Incomplete Krätzel Function Model of Leaky Aquifer and Alike Functions

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Abstract—**In this note our aim is to derive certain important properties of a general mathematical model given in the form of an incomplete Krätzel function which contains as sub–models the generalized leaky aquifer function, the van't Hoff thermal analysis temperature integral and the so–called thermonuclear integral among others.**

Index Terms—**Leaky aquifer function, Krätzel function, incomplete Krätzel function, Laguerre–type inequality, temperature integral, thermonuclear integral, Turán–type inequality, upper incomplete gamma function, generalized incomplete gamma function, log–convexity, bilateral bounding inequalities.**

I. Introduction

SEVERAL mathematical models possess integral representations which kernel contains exponential kernel function of rational argument, *viz.*

$$\mathscr{K}_{\alpha,\beta}^{a,b}(x) = \exp\left\{-ax^\alpha - bx^{-\beta}\right\}, \tag{1}$$

where $a, b \geq 0; \alpha \in \mathbb{R}^+, \beta \in \mathbb{R}$ and the integration domain is inside the positive real half–axis, but instead of the massive notation $\mathscr{K}_{\alpha,\beta}^{a,b}$ we will signify this kernel by convention as $\mathscr{K}_{\alpha,\beta}$ any time when parameters a, b are not specified. The first of these concepts is the *leaky aquifer function* [1], [2]

$$\mathscr{W}(a,b) = \int_1^\infty \mathscr{K}_{1,1}(x)\,\frac{\mathrm{d}x}{x}\,,$$

which can be rewritten into [3]

$$\mathscr{W}(a,b) = \int_{\sqrt{\frac{a}{b}}}^\infty \exp\left\{-\sqrt{ab}(x+x^{-1})\right\}\frac{\mathrm{d}x}{x};$$

we point out that $\mathscr{W}(a,b)$ approaches $2K_0(2\sqrt{ab})$ when $ab^{-1} \to 0^+$ (here K_0 denotes the modified Bessel function of the second kind of zero order). The denomination is coming from hydrology, since $\mathscr{W}$ occurs in water levels, in pumped aquifer systems with finite transmissivity, and leakage modeling.

The plausible extension of the leaky aquifer function $\mathscr{W}(a,b)$ reads as follows

$$\mathscr{W}_\nu(a,b;\alpha,\beta) = \int_{\alpha\beta^{-1}}^\infty x^\nu\,\mathscr{K}_{\alpha,\beta}^{a,b}(x)\,\frac{\mathrm{d}x}{x}\,, \tag{2}$$

which we call *generalized leaky aquifer function*; we notice that actually $\mathscr{W}(a,b) = \mathscr{W}_0(a,b;1,1)$. The range of the newly introduced parameters is

$$\mathsf{R}^\star = \left\{(\alpha,\beta,\nu) \in \mathbb{R}^3 \colon \alpha\beta > 0\right\}.$$

The kernel $\mathscr{K}_{1,1}(x)$ also builds the *temperature integral* in several thermal analysis models among others the ones introduced by Arrhenius, van't Hoff and Kooij, compare [4, p. 84, Table 1]. The most exhaustive from these concepts is van't Hoff's model II [5], [4, p. 84, Table 1, (e)], which reads in our setting

$$p_B\left(\frac{E}{RT}\right) = A_0 \int_{T_s}^\infty x^B \mathscr{K}_{1,1}(x)\mathrm{d}x\,,$$

where E, A_0, R stand for the energy of activation, the pre–exponential factor and the gas constant respectively, while T and $T_s = T_s(t) \geq 0$ denotes the absolute temperature and the time–dependent initial temperature when the reaction starts. Let us mention that the Arrhenius and Kooij equations/models are sub–models of the van't Hoff's [4, p. 84 *et seq.*].

The next important use of the kernel (1) occurs in modeling the reaction rates for thermal particles, having in mind the average speed in the Maxwell–Boltzmann distribution:

$$\langle\sigma\rangle = M_0 \int_0^\infty E\sigma(E)\exp\left\{-\frac{E}{kT}\right\}\mathrm{d}E, \qquad M_0 > 0\,,$$

relative to the functional $\sigma(E)$, where $kT = 0.86$ keV.

In quantum mechanics application we expect

$$\sigma(E) = \frac{S(E)}{E}\,\exp\left\{-bE^{-\frac{1}{2}}\right\}\,,$$

where $S(E)$ stands for the *astrophysical S-factor*, which encodes nuclear contribution to reaction and it is slowly varying in E. All together result in the *thermonuclear rates integral* (in short thermonuclear integral):

$$\langle\sigma\rangle = M_0 \int_0^\infty S(x)\,\mathscr{K}_{1,\frac{1}{2}}(x)\,\mathrm{d}x\,.$$

In the case when the S–factor behaves according to the power law, that is $S(E) \propto E^B$, we have

$$\langle\sigma\rangle_B = M_1 \int_0^\infty x^B\,\mathscr{K}_{1,\frac{1}{2}}(x)\,\mathrm{d}x\,.$$

Now, we recognize the similarities with the leaky aquifer and van't Hoff model. Different kind thermonuclear integrals were studied for instance in [6, p. 728] and in the important recently issued monograph [7] by Mathai *et al.*, we suggest to consult the extensive references lists therein too.

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However, integrals involving kernel $\mathscr{K}_{\alpha,\beta}^{a,b}(x)$ occur in the so–called (*complete*) *Krätzel function* [8], [9], [10], which is defined for $u > 0$, $\rho \in \mathbb{R}$ and $\nu \in \mathbb{C}$, being such, that $\Re(\nu) < 0$ for $\rho \leq 0$. This special function reads

$$Z_\rho^\nu(u) = \int_0^\infty x^{\nu-1} \mathrm{e}^{-x^\rho - ux^{-1}}\, \mathrm{d}x. \tag{3}$$

The main difference between the leaky aquifer, the temperature and thermonuclear integrals from one, and the Krätzel integral (3) from another side is in the integration domain, which don't always coincide. To cover this disadvantage we introduce the *incomplete Krätzel function* concept. In this note our aim is to study this approach.

II. On the incomplete Krätzel function

Let us denote $\mathcal{I}_A(x)$ the indicator function of the set A, that is $\mathcal{I}_A(x) = 1$ when $x \in A$, and $= 0$ elsewhere.

Definition 1: The incomplete Krätzel function $\zeta_\rho^\nu(\cdot; A)$ with respect to the set $A \subseteq R$ which possesses positive Lebesgue measure $\lambda(A \cap \mathbb{R}^+) > 0$, one defines as the convolution type integral

$$\begin{aligned} \zeta_\rho^\nu(u; A) &= \int_0^\infty x^{\nu-1} \mathscr{K}_{\rho,1}^{1,u}(x)\, \mathcal{I}_A(x)\, \mathrm{d}x \\ &= \int_{A\cap\mathbb{R}^+} x^{\nu-1} \mathscr{K}_{\rho,1}^{1,u}(x)\, \mathrm{d}x\,. \end{aligned} \tag{4}$$

Obviously $A \supseteq \mathbb{R}^+$ implies $\zeta_\rho^\nu(u; A) \equiv Z_\rho^\nu(u)$, moreover when $\lambda(A \cap \mathbb{R}^+) = 0$, the function $\zeta_\rho^\nu(u; A)$ terminates. In turn, for our purposes we should discuss all parameter constraints in (4) separately.

Baricz *et al.* [9, p. 718, Theorem 1] established a set of results for $Z_\rho^\nu(u)$, among others three–term recurrence relation, complete monotonicity of $u \mapsto Z_\rho^\nu(u)$ and log–convexity of $\nu \mapsto Z_\rho^\nu(u)$. It immediately arises the question whether the same results hold for $\zeta_\rho^\nu(u; A)$.

Theorem 1: For all $\nu, \rho \in \mathbb{R}$, $u > 0$; and for all measurable $A \subseteq R$ with the positive Lebesque measure $\lambda(A \cap \mathbb{R}^+) > 0$ we have:

- **(i)** *The incomplete Krätzel function $\zeta_\rho^\nu(u; A)$ satisfies the three–term recurrence relation*
$$\nu\zeta_\rho^\nu(u; A) - \rho\zeta_\rho^{\nu+\rho}(u; A) + u\zeta_\rho^{\nu-1}(u; A) = 0.$$
- **(ii)** *The function $u \mapsto \zeta_\rho^\nu(u; A)$ is log–convex on $\mathbb{R}^+$.*
- **(iii)** *The function $\nu \mapsto \zeta_\rho^\nu(u; A)$ is log–convex on $\mathbb{R}$.*

Proof: As to the proof of **(i)** by integration by parts we immediately have

$$\begin{aligned} \nu\zeta_\rho^\nu(u; A) &= \int_{A\cap\mathbb{R}^+} \nu x^{\nu-1} \exp\left\{-x^\rho - ux^{-1}\right\} \mathrm{d}x \\ &= \int_{A\cap\mathbb{R}^+} x^\nu \left(\rho x^{\rho-1} - \frac{u}{2}\right) \exp\left\{-x^\rho - ux^{-1}\right\} \mathrm{d}x \\ &= \rho\zeta_\rho^{\nu+\rho}(u; A) - u\zeta_\rho^{\nu-1}(u; A). \end{aligned}$$

We note that **(ii)** can be verified by using Hölder's inequality, repeating completely the proof of the same fact for the complete Krätzel function in [9, p. 719], that is that

$$\zeta_\rho^{\alpha\nu_1+(1-\alpha)\nu_2}(u; A) \leq \left[\zeta_\rho^{\nu_1}(u; A)\right]^\alpha \left[\zeta_\rho^{\nu_2}(u; A)\right]^{1-\alpha}$$

holds for all $\alpha \in [0, 1]$; $\nu_1, \nu_2, \rho \in \mathbb{R}$ and $u > 0$, i.e. the function $\nu \mapsto Z_\rho^\nu(u)$ is log-convex on $\mathbb{R}$.

Next as to **(iii)**, we deduce also by the Hölder inequality applied for the linear combination inside argument that

$$\zeta_\rho^\nu(\alpha u_1 + (1-\alpha)u_2; A) \leq \left[\zeta_\rho^\nu(u_1; A)\right]^\alpha \left[\zeta_\rho^\nu(u_2; A)\right]^{1-\alpha}$$

is valid for all

$$\alpha \in [0, 1], \quad \nu, \rho \in \mathbb{R}; \quad u_1, u_2 > 0,$$

so $\zeta_\rho^\nu(u; A)$ is log-convex on $\mathbb{R}^+$ with respect to u. □

Theorem 2: For all $a, b > 0; (\alpha, \beta, \nu) \in \mathsf{R}^\star; B \in \mathbb{R}$ we have

$$\mathscr{W}_\nu(a, b; \alpha, \beta) = \frac{1}{\beta\, a^{\frac{\nu}{\alpha}}}\, \zeta_{\frac{\alpha}{\beta}}^{\frac{\nu}{\beta}} \left(a^{\frac{\beta}{\alpha}}\, b; ([a^{\frac{1}{\alpha}}\, \alpha\beta^{-1}]^\beta, \infty)\right) \tag{5}$$

$$p_B\left(\frac{E}{RT}\right) = \frac{A_0}{a^{B+1}}\, \zeta_1^{B+1}\big(ab; (aT_s, \infty)\big)\,. \tag{6}$$

Moreover, for all $a, b \geq 0; B > -1$ there holds

$$\langle\sigma\rangle_B = \frac{2M_1}{a^{B+1}}\, Z_2^{2B+2}(b\sqrt{a})\,, \tag{7}$$

while for $B \leq -1$ the Krätzel integral has to be in the Cauchy's principal value sense used.

Proof: Setting the substitution $a^{\frac{1}{\alpha}}x \mapsto x$ in (4) we clearly get (5). The parameter space regarding β has been reduced since according to (3) $\nu = 1$ excludes non–positive β–values.

Next, as to (6), the same substitution applies with $\alpha = 1$. Because $aT_s > 0$, the temperature integral converges for all real B.

Finally, as to (7), the positive expression $b\sqrt{a}$ controls the behavior of the integrand at zero. The rest is clear. □

Remark 1: Obviously, the leaky aquifer function's incomplete Krätzel function description becomes

$$\mathscr{W}(a, b) = \zeta_1^0(a\,b; (a, \infty))\ . \qquad \blacksquare$$

The compilation of *Theorem 1* and *Theorem 2* results in

Theorem 3: For all $(\alpha, \beta, \nu) \in \mathsf{R}^\star$, $a, b > 0$ we have:

- **(iv)** *The generalized leaky aquiver function satisfies the three–term recurrence relation*
$$\nu a^{\frac{\nu}{\alpha}} \mathscr{W}_\nu - \alpha a^{\frac{\nu}{\alpha}+1} \mathscr{W}_{\nu+\alpha} + a^{\frac{\nu+\beta}{\alpha}-1} b\, \beta \mathscr{W}_{\nu-\alpha} = 0\,,$$
when $\nu \geq \min\{0, |\alpha|\}$, where
$$\mathscr{W}_\nu = \mathscr{W}_\nu(a, b; \alpha, \beta)\,.$$
- **(v)** *The function $u = a^{\frac{\beta}{\alpha}}b \mapsto \mathscr{W}_\nu(a, b; \alpha, \beta) = \varphi_1(u)$ is log–convex on $\mathbb{R}^+$.*
- **(vi)** *The function $v = \nu\beta^{-1} \mapsto \mathscr{W}_\nu(a, b; \alpha, \beta) = \varphi_2(v)$ is log–convex on $\mathbb{R}_0^+$.*

Proof: As to the proof of these claims it is enough to compare *Theorem 1* and (5) from *Theorem 2*. □

Remark 2: It is clear that for leaky aquifer function $\mathscr{W}(a, b)$ no such recurrence identity exists, since ν has to be different of zero. However there are no obstacles for the log–convexity of $\mathscr{W}(a, b)$ with respect to the argument ab on $\mathbb{R}^+$. ■

Theorem 4: For all $a, b > 0$, and $B > 0$ we have:

(vii) *The van't Hoff temperature integral $p_B(\cdot)$ satisfies the three–term recurrence relation*

$$a(B+1)\, p_B - a^2\, p_{B+1} + ab\, p_B = 0\,.$$

(viii) *The function $ab \mapsto p_B = \psi_1(ab)$ is log–convex on $\mathbb{R}^+$.*

(ix) *The function $B \mapsto p_B$ is log–convex on $(-1, \infty)$.*

Proof: Infer by *Theorem 1* and (6). □

Remark 3: To close this section, we remark that being the thermonuclear function up to a constant equal to a Krätzel function (compare (7)), we refer to the direct use of the Theorem 1 in establishing its adequate recurrence and log–convexity properties. ■

III. Bilateral bounding inequalities

It is known that there exist bounds for the complete Krätzel function, see [9, p. 718, Theorem 1 (f)]. So it is worth to establish bilateral bounding inequalities for its incomplete counterpart $\zeta_\rho^\nu(u;A)$. First we introduce the *generalized incomplete gamma function* in the form of the integral [11]

$$\Gamma(a,z,w) = \int_z^w x^{a-1}\, \mathrm{e}^{-x}\, \mathrm{d}x\,,$$

which is an entire function of a for all fixed $z, w \in \mathbb{C}$. We remark that this function is in-built having the *Mathematica* code `Gamma(a, z, w)`. For $w = \infty$, $\Gamma(a,z,w)$ reduces to the *upper incomplete gamma function* in notation $\Gamma(a,z)$; the associated code is `Gamma(a, z)`, while when additionally $z = 0$, we get the 'classical' Euler gamma function $\Gamma(a)$.

Theorem 5: Denote $S^\alpha = \{x^\alpha\colon x \in S, \alpha > 0\}$. Then for all $\rho > 0, \Re(\nu) > 0$, $u > 0$, and $A \subseteq R_0^+$ we have

$$\exp\left\{-\frac{u}{\inf(A)}\right\} <^* \frac{\rho\zeta_\rho^\nu(u;A)}{\Gamma\left(\dfrac{\nu}{\rho}, m, M\right)} <^* \exp\left\{-\frac{u}{\sup(A)}\right\}\,,$$

where

$$m = \inf(A^\rho), \qquad M = \sup(A^\rho)\,.$$

If A is open $<^*$ means $<$, when it is closed, $<^*$ denotes the relation $\le$.

Proof: Let us minimize and maximize the exponential kernel $\mathscr{K}_{\rho,1}^{1,u}(x)$ when $x \in A$. For the sake of simplicity assume that A is an open set. It follows that

$$\exp\left\{-x^\rho - \frac{u}{\inf(A)}\right\} < \mathscr{K}_{\rho,1}^{1,u}(x) < \exp\left\{-x^\rho - \frac{u}{\sup(A)}\right\}\,.$$

Applying the upper bound estimate to the incomplete Krätzel function $\zeta_\rho^\nu(u;A)$ *via* (4), we get

$$\mathrm{e}^{\frac{u}{\sup(A)}}\, \zeta_\rho^\nu(u;A) < \int_A x^{\nu-1} \mathrm{e}^{-x^\rho}\, \mathrm{d}x = \frac{1}{\rho}\int_{A^\rho} x^{\frac{\nu}{\rho}-1} \mathrm{e}^{-x}\, \mathrm{d}x,$$

which proves the claimed upper bound. □

Applying the two–sided bounding inequality to the representations of leaky aquifer function and the temperature integral exposed in Theorem 2 we arrive at

Corollary 5.1: For all $a, b \ge 0$; $\alpha, \beta > 0$; and $B+1 > 0$ we have

$$\exp\left\{-b\left(\frac{\beta}{\alpha}\right)^\beta\right\} < \frac{\alpha a^{\frac{\nu}{\alpha}}\, \mathscr{W}_\nu(a,b;\alpha,\beta)}{\Gamma\left(\dfrac{\nu}{\alpha}, a\left(\dfrac{\beta}{\alpha}\right)^\alpha\right)} < 1\,;$$

$$\frac{\Gamma(0,a)}{\alpha\, \mathrm{e}^b} < \mathscr{W}(a,b) < \frac{\Gamma(0,a)}{\alpha}\,;$$

$$\exp\left\{-\frac{b}{T_s}\right\} < \frac{a^{B+1}\, p_B\left(\dfrac{E}{RT}\right)}{A_0\, \Gamma(B+1, aT_s)} < 1\,.$$

Remark 4: Being $m = 0, M = \infty$ in the case of the thermonuclear integral, *Theorem 5* gives uniform two–sided bounds

$$0 < \langle\sigma\rangle_B < \frac{a^{B+1}}{2M_1}\, \Gamma(B+1)\,,$$

restricting $B > -1$. ■

IV. Further inequality results

The complete Krätzel function is log–convex, completely monotone and it satisfies a Laguerre–type inequality [9, p. 718, Theorem 1]. The latter results we can translate *pro primo* to the incomplete Krätzel function introduced here by *Definition 1*, regarding to the domain A. Indeed, since for all $\nu, \rho \in \mathbb{R}$, and $u > 0$ we have

$$\frac{\mathrm{d}^n}{\mathrm{d}u^n}\zeta_\rho^\nu(u;A) = (-1)^n\, \zeta_\rho^{\nu-n}(u;A), \qquad n \in \mathbb{N}_0\,,$$

and this differentiation property implies the inequality

$$\begin{aligned}\left\{\left[\zeta_\rho^\nu(u;A)\right]^{(n)}\right\}^2 &- \left[\zeta_\rho^\nu(u;A)\right]^{(n-1)} \left[\zeta_\rho^\nu(u;A)\right]^{(n+1)} \\ &= \left[\zeta_\rho^{\nu-n}(u;A)\right]^2 - \zeta_\rho^{\nu-1}(u;A)\zeta_\rho^{\nu+1}(u;A) \le 0\,,\end{aligned}$$

which holds true by the log–convexity of $\zeta_\rho^\nu(u;A)$ (compare *Theorem 2*, **(ii)**), the Laguerre–type inequality

$$\left\{\left[\zeta_\rho^\nu(u;A)\right]^{(n)}\right\}^2 \le \left[\zeta_\rho^\nu(u;A)\right]^{(n-1)} \left[\zeta_\rho^\nu(u;A)\right]^{(n+1)}$$

is proved.

Pro secundo, as to the same result for the leaky aquifer function, we only need to connect $\zeta_\rho^\nu(u;A)$ and $\mathscr{W}_\nu$ by *Theorem 2*, (5). Thus, having in mind that for all $n \in \mathbb{N}_0$ and for all $(\alpha,\beta,\nu) \in \mathsf{R}^\star$ and $a, b > 0$ there holds

$$\frac{\mathrm{d}^n\, \mathscr{W}_\nu}{\mathrm{d}\big(a^{\frac{\beta}{\alpha}} b\big)^n} = (-1)^n\, \mathscr{W}_{\nu-n}\,, \tag{8}$$

denoting $\mathscr{W}_\nu = \mathscr{W}_\nu(a,b;\alpha,\beta)$. This implies the validity of the next result.

Theorem 6: For all $n \in \mathbb{N}_0$ and for all $(\alpha,\beta,\nu) \in \mathsf{R}^\star$, $a, b > 0$ we have the Laguerre–type inequality

$$\mathscr{W}_\nu^{(n)} \le \sqrt{\mathscr{W}_\nu^{(n-1)}\, \mathscr{W}_\nu^{(n+1)}}\,.$$

Moreover, $\mathscr{W}_\nu(a,b;\alpha,\beta)$ is completely monotonic function with respect to $u = a^{\frac{\beta}{\alpha}} b$ on $u \in \mathbb{R}^+$.

Proof: The complete monotonicity issue is the consequence of (8) and $\mathscr{W}_\nu > 0$. □

A Turán–type inequality for the incomplete Krätzel function reads as follows:

$$\big[\zeta_\rho^\nu(u;A)\big]^2 \le \zeta_\rho^{\nu-h}(u;A)\,\zeta_\rho^{\nu+h}(u;A)\,;$$

we skip the proving procedure since it follows the same lines that the one exposed in [9]. However, also by the key relationship (5) of *Theorem 2* we deduce the following result.

Theorem 7: For all $h \ge 0$; $(\alpha,\beta,\nu) \in \mathsf{R}^\star$, $a,b > 0$ *we have the Turán–type inequality*

$$\mathscr{W}_\nu \le \sqrt{\mathscr{W}_{\nu-h}\,\mathscr{W}_{\nu+h}}\,.$$

Proof: By a direct calculation we have

$$\mathscr{W}_\nu^2 - \mathscr{W}_{\nu-h}\,\mathscr{W}_{\nu+h} = \frac12 \int_{(\frac{\alpha}{\beta},\infty)^2} (xy)^{\nu-1}\,\mathscr{K}_{\alpha,\beta}^{a,b}(x) \times \mathscr{K}_{\alpha,\beta}^{a,b}(y)\left\{2-\left(\frac{x}{y}\right)^h - \left(\frac{y}{x}\right)^h\right\}\mathrm{d}x\mathrm{d}y\,.$$

Based on the elementary inequality $a + a^{-1} \ge 2$, $a > 0$, the expression in the curly brackets is non–positive. Therefore, so does the double integral and the left–hand side expression. The proof is complete. □

V. Final remarks, applications. Discussion

A. The connection between the leaky aquifer function $\mathscr{W}(a,b)$ and the upper incomplete gamma function is evident, as it was pointed out by Harris [2], [6], [12], and [13]. However, Temme [13] worked out an exhaustive procedure for covering the calculation problems by proposing a numerical quadrature method.

The leaky aquifer area and applications are well organized. Namely Bruce Hunt with University of Canterbury, New Zealand developed computer programs, and in-built routines connecting pipe networks, groundwater analysis tools, fluid mechanics approach and/or additional teaching materials [14] for the underlying mathematics from one side, and further research papers from the other side, consult e.g. [15], [16], [17], [18], the references therein and his homepage [14]. For further specific information about leakage and different kind aquifers we are referred to Jacob Bear's homepages [19], [20] who is with the Faculty of Civil Engineering, Technion–Israel Institute of Technology, Haifa.

B. Further generalizations are prospective in the sense that the double–exponential kernel $\mathscr{K}_{\alpha,\beta}^{a,b}(x)$ can be replaced with the so–called *pathway kernel* [7, Section 5], [21] that is

$$\mathrm{e}^{-ax^\alpha}\,\left(1+(1-q)x^{-\beta}\right)^{-\frac{b}{1-q}} \underset{q\to1}{\longrightarrow} \mathscr{K}_{\alpha,\beta}^{a,b}(x)$$

point-wise with respect to x on the tacitly unified integration domain

$$\left(\frac{\alpha(1-q)}{\beta},\,\frac{\beta}{\alpha(1-q)}\right)\,,$$

frequently used in the Fourier transform studies in the L_2 functions class, see for instance [22, p. 50 *et seq.*]. By these transformations we arrive at the *pathway generalized leaky aquifer integral*:

$$\mathscr{W}_\nu^q(a,b;\alpha,\beta) = \int_{\frac{\alpha(1-q)}{\beta}}^{\frac{\beta}{\alpha(1-q)}} \frac{x^{\nu-1}\,\mathrm{e}^{-ax^2}\,\mathrm{d}x}{\left[1+(1-q)x^{-\beta}\right]^{\frac{b}{1-q}}}\,,$$

which obviously restores the generalized leaky aquifer function (2) as $q \to 1$. Whereas this kind appearance will be exploited in some future work.

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References

[1] M.S. Hantush, C.E. Jacob, "Non-steady radial flow in an infinite leaky aquifer," *Trans. Amer. Geophys. Union* **36**, pp. 95100, 1955.

[2] F.E. Harris, "New approach to calculation of the leaky aquifer function," *Internat. J. Quantum Chem.* **6**, pp. 913-916, 1997.

[3] R. Terras, "A Miller algorithm for an incomplete Bessel function," *J. Comput. Phys.* **39(1)**, pp. 233–240, 1981.

[4] J.H. Flynn, "The 'Temperature Integral' - its use and abuse", *Termochimica Acta* **300**, pp. 83–92, 1997.

[5] J.H. van't Hoff, "Influence of temperature on velocity in dilute homogeneous systems," in: *Vorlesungen über theoretische und physikalische Chemie*, (Translation R.A. Lechfeldt), Edward Arnold Publisher, London, Part 1, Chemical Dynamics 1899, pp. 230–235.

[6] M.A. Chaudhry, S.M. Zubair, "Extended incomplete gamma functions with applications," *J. Math. Anal. Appl.* **274(2)**, pp. 725–745, 2002.

[7] A.M. Mathai, R.K. Saxena, H.J. Haubold, *The H-function. Theory and Applications*, New York: Springer, 2010.

[8] E. Krätzel, "Integral transformations of Bessel type," in: *Generalized Functions and Operational Calculus*, Proc. Conf. Varna 1975, Sofia: Bulg. Acad. Sci., pp. 148–155, 1979.

[9] Á. Baricz, D. Jankov, T. K. Pogány, "Turán type inequalities for Krätzel functions," *J. Math. Anal. Appl.* **388(2)**, pp. 716–724, 2012.

[10] A.A. Kilbas, L. Rodríguez–Germá, M. Saigo, R.K. Saxena, J.J. Trujillo, "The Krätzel function and evaluation of integrals," *Comput. Math. Appl.* **59(5)**, pp. 1790–1800, 2010.

[11] http://functions.wolfram.com/GammaBetaErf/Gamma3/

[12] F.E. Harris, "Incomplete Bessel, generalized incomplete gamma, or leaky aquifer functions," *J. Comp. Appl. Math.* **215**, pp. 260-269, 2008.

[13] N.M. Temme, "The leaky aquifer function revisited," *Internat. J. Quantum Chem.* **109(3)**, pp. 2826–2830, 2009.

[14] http://www.civil.canterbury.ac.nz/staff/bhunt.shtml

[15] B. Hunt, "Characteristics of unsteady flow to wells in unconfined and semi-confined aquifers," *Journal of Hydrology* **325(1–4)**, pp. 154–163, 2006.

[16] B. Hunt, "Stream depletion for streams and aquifers with finite widths," *ASCE Journal of Hydrologic Engineering* **13(2)**, pp. 80–89, 2008.

[17] B. Hunt, D. Scott, "An extension of the Hantush and Boulton solutions," *ASCE Journal of Hydrologic Engineering* **10(3)**, pp. 223–236, 2005.

[18] B. Hunt, D. Scott, "Flow to a well in a two-aquifer system," *ASCE Journal of Hydrologic Engineering* **12(2)**, pp. 146–155, 2007.

[19] http://www.interpore.org/ref-mat_pub/mgfc-course/mgfcclas.html

[20] http://www.interpore.org/ref-mat_pub/mgfc-course/jbear.html

[21] A.M. Mathai, H.J. Haubold, "Pathway model, superstatistics, Tsallis statistics and a generalized measure of entropy," *Physica A* **375**, pp. 110–122, 2007.

[22] M.M. Dzhrbashyan, *Integral Transforms and Representations of Functions in the Complex Domain*, Moscow: Nauka Publisher, 1966. [Russian]