Bounds for the symmetric difference of generalized Marcum Q-functions

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***Abstract*—Recently, an approximation for large values of a and b for the symmetric difference of Marcum Q-functions $Q_\nu(a,b)$ was given in [1] in the case of integer order, i.e. when $\nu = n \in \mathbb{N}$. Motivated by this result, in this note we study the symmetric difference of Marcum Q-functions $Q_\nu(a,b)$ of real order $\nu \geq 1$ for the parameters $a > b > 0$. Our aim is to use some of the lower and upper bounds of the Marcum Q-function that appear in the literature to obtain some tight bounds for the symmetric difference. Another approach, presented in this note, is to investigate the difference via closed forms of the Marcum Q-function.**

***Index Terms*—Symmetric difference of Marcum Q-functions; lower and upper bounds; approximations.**

I. Introduction

The generalized Marcum Q-function is defined by

$$Q_\nu(a,b) = \frac{1}{a^{\nu-1}} \int_b^\infty t^\nu e^{-\frac{(t^2+a^2)}{2}} I_{\nu-1}(at)\,\mathrm{d}t, \qquad (1)$$

where $b \geq 0$ and a, $\nu > 0$ are real numbers. I_ν denotes the modified Bessel function of the first kind of order ν, defined by

$$I_\nu(x) = \sum_{n\geq 0} \frac{(x/2)^{2n+\nu}}{n!\Gamma(\nu+n+1)}.$$

This function was introduced over 60 years ago and it is the subject of many papers and studies ever since. Initially it was used in the study of target detection probability in radar communications, see [2] and [3], where ν denotes the number of independent samples of the output of a detector, being an integer. Nowadays it has many important role in other fields as well, and the study of its behavior for real parameters is required. Some of its applications appear in communication theory, for instance in the analysis of bit error performance (see [4]) in the characterization of quadratic forms in complex Gaussian random variables and in the energy detection of unknown signals over fading channels.

In the literature sometimes the study of the symmetric difference

$$\Delta Q_\nu(a,b) = Q_\nu(a,b) - Q_\nu(b,a) \qquad (2)$$

is required. This is the case for example when in [4] the authors characterize the bit error rate performance (BER) of digital communication systems with quadratic-form receivers. Since this function and its symmetric difference has a rather complicated nature, many papers (see [5], [6]) have studied its behavior due to monotonicity or log-concavity properties, closed-form expressions, aiming to derive lower and upper bounds.

In [1] the authors approximated the symmetric difference (2) for $\nu \in \mathbb{N}$ and large values of $a > b \geq 0$ by

$$\Delta Q_\nu(a,b) \approx 1 - \left[\left(\frac{a}{b}\right)^{\nu-\frac{1}{2}} + \left(\frac{b}{a}\right)^{\nu-\frac{1}{2}}\right] Q(a-b), \qquad (3)$$

where $Q(\cdot)$ is the Gaussian Q-function, expressed by

$$Q(a) = \frac{1}{\sqrt{2\pi}} \int_a^\infty e^{-\frac{t^2}{2}}\,\mathrm{d}t \qquad (4)$$

Fig. 1 and 2 show us that although approximation (3) is a quite good one for big values of a and b, it is not satisfying for small values.

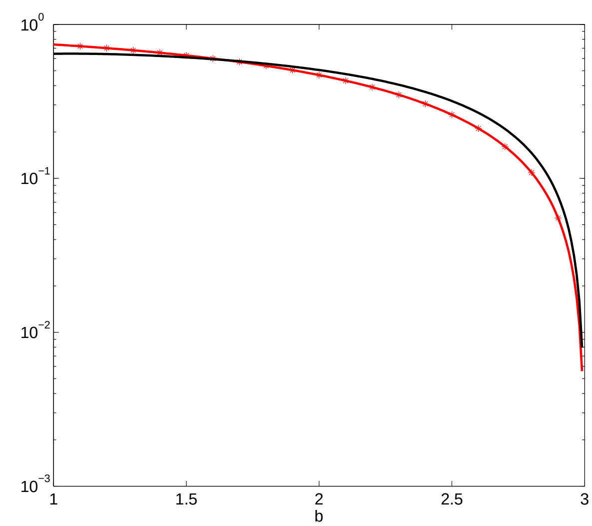

Fig. 1. Symmetric difference of the generalized Marcum Q-functions for $a = n = 3$. 'x': exact; straight line: approximation (3)

Motivated by the above result, in the next sections we present some lower and upper bounds based on the main results in [5], [6]. The advantage of this approach is that we are able to establish bounds also in the case of real valued parameter ν. We also present some closed forms for the symmetric difference by using some known results from the literature.

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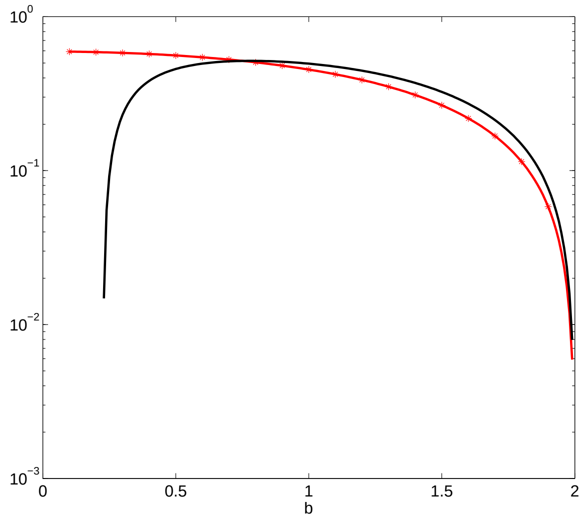

Fig. 2. Symmetric difference of the generalized Marcum Q-functions for $a = n = 2$. 'x': exact; straight line: approximation (3)

II. Bounds for the symmetric difference of Marcum Q-functions for integer parameter

In this section we present some upper and lower bounds for the symmetric difference of Marcum Q-functions in the case of integer parameter (i.e. $\nu = n \in \mathbb{N}$). Our computations are based on the bounds presented in [6], where the authors distinguished the cases $b \geq a$ and $a > b$, but providing lower and upper bounds for the Marcum Q-function $Q_n(a,b)$ in both cases. These bounds, [6, eq 2.7-2.9, 2.13, 2.19, 2.20, 2.23], are given in terms of the next expressions

$$\begin{aligned} \mathcal{I}^k_{\sinh}(ab) &= \frac{I_k(ab)}{2(ab)^k \sinh(ab)}, \\ \mathcal{I}^k_{\cosh}(ab) &= \frac{I_k(ab)}{2(ab)^k \cosh(ab)}, \\ \mathcal{I}^k_{e}(a,b) &= \frac{a I_k(ab)}{b^k e^{ab}}, \end{aligned} \tag{5}$$

where I_k stands for the modified Bessel function of the first kind, $k \in \{2, 3, \dots\}$, and

$$\begin{aligned} \mathcal{A}^{\pm}_{n,j}(a,b) &= A_{n-j-1}(a-b) \pm (-1)^j A_{n-j-1}(a+b) \\ \mathcal{B}^{\pm}_{n,j}(a) &= B_{n-j-1}(a) \pm (-1)^j B_{n-j-1}(-a), \end{aligned} \tag{6}$$

where $n \in \mathbb{N}$, $j \in \{0, 1, \dots n-1\}$, and A, B are defined by

$$A_k(\alpha) = \int_\alpha^\infty t^k e^{-\frac{t^2}{2}}\, \mathrm{d}t, \qquad B_k(\alpha) = \int_{-\alpha}^{b-\alpha} t^k e^{-\frac{t^2}{2}}\, \mathrm{d}t \tag{7}$$

Let us consider a and b such that $a > b > 0$. To derive a lower bound for the symmetric difference $\Delta Q_n(a,b)$, we use the above mentioned bounds, more precisely we subtract an upper bound of $Q_n(b,a)$ from a lower bound of $Q_n(a,b)$. This way we get that

$$\begin{aligned} \Delta Q_n(a,b) \geq 1 &- b^n \mathcal{I}^{n-1}_{\sinh}(ab) \sum_{j=0}^{n-1} C^j_{n-1} a^j \mathcal{B}^{-}_{n,j}(a) \\ &- \mathcal{I}^{n-1}_{\cosh}(ab) \sum_{j=0}^{2n-1} C^j_{2n-1} b^j \mathcal{A}^{+}_{2n,j}(a,b). \end{aligned} \tag{8}$$

Similarly, to get an upper bound, we subtract a lower bound of $Q_n(b,a)$ from an upper bound of $Q_n(a,b)$, getting that

$$\begin{aligned} \Delta Q_n(a,b) \leq 1 &- \mathcal{I}^{n-1}_{\cosh}(ab) \sum_{j=0}^{2n-1} C^j_{2n-1} a^j \mathcal{B}^{+}_{2n,j}(a) \\ &- \mathcal{I}^{n-1}_{e}(a,b) \sum_{j=0}^{n-1} C^j_{n-1} b^j A_{n-j-1}(a-b), \end{aligned} \tag{9}$$

and

$$\begin{aligned} \Delta Q_n(a,b) \leq 1 &- \mathcal{I}^{n-1}_{\cosh}(ab) \sum_{j=0}^{2n-1} C^j_{2n-1} a^j \mathcal{B}^{+}_{2n,j}(a) \\ &- a^n \mathcal{I}^{n-1}_{\sinh}(ab) \sum_{j=0}^{n-1} C^j_{n-1} b^j \mathcal{A}^{-}_{n,j}(a,b). \end{aligned} \tag{10}$$

These bounds are illustrated on Fig. 3 for $n = 1$, $a = 2$ and $b \in [0.1, 0.5]$:

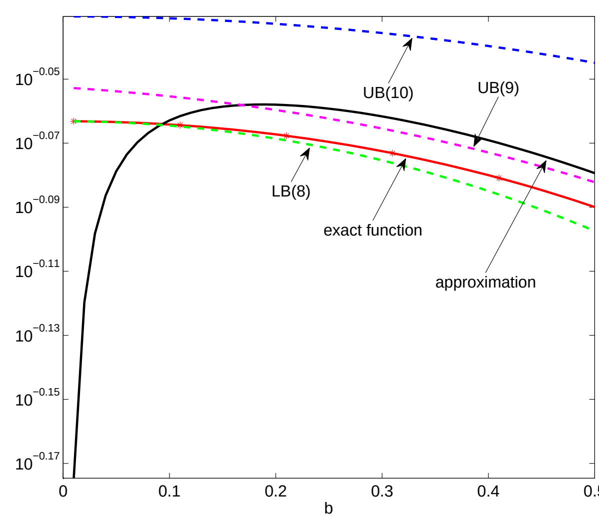

Fig. 3. Symmetric difference of the generalized Marcum Q-functions for $n = 1, a = 2$. 'x': exact; straight line: approximation (3); dashed lines: lower and upper bounds (8)-(10)

Moreover, for $n \in \{2, 3, 4, \dots\}$, the following bounds hold true as well, being slightly tighter than the previous ones

$$\begin{aligned} \Delta Q_n(a,b) \geq 1 &- b^n \mathcal{I}^{n-1}_{\sinh}(ab) \sum_{j=0}^{n-1} C^j_{n-1} a^j \mathcal{B}^{-}_{n,j}(a) \\ &- a \mathcal{I}^{n-1}_{\sinh}(ab) \sum_{j=0}^{2n-2} C^j_{2n-2} b^j \mathcal{A}^{+}_{2n,j+1}(a,b), \end{aligned} \tag{11}$$

$$\begin{aligned} \Delta Q_n(a,b) \leq 1 &- b \mathcal{I}^{n-1}_{\sinh}(ab) \sum_{j=0}^{2n-2} C^j_{2n-2} a^j \mathcal{B}^{+}_{2n,j+1}(a) \\ &- \mathcal{I}^{n-1}_{e}(a,b) \sum_{j=0}^{n-1} C^j_{n-1} b^j A_{n-j-1}(a-b), \end{aligned} \tag{12}$$

$$\Delta Q_n(a,b) \le 1 - b\mathcal{I}_{\sinh}^{n-1}(ab)\sum_{j=0}^{2n-2} C_{2n-2}^{j} a^j \mathcal{B}_{2n,j+1}^{+}(a) \\ -a^n \mathcal{I}_{\sinh}^{n-1}(ab)\sum_{j=0}^{n-1} C_{n-1}^{j} b^j \mathcal{A}_{n,j}^{-}(a,b). \tag{13}$$

See Fig. 4 for bounds (11)-(13).

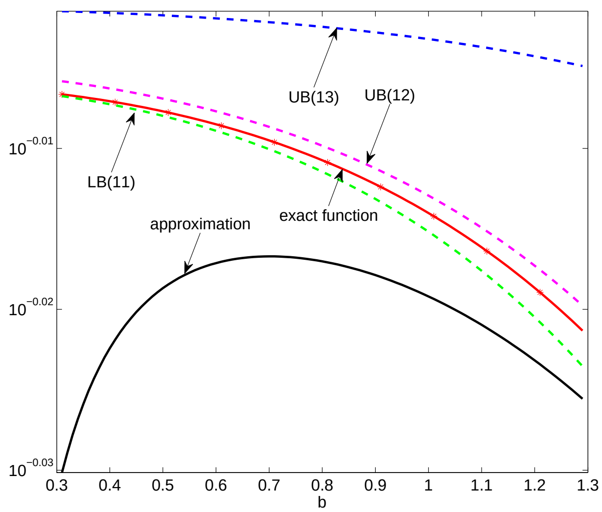

Fig. 4. Symmetric difference of the generalized Marcum Q-functions for $n = 3, a = 4$. 'x': exact; straight line: approximation (3); dashed lines: lower and upper bounds (11)-(13)

III. Bounds for the symmetric difference of Marcum Q-functions for real parameter $\nu \ge 1$

In this section first we present some similar inequalities to (8)-(13) that hold true for real values of the parameter ν. These inequalities are based on the upper and lower bounds for the generalized Marcum Q-function $Q_\nu(a,b)$ presented in [6].

Considering the previously introduced notations in (5) and the following ones for $\lambda \ge 0$

$$\mathcal{G}_\lambda^{\pm} = \int_{-a}^{b-a} (t+a)^\lambda e^{-\frac{t^2}{2}}\,\mathrm{d}t \pm \int_{a}^{b+a} (t-a)^\lambda e^{-\frac{t^2}{2}}\,\mathrm{d}t, \\ \mathcal{H}_\lambda^{\pm} = \int_{a-b}^{\infty} (t+b)^\lambda e^{-\frac{t^2}{2}}\,\mathrm{d}t \pm \int_{a+b}^{\infty} (t-b)^\lambda e^{-\frac{t^2}{2}}\,\mathrm{d}t, \tag{14}$$

we have the following bounds

$$\Delta Q_\nu(a,b) \ge 1 - b^\nu \mathcal{I}_{\sinh}^{\nu-1}(ab)\cdot\mathcal{G}_{\nu-1}^{-} - \mathcal{I}_{\cosh}^{\nu-1}(ab)\cdot\mathcal{H}_{2\nu-1}^{+}, \tag{15}$$

$$\Delta Q_\nu(a,b) \le 1 - \mathcal{I}_{\cosh}^{\nu-1}(ab)\cdot\mathcal{G}_{2\nu-1}^{+} \\ - \mathcal{I}_{e}^{\nu-1}(a,b)\int_{a-b}^{\infty}(t+b)^{\nu-1}e^{-\frac{t^2}{2}}\,\mathrm{d}t, \tag{16}$$

$$\Delta Q_\nu(a,b) \le 1 - \mathcal{I}_{\cosh}^{\nu-1}(ab)\cdot\mathcal{G}_{2\nu-1}^{+} - a^\nu \mathcal{I}_{\sinh}^{\nu-1}\cdot\mathcal{H}_{\nu-1}^{-}. \tag{17}$$

Fig. 5 illustrates these bounds, however, computing them for given parameters is expensive. Another problem occurs when we have to compute the exact values of the symmetric difference of generalized Marcum Q-functions in this case, for real values of ν, since the built-in functions in commercial mathematical software packages usually deal only with the integer case. Thus computing $\Delta Q_\nu(a,b)$ is likewise costly.

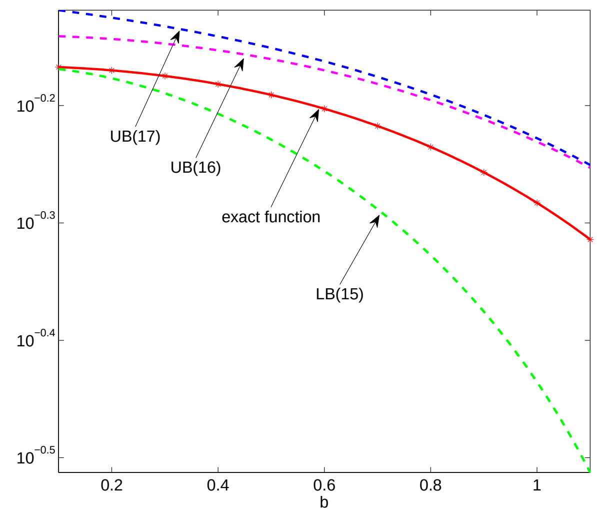

Fig. 5. The symmetric difference of generalized Marcum Q-functions for $a = 2, \nu = 1.9$

To overcome these problems, we present another possible approach to establish bounds. To do this, the key idea is to consider a closed-form expression of $Q_\nu(a,b)$ that is simpler to work with. These kind of forms are proposed in [5], [7] for the case when $\nu + \frac{1}{2} \in \mathbb{N}$. For instance, we consider the following closed-form expression of $Q_\nu(a,b)$ for $a, b > 0$ and $\nu + \frac{1}{2} \in \mathbb{N}$:

$$Q_\nu(a,b) = \frac{1}{2}\mathrm{erfc}\left(\frac{b+a}{\sqrt{2}}\right) + \frac{1}{2}\mathrm{erfc}\left(\frac{b-a}{\sqrt{2}}\right) \\ + \frac{1}{\sqrt{2\pi ab}} e^{-\frac{a^2+b^2}{2}} \sum_{k=0}^{\nu-\frac{3}{2}} S_k \left(\frac{b}{a}\right)^{k+\frac{1}{2}}. \tag{18}$$

Here erfc denotes the complementary error function, i.e.

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{t^2}\,\mathrm{d}t$$

and S_k stands for

$$\sum_{r=0}^{k} \frac{(k+r)!}{r!(k-r)!(2ab)^r}[(-1)^r e^{ab} + (-1)^{k+1}e^{-ab}].$$

To study the general non-integer case when $\nu + \frac{1}{2}$ is not necessarily an integer, we construct ν_1 and ν_2 as it is proposed in [5, VI/B/1], namely such that

$$\nu_1 = \min\left\{\nu \le \mu : \mu + \frac{1}{2} \in \mathbb{N}\right\}$$

$$\text{and } \nu_2 = \max\left\{\nu \ge \mu : \mu + \frac{1}{2} \in \mathbb{N}\right\}.$$

These orders are simply

$$\nu_1 = \left\lfloor \nu + \frac{1}{2} \right\rfloor + \frac{1}{2}, \qquad \nu_2 = \left\lfloor \nu - \frac{1}{2} \right\rfloor + \frac{1}{2}.$$

This leads to the desired inequalities due to [5, Theorem 3(b)], which claims the log-concavity of $\nu \mapsto Q_\nu(a,b)$ on $[1,\infty)$. Thus we have that

$$Q_\nu(a,b) \geq Q_{\nu_1}(a,b)^{\nu-\nu_2} Q_{\nu_2}(a,b)^{\nu_1-\nu}, \qquad \nu \geq \frac{3}{2}.$$

Indeed, since $\nu_1(\nu-\nu_2)+\nu_2(\nu_1-\nu) = \nu(\nu_1-\nu_2) = \nu$, the above inequality is clear. This way we have a lower bound. Using the upper bounds [5, (55), (56)] which are also consequences of the log-concavity in ν, we get that

$$Q_\nu(a,b) \leq \frac{Q_{\nu_1}(a,b)^{\nu_1-\nu+1}}{Q_{\nu_1+1}(a,b)^{\nu_1-\nu}},\ Q_\nu(a,b) \leq \frac{Q_{\nu_2}(a,b)^{\nu-\nu_2+1}}{Q_{\nu_2-1}(a,b)^{\nu-\nu_2}}.$$

Thus, the upper bounds of the symmetric difference that we get this way are

$$\Delta Q_\nu(a,b) \leq \frac{Q_{\nu_1}(a,b)^{\nu_1-\nu+1}}{Q_{\nu_1+1}(a,b)^{\nu_1-\nu}} - Q_{\nu_1}(b,a)^{\nu-\nu_2} Q_{\nu_2}(b,a)^{\nu_1-\nu}, \tag{19}$$

$$\Delta Q_\nu(a,b) \leq \frac{Q_{\nu_2}(a,b)^{\nu-\nu_2+1}}{Q_{\nu_2-1}(a,b)^{\nu-\nu_2}} - Q_{\nu_1}(b,a)^{\nu-\nu_2} Q_{\nu_2}(b,a)^{\nu_1-\nu}, \tag{20}$$

and the lower bounds are

$$\Delta Q_\nu(a,b) \geq Q_{\nu_1}(a,b)^{\nu-\nu_2} Q_{\nu_2}(a,b)^{\nu_1-\nu} - \frac{Q_{\nu_1}(b,a)^{\nu_1-\nu+1}}{Q_{\nu_1+1}(b,a)^{\nu_1-\nu}}, \tag{21}$$

$$\Delta Q_\nu(a,b) \geq Q_{\nu_1}(a,b)^{\nu-\nu_2} Q_{\nu_2}(a,b)^{\nu_1-\nu} - \frac{Q_{\nu_2}(b,a)^{\nu-\nu_2+1}}{Q_{\nu_2-1}(b,a)^{\nu-\nu_2}}. \tag{22}$$

These bounds are clearly simpler expressions than the ones in (15)-(17). They are illustrated on fig. 6.

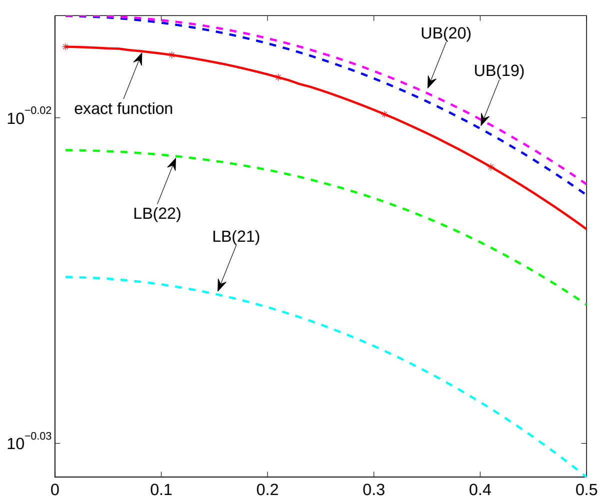

Fig. 6. Symmetric difference of generalized Marcum Q-functions for $\nu = 1.7, a = 3$. 'x': exact; dashed lines: bounds in (19)-(22).

IV. Closed forms of the symmetric difference

In the previous two sections we showed some possible ways to give lower and upper bounds for the symmetric difference of the generalized Marcum Q-functions for both integer and real values of ν. In this section we would like to point out some other results from the literature that offer further directions in the study of the considered symmetric difference. For instance in [8, eq. 6] the authors give an estimation for the generalized Marcum Q-function for arbitrary order ν

$$Q_\nu(a,b) \approx \sum_{i=0}^{k} \frac{(k+i)!k^{1-2i}a^{2i}2^{-i}\Gamma(\nu+i, b^2/2)}{i!(k-i)!\Gamma(\nu+i)a^{a^2/2}}. \tag{23}$$

Fig. 7 and 8 illustrate the exact symmetric difference, the approximation in (3), and the approximation by using (23), respectively, with black and dashed lines.

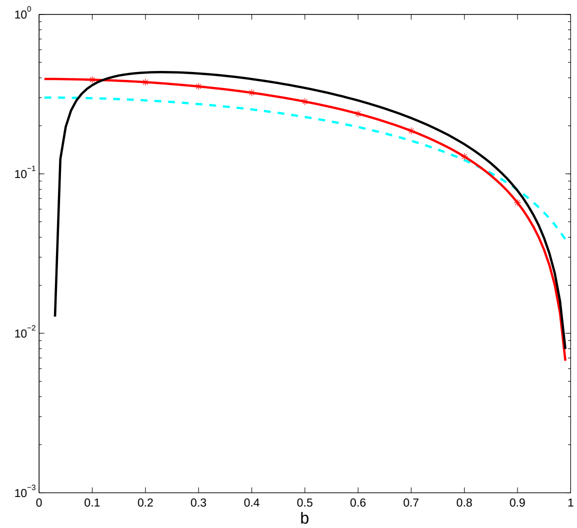

Fig. 7. Symmetric difference of generalized Marcum Q-functions for $a = 1, \nu = 1$, 'x': exact; black line: approximation in (3); dashed line: by using the approximation in (23)

It seems that the approximation of the symmetric difference by using (23) is a better one for small values of a and b, however, the approximation (3) is most efficient for the larger values.

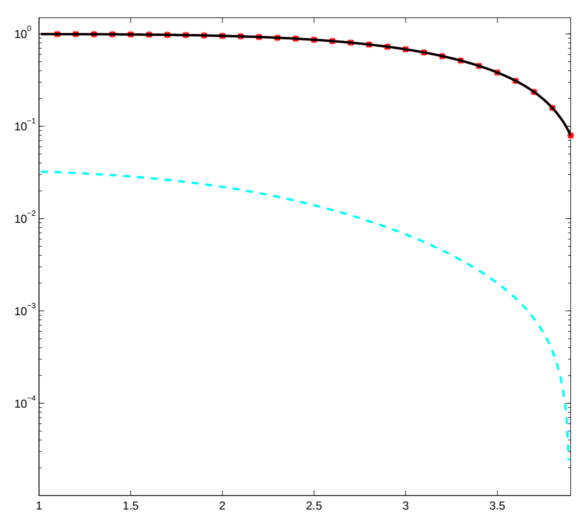

Fig. 8. Symmetric difference of generalized Marcum Q-functions for $a = 1, \nu = 1$, 'x': exact; black line: approximation in (3); dashed line: by using the approximation in (23)

Other studies reveal different forms of the Marcum Q-function, involving for example the upper incomplete gamma function $\Gamma(\cdot,\cdot)$ or some polynomial.

First, let us consider [9, eq 40], that gives us the following form of the generalized Marcum Q-function

$$Q_\nu(a,b) = e^{-\frac{a^2}{2}} \sum_{n\geq 0} \frac{1}{n!}\left(\frac{a^2}{2}\right)^n \frac{\Gamma\left(\nu+n, \frac{b^2}{2}\right)}{\Gamma(\nu+n)}, \qquad (24)$$

for all $\nu > 0$ real numbers. This result appears also in [10, pp. 140]. Both studies conclude that $\nu \mapsto Q_\nu(a,b)$ is strictly increasing on $(0,\infty)$, but more importantly, it gives another form of $Q_\nu(a,b)$, which is a bit easier to work with. By using (24) we can give the following exact form for the symmetric difference $\Delta Q_\nu(a,b)$

$$\sum_{n\geq 0} \frac{a^{2n}e^{-\frac{a^2}{2}}\Gamma\left(\nu+n,\frac{b^2}{2}\right) - b^{2n}e^{-\frac{b^2}{2}}\Gamma\left(\nu+n,\frac{a^2}{2}\right)}{n!2^n\Gamma(\nu+n)}.$$

In [11, eq. 8] the authors give the following series representation of the Marcum Q-function for $a,\nu>0$ and $b\geq 0$

$$Q_\nu(a,b) = 1 - \sum_{n\geq 0}(-1)^n e^{-\frac{a^2}{2}} \frac{L_n^{\nu-1}\left(\frac{a^2}{2}\right)}{\Gamma(\nu+n+1)}\left(\frac{b^2}{2}\right)^{n+\nu},$$

where L_n^ν is the generalized Laguerre polynomial of degree n and order ν defined by

$$L_n^\nu(x) = \sum_{k=0}^{n} \frac{\Gamma(n+\nu+1)}{\Gamma(k+\nu+1)\Gamma(n-k+1)}\frac{(-x)^k}{k!}.$$

This representation opens up yet another door to investigate the symmetric difference, which can be rewritten as

$$\Delta Q_\nu(a,b) = \sum_{n\geq 0}\frac{(-1)^n\Delta L_n^{\nu-1}(a,b)}{2^{n+\nu}\Gamma(\nu+n+1)},$$

where

$$\Delta L_n^{\nu-1}(a,b) = \frac{a^{2n+2\nu}L_n^{\nu-1}\left(\frac{b^2}{2}\right)}{e^{\frac{b^2}{2}}} - \frac{b^{2n+2\nu}L_n^{\nu-1}\left(\frac{a^2}{2}\right)}{e^{\frac{a^2}{2}}}.$$

V. Conclusions

In this note we presented some results about the generalized Marcum Q-function and its symmetric difference, deriving upper and lower bounds for the difference. Although, by using (18) we obtained good bounds, we remark that some other forms of the generalized Marcum Q-function are presented, as the reader can see in the last section.

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