the minimal number of intersections, we claim that this is a pseudoline arrangement.

Assume on the contrary, that there are two curves ℓ_A and ℓ_B intersecting (at least) twice. Let two consecutive (in the x -order) intersection points be p and q , where p has smaller x -coordinate than q . Without loss of generality, ℓ_A is above ℓ_B close to the left of p and close to the right of q , while ℓ_A is below ℓ_B in the open vertical strip between p and q . This structure is usually called a lens, and we want to eliminate it in a standard way, decreasing the number of intersections. We can change the part of ℓ_A and ℓ_B to the left of p (and to the right from the intersection p' next to and left of p if there is any) and change their drawing locally around p (and p' if it exists) such that we get rid of the intersection at p , see Figure 1. If there are no points of S between ℓ_A and ℓ_B and to the left of p (and to the right of p'), then this redrawing does not change the hyperedges defined by ℓ_A and ℓ_B , so we get a representation of \mathcal{H} with less intersections, a contradiction. Thus there is a point $(p' <) a < p$ below ℓ_A and above ℓ_B . Similarly, there must be a point $p < b < q$ above ℓ_A and below ℓ_B and finally a point $q < c$ below ℓ_A and above ℓ_B , otherwise we could redraw the pseudolines with less intersections. These three points $a < b < c$ contradict the ABA-freeness of \mathcal{H} as by the definition of the pseudolines, $b \in A \setminus B$ and $a, c \in B \setminus A$.

We remark that similarly, a $(t + 2)$ -as-free hypergraph corresponds to an arrangement of graphic curves that intersect at most t times.

B Proof of Theorem 22

Proof. Our proof is completely about the abstract setting, yet it translates naturally to the geometric setting, also the figures illustrate the geometric interpretations.

By definition there exists an ABA-free \mathcal{F} such that $\mathcal{H} \subset \mathcal{F} \cup \bar{\mathcal{F}}$. Call $\mathcal{U} = \mathcal{H} \cap \mathcal{F}$ the upsets and $\mathcal{D} = \mathcal{H} \cap \bar{\mathcal{F}}$ the downsets, observe that both \mathcal{U} and \mathcal{D} are ABA-free.

Further, the unskippable vertices of \mathcal{U} (resp. \mathcal{D}) are called top (resp. bottom) vertices. The top and bottom vertices are called the unskippable vertices of \mathcal{H} . Recall that by adding these unskippable vertices as one-element edges to \mathcal{H} , \mathcal{H} remains to be a pseudohalfplane-hypergraph, as we can extend \mathcal{F} and $\bar{\mathcal{F}}$ with the appropriate hyperedge (this is a convenient way of thinking about top/bottom vertices in the geometric setting, as seen later in the figures).

Observation 29 *If x is top and X is a downset and $x \in X$, then X contains all vertices that are bigger or all vertices that are smaller than x . The same holds if x is bottom, X is an upset and $x \in X$.*

Lemma 30 *If \mathcal{H} is a finite Sperner pseudohalfplane-hypergraph, then any minimal hitting set of \mathcal{H} that contains only unskippable vertices is 2-shallow.*

Proof. Let R be a minimal hitting set of unskippable vertices. Suppose for a contradiction that $\{a, b, c\} \subset R \cap X$ and $a < b < c$ for some $X \in \mathcal{H}$. Without loss of generality, suppose that b is top. As R is minimal, let B be a set for which $B \cap R = \{b\}$. From Observation 29 it follows that B is an upset.

First suppose that X is an upset. As $B \not\subset X$, take a $b_2 \in B \setminus X$. As B and X are both upsets and thus have the ABA-free property, we have $b_2 < a$ or $c < b_2$. Without loss of generality, we can suppose $c < b_2$. If c is top, $\{c\}$ and B violate ABA-freeness. See Figure 2a. If c is bottom, then using Observation 29, X contains all the vertices that are smaller than c . Take a set $A \not\subset X$ for which $A \cap R = \{a\}$. This set must contain an $a_2 \in A \setminus X$ and so we must have $c < a_2$. If A is an upset, as it does not contain b and recall $a < b < a_2$, A and $\{b\}$ violate ABA-freeness. See Figure 2b. If A is a downset, as it does not contain c and recall $a < c < a_2$, A and $\{c\}$ violate ABA-freeness, both cases lead to a contradiction.

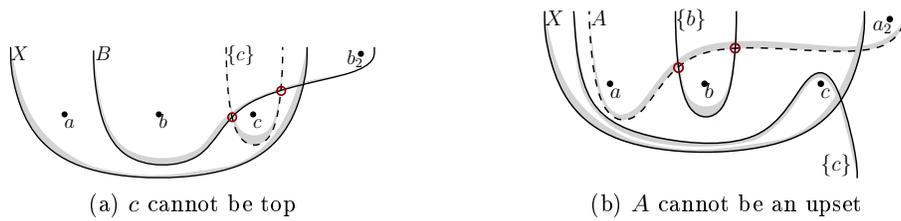


Fig. 2: Proof of Lemma 30

The case when X is a downset is similar. Using Observation 29 for X and $\{b\}$ we can suppose without loss of generality that X contains all vertices that are smaller than b . Take a set $A \not\subset X$ for which $A \cap R = \{a\}$ and an $a_2 \in A \setminus X$. As X contains all vertices smaller than b , we have $b < a_2$. A cannot be an upset, as then it would contain b , so it is a downset. If $b < a_2 < c$, then A and X would violate ABA-freeness, thus we must have $c < a_2$. This means c cannot be bottom, so it is top. Using Observation 29, X contains all the vertices that are smaller than c . But then $B \setminus X$ must have an element that is bigger than c , contradicting the ABA-freeness of B and $\{c\}$.

From here the rest of the proof is the same. Our algorithm to give a k -coloring of the vertices of \mathcal{H} such that every pseudohalfplane of size at least $2k - 1$ contains all k colors is as follows. Using Proposition 21, it is enough to consider pseudohalfplanes of size exactly $2k - 1$. Apply Lemma 30 to select a 2-shallow hitting set R and color its vertices with the first color. Delete these vertices and apply induction on k .

C Proof of Proposition 24

Using Definition 20, let \mathcal{F} be an ABA-free hypergraph that represents the original pseudohalfplane-hypergraph, that is, every pseudohalfplane is equal to a set $F \in \mathcal{F}$ or to a set $F \in \bar{\mathcal{F}}$. Using Proposition 17, the dual of \mathcal{F} is an ABA-free hypergraph, \mathcal{F}^* , whose vertices $\{v_F : F \in \mathcal{F}\}$ correspond to the edges of \mathcal{F} (ordered in some way) and whose edges $\{f_p : p \in S\}$ correspond to the vertices of \mathcal{F} , with incidence relations preserved, i.e., $f_p = \{v_F : p \in F\}$. Now we add also the set of vertices $\{v_{\bar{F}} : \bar{F} \in \bar{\mathcal{F}}\}$ corresponding to the edges of $\bar{\mathcal{F}}$. Each vertex $v_{\bar{F}}$ is put right after vertex v_F in the order. The edges change in the following way. As in \mathcal{F} we have $p \notin F \in \mathcal{F}$ if and only if $p \in \bar{F}$, in the dual the corresponding edge is $h_p = \{v_F : p \in F\} \cup \{v_{\bar{F}} : p \in \bar{F}\}$ contains exactly one of v_F and $v_{\bar{F}}$. First, without loss of generality, we can suppose that for every $A \in \mathcal{F}$, \mathcal{H} contains at least one of $A \in \mathcal{F}$ and $\bar{A} \in \bar{\mathcal{F}}$, otherwise we can delete A from \mathcal{F} too. Further, we can suppose that \mathcal{H} contains exactly one of $A \in \mathcal{F}$ and $\bar{A} \in \bar{\mathcal{F}}$, as if it contains both, we can add another copy A' of A to \mathcal{F} (\mathcal{F} is then a(n) ABA-free) multihypergraph) and regard \bar{A} as \bar{A}' . This way it can never happen that both $A \in \mathcal{H}$ and $\bar{A} \in \mathcal{H}$, thus in the dual only one of the corresponding vertices are present. Thus, we can relabel to w_A the one vertex that is present in \mathcal{H} among v_A and $v_{\bar{A}}$. After the relabeling we have $V = \{w_A : A \in \mathcal{F}\}$. Denote by X the set of vertices of V for which $w_A = v_{\bar{A}}$. Now an arbitrary edge $h_p = \{v_F : p \in F\} \cup \{v_{\bar{F}} : p \in \bar{F}\} = \{w_F : p \in F \cap \bar{X}\} \cup \{w_F : p \in \bar{F} \cap X\} = f'_p \Delta X$, where Δ denotes the symmetric difference of two sets and $f'_p = \{w_F : v_F \in f_p\}$, i.e., f_p injected in the natural way into the relabeled set V . These f'_p define the same (up to this projection) ABA-free hyperedge \mathcal{G} as \mathcal{F}^* . \square

D Proof of Theorem 27

The proof of this result follows again closely the argument of [26]. We note that the next few statements can also be proved using the geometric representation, but here we develop further our completely abstract approach. The reason for this is to demonstrate the power of our method, hoping that in the future it enables attacking completely different problems as well.

Lemma 31 *Suppose \mathcal{H} is an ordered signed pseudohalfplane-hypergraph on vertex set S defined by the ABA-free hypergraph \mathcal{F} and X , i.e., the edges of \mathcal{H} are $\{F\Delta X \mid F \in \mathcal{F}\}$ and $\{\bar{F}\Delta X \mid F \in \mathcal{F}\}$. Take a partition of its vertices $S = (Y, Z)$ such that the vertices in Y precede the ones in Z . If we order the vertices as (Z, Y) (keeping the order inside Y and Z unchanged), then we also get an ordered signed pseudohalfplane-hypergraph, \mathcal{H}' . Moreover, \mathcal{H}' is defined by the ABA-free hypergraph $\mathcal{F}' = \mathcal{F}\Delta Y = \{F\Delta Y \mid F \in \mathcal{F}\}$ and $X' = X\Delta Y$.*

Proof. As $F\Delta Y\Delta X' = F\Delta Y\Delta X\Delta Y = F\Delta X$, the hyperedges of \mathcal{H} and \mathcal{H}' are indeed the same. So we only need that \mathcal{F}' is ABA-free but this is exactly what Lemma 7 states.

Lemma 32 [*Helly's theorem for pseudohalfplanes*] *If any three hyperedges of a pseudohalfplane-hypergraph intersect, then we can add a vertex contained in all the pseudohalfplanes of the arrangement.*

Proof. We prove the dual statement, as it will be more convenient. That is, suppose that we are given a signed pseudohalfplane-hypergraph \mathcal{H} , such that all its edges have a positive sign, i.e., \mathcal{H} has a representing ABA-free \mathcal{F} and vertex set X such that for every $H \in \mathcal{H}$ there is an $F \in \mathcal{F}$ such that $H = F\Delta X$. Then if for any three vertices there exists an edge that contains all three of them, then we can add the hyperedge \bar{X} to \mathcal{F} such that it stays ABA-free. This is indeed the dual equivalent of the statement, as $\bar{X}\Delta X \in \mathcal{H}$ contains all the vertices.

For a contradiction, suppose that \bar{X} and some $F \in \mathcal{F}$ violate ABA-freeness because of some x, y, z vertices. In this case $x, y, z \in \bar{X}\Delta F$ and then $F\Delta X \in \mathcal{H}$ avoids all of x, y, z . Also, by our assumption, there exists another edge $G\Delta X$ which contains all of x, y, z , thus G and \bar{X} contain the same subset of x, y, z . Thus $F, G \in \mathcal{F}$ contain an ABA-sequence on the vertices x, y, z as F, \bar{X} contains an ABA-sequence on x, y, z , a contradiction.

Applying this to the complementers of the pseudohalfplanes we get the following.

Corollary 33 *Given a pseudohalfplane-hypergraph, either there are already three hyperedges that cover all the vertices, or we can add a vertex which is in none of the hyperedges.*

Lemma 34 *If all the hyperedges of a signed pseudohalfplane-hypergraph \mathcal{H} avoid some vertex p in S , then \mathcal{H}' , the dual hypergraph of \mathcal{H} , is an unsigned pseudohalfplane-hypergraph.*

Proof. Start with a representation of \mathcal{H} : an ABA-free hypergraph \mathcal{F} and a point set X such that every H equals $F\Delta X$ for some $F \in \mathcal{F}$ or $F \in \bar{\mathcal{F}}$. Apply Lemma 31 with Y being the vertices before p , this way we get a representation of \mathcal{H} in which p is the first point. Take \mathcal{H}' , the dual of \mathcal{H} , with representation \mathcal{F}' and X' . In \mathcal{H}' , the set corresponding to p is $H_p = F_p\Delta X'$ for some $F_p \in \mathcal{F}'$, where we can choose the representation such that F_p is the smallest set of \mathcal{F}' (as p was the smallest point of \mathcal{F}). Now apply Lemma 6 to get another representation of \mathcal{H}' in which the points of F_p are at the beginning in the order. As p was a point that was in none of the edges of \mathcal{H} , in the dual H_p contains no points and so $F_p = H_p\Delta X' = \emptyset\Delta X' = X'$. Now apply again Lemma 31 to \mathcal{F}' with $Y = X'$. We get a representation of \mathcal{H}' in which the sign set of the points is $X'' = X'\Delta X' = \emptyset$, that is, \mathcal{H}' is an unsigned pseudohalfplane-hypergraph.

Now we are ready to prove Theorem 27 which implies Theorem 23. Again our proof follows closely the proof of Smorodinsky and Yuditsky [26].

Proof. If there is a set \mathcal{H}_k of at most 3 pseudohalfplanes covering every point, then these form a 3-shallow hitting set. Otherwise, by Corollary 33 we can add a point that is in none of the pseudohalfplanes. Note that adding a point makes our problem only harder, i.e., if we find a shallow hitting set (which is a collection of pseudohalfplanes) then it is a shallow hitting set in the original problem too.

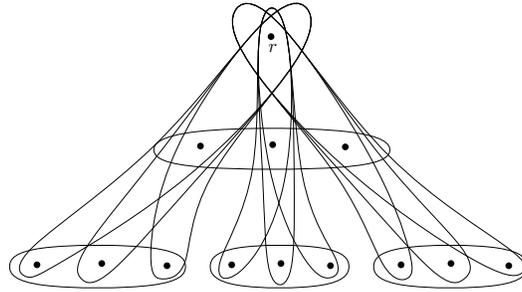
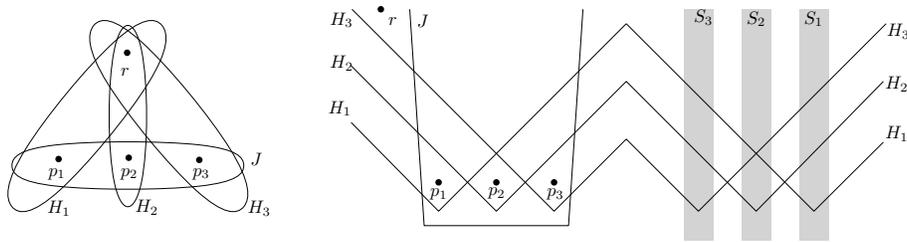
In this case, we can dualize according to Lemma 34 to get an unsigned pseudohalfplane-hypergraph. The minimal elements of this hypergraph form a Sperner system, so by Lemma 30 they have a 2-shallow hitting set.

Remark 3. The above argument would also hold for pseudohalfspheres if an analogue of Helly's theorem was true for them, which is probably the case with some weaker constant, just like for halfspheres.

E Small epsilon-nets for pseudohalfplanes

Here we briefly mention the consequences of our results to ϵ -nets of hypergraphs defined by pseudohalfplanes. We omit proofs as they are not hard and can be obtained exactly as the corresponding results in [26].

Let $\mathcal{H} = (V, E)$ be a hypergraph where V is a finite set. Let $\epsilon \in (0, 1]$ be a real number. A subset $N \subseteq V$ is called an ϵ -net if for every hyperedge $S \in E$ such that $|S| \geq \epsilon|V|$, we also have $S \cap N \neq \emptyset$, i.e., N is a hitting set for all "large" hyperedges. It is known that hypergraphs with VC-dimension d have small ϵ -nets (of size $O(d/\epsilon \log(1/\epsilon))$) [10] and in general this is best possible [16]. However, for geometric hypergraphs this is usually not optimal, in particular for halfplanes the following is true. Consider a hypergraph $\mathcal{H} = (P, E)$ where P is a finite set of points in the plane and $E = \{P \cap H \mid H \text{ is a halfplane}\}$. For this hypergraph there is an ϵ -net of size $2/\epsilon - 1$ for every ϵ [27,26]. Theorem 22 implies that the same bound holds if the hypergraph is defined by pseudohalfplanes instead of halfplanes. Also, for the dual hypergraph $\bar{\mathcal{H}}$ Theorem 23 implies that for $\epsilon \leq 2/3$ there exists an ϵ -net of size $2/\epsilon$. Note that our results are in fact stronger as

Fig. 3: \mathcal{H}_3 Fig. 4: \mathcal{H}'_2 and its realization with pseudoparabolas (for $k = 3$)

in the appropriate polychromatic coloring each color class intersects all large enough hyperedges, thus we get a partition of the vertices into ϵ -nets (and at least one of them is a small ϵ -net by the pigeonhole principle).

F ABAB, bottomless rectangles and more

F.1 ABAB-free hypergraphs that are not two-colorable

We show that there are ABAB-free hypergraphs that do not have a proper 2-coloring. We prove this by ordering the vertices of a non-2-colorable hypergraph \mathcal{H}_k in a tricky way to give an ABAB-free hypergraph. First we define this hypergraph \mathcal{H}_k often used in counterexamples, e.g., [20].

Definition 35 Let G_k be the complete k -ary tree of depth k , i.e., the rooted tree such that its root r has k children, each vertex of G_k in distance at most $k - 2$ from r has k children and the vertices in distance $k - 1$ from r are the leaves (without children).

\mathcal{H}_k is the k -uniform hypergraph which has two types of hyperedges. First, for every non-leaf vertex the set of its children form an edge. Second, the vertices of every descending path starting in r and ending in a leaf form an edge.

It is easy to see that \mathcal{H}_k is not two-colorable. Now we show how to realize \mathcal{H}_k such that its vertices correspond to points in the plane and its hyperedges

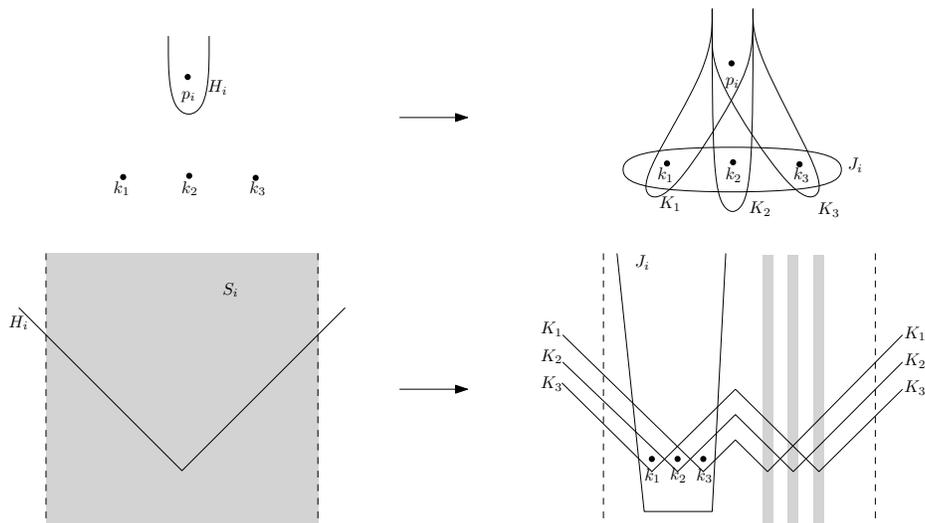


Fig. 5: Recursive realization of \mathcal{H}'_l : adding k children to a leaf

correspond to the points *above* pseudoparabolas (simple curves such that any two intersect at most *twice*). This implies that the x -coordinates define an ordering of the vertices of \mathcal{H}_k showing that \mathcal{H}_k is ABAB-free. We fix k and define \mathcal{H}'_l (resp. G'_l) to be the hypergraph (resp. graph) induced by \mathcal{H}_k (resp. G_k) and the subset of the vertices that are in distance at most $l - 1$ from the root r in G_k (\mathcal{H}'_l is a simple hypergraph, i.e., if multiple edges induce the same edge, we take it only once). Thus in particular G'_1 has one vertex and no edges while \mathcal{H}'_1 has one vertex and one edge containing it, while $\mathcal{H}'_k = \mathcal{H}_k$ and $G'_k = G_k$. Note that in G'_l every non-leaf vertex has k children, and \mathcal{H}'_l has hyperedges of size l corresponding to descending paths (which we usually denote by H_i for some i) and hyperedges of size k corresponding to the set of children of some vertex (which we usually denote by J_i for some i). See Figure 3.

In our realization, to simplify the presentation, points corresponding to vertices will be denoted with the same label, and similarly hyperedges and the corresponding pseudoparabolas will have the same label.

We will recursively realize \mathcal{H}'_l , for an illustration see Figure 5. We additionally maintain that each hyperedge (pseudoparabola) H_i corresponding to a descending path has a vertical strip S_i associated to it, such that inside S_i there are no points and H_i has the lowest boundary (thus no other hyperedge intersects H_i inside S_i). For $l = 1$, this is trivial to do as \mathcal{H}'_1 has one vertex and one edge containing this vertex. For $l = 2$, Figure 4 shows a way to achieve this (for $k = 3$). Now suppose that for some l we have \mathcal{H}'_l and we want to construct \mathcal{H}'_{l+1} . Take the construction of \mathcal{H}'_l , and for each hyperedge H_i corresponding to a descending path P_i with endvertex p_i , do the following. First make k vertically translated copies of H_i very close to each other. Denote these by K_1, K_2, \dots, K_k . Next, using

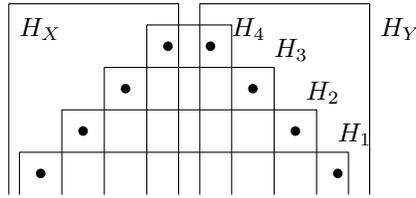


Fig. 6: A Sperner bottomless rectangle family without a shallow hitting set

these k copies of H_i , realize \mathcal{H}'_2 (except the root r) in an appropriately small area inside S_i , by adding k more points k_1, k_2, \dots, k_k such that for every i , k_i is above K_i and below every other pseudoparabola. These points correspond to the children of p_i . Finally, define the pseudoparabola J_i , which corresponds to the hyperedge containing all the k_i 's but no other vertex, as a parabola very close to the vertical strip containing the k_i 's. For each i , the vertical strip that belongs to K_i in the inner copy of \mathcal{H}'_2 is the strip corresponding to the descending hyperedge that ends at k_i . Therefore all properties are maintained, and by repeating the above procedure for each of the leafs p_i of \mathcal{H}'_1 we get a realization of \mathcal{H}'_{t+1} . \square

We are not aware of any nice characterization for the dual of ABAB-free hypergraphs, like we had for ABA-free hypergraphs in Proposition 17.

F.2 Bottomless rectangles and balanced colorings

Every hypergraph given by a set of points and a collection of bottomless rectangles is ABAB-free, but not necessarily ABA-free. In fact, it is not hard to see that such hypergraphs would correspond exactly to “aBAb”-free hypergraphs, which can be defined similarly to Definition 1 as follows.

Definition 36 *A hypergraph whose vertices are real numbers is aBAb-free if for any two of its hyperedges, A and B , and vertices $x_1 < x_2 < x_3 < x_4$ it does not hold that $x_1 \in A$, $x_2 \in B \setminus A$, $x_3 \in A \setminus B$, $x_4 \in B$.*

It was shown in [1] that any finite set of points can be colored with k colors such that any bottomless rectangle that contains at least $3k - 2$ points contains a point of every color. Unfortunately, we were not able to prove this using our methods, because Sperner bottomless rectangle families do not have a shallow hitting set, as shown by the following example.

Example 37 *Consider the set of points $X = \{(i, i) \mid i = 1..k\}$ and $Y = \{(k + i, k + 1 - i) \mid i = 1..k\}$ and the bottomless rectangle family that consists of the following.*

- I. A rectangle H_X containing X .
- II. A rectangle H_Y containing Y .
- III. Rectangles H_i containing (i, i) and $(2k + 1 - i, i)$ for $i = 1..k$.

Any hitting set for the H_i rectangles contains $k/2$ points from X or Y , thus it is not $(k/2 - 1)$ -shallow for H_X or H_Y (for an illustration for $k = 4$ see Figure 6).

Instead of shallow hitting sets, we can ask whether a k -coloring exists for any Sperner bottomless rectangle family that satisfies a certain nice property, that can be achieved by repeatedly finding c -shallow hitting sets and making each of them a separate color class. In the proofs in earlier sections, after k shallow hitting sets were found and colored to different colors, we did not care about the remaining points, they were colored arbitrarily. Instead, we could find a $(k+1)$ -st shallow hitting set for the remaining points and use the first color for them, then the second color for the $(k+2)$ -nd shallow hitting set, and so on, until there are no more points left. In general in the i -th step the shallow hitting set is colored with color $i \pmod k$, where color 0 and color k denote the same color. This way we achieve a coloring that is not just polychromatic, but also has the following *balanced* property.

Definition 38 *We say that a k -coloring is c -balanced if for any given set (hyperedge) of our family denoting the sizes of any two color classes in it by n_1 and n_2 , then we have $n_1 \leq c(n_2 + 1)$.*

As we have seen above, if a family has a c -shallow hitting set, then it also has a c -balanced k -coloring for any k . For uniform families, a converse also holds; if every set has size n , then any color class of a c -balanced n/c -coloring is a c^2 -shallow hitting set. For non-uniform families, however, these notions can differ, so it is natural to ask the following.

Problem 39 *Is there a balanced coloring for any family of bottomless rectangles?*

Example 37 generalizes easily to other families, such as the translates or homothets of a convex polygon, so there is not much hope to achieve shallow hitting sets for other interesting planar families. We do not, however, know whether a balanced coloring exists for the above families.