

On decoupling inequalities and percolation of excursion sets of the Gaussian free field

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Abstract

We prove decoupling inequalities for the Gaussian free field on \mathbb{Z}^d , $d \geq 3$. As an application, we obtain exponential decay (with logarithmic correction for $d = 3$) of the connectivity function of excursion sets for large values of the threshold.

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1 Correlation and decoupling inequalities

Let us denote by $\|x\|$ the Euclidean norm of $x \in \mathbb{Z}^d$. The Gaussian free field (GFF) on \mathbb{Z}^d , $d \geq 3$, is a centered Gaussian field $\varphi = (\varphi_x)_{x \in \mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$ under the probability measure \mathbb{P} with covariance $\mathbb{E}(\varphi_x \varphi_y) = g(x, y)$ for all $x, y \in \mathbb{Z}^d$, where $g(\cdot, \cdot)$ denotes the Green function of the simple random walk on \mathbb{Z}^d . The random field φ exhibits long-range correlations, since

$$c_g \|x - y\|^{2-d} \leq g(x, y) \leq C_g \|x - y\|^{2-d}, \quad x \neq y \quad (1.1)$$

for some $0 < c_g(d) \leq C_g(d) < \infty$, see [6, Theorem 1.5.4].

The goal of this section is to quantify the dependence between configurations of the GFF, supported on disjoint (and, usually, distant) sets. Decoupling inequalities of this kind are useful tools in the study of percolation of the excursion sets of the GFF (see [10, Proposition 2.2] and [2, Lemma 2.6]) and the vacant set of random interacements (see [13, Theorem 2.6]). When starting to work on the subject of the present paper, our initial aim was to find the GFF counterpart of the decoupling inequalities proved for random interacements in [8, Theorem 1.1] (see Remark 1.5 for further discussion), but our Theorem 1.2 actually gives a stronger, conditional form of decoupling inequalities for events defined in terms of the GFF. In Section 2 we give an application of the decoupling inequalities stated in this section.

If $K, K' \subseteq \mathbb{Z}^d$, we define

$$\text{dist}(K, K') = \min_{x \in K, y \in K'} \|x - y\|, \quad \text{diam}(K) = \sup_{x, y \in K} \|x - y\|.$$

With a slight abuse of notation, for $K \subset \mathbb{Z}^d$ we write $\varphi_K := (\varphi_x)_{x \in K}$ for the GFF restricted to K . We say that a (measurable) function $f : \mathbb{R}^{\mathbb{Z}^d} \rightarrow \mathbb{R}$ is supported on K if for any $\eta, \eta' \in \mathbb{R}^{\mathbb{Z}^d}$ such that $\eta_K = \eta'_K$ it holds that $f(\eta) = f(\eta')$. Let us also define

$$\text{Cov}(f_1(\varphi), f_2(\varphi)) = \mathbb{E}(f_1(\varphi)f_2(\varphi)) - \mathbb{E}f_1(\varphi)\mathbb{E}f_2(\varphi).$$

If K is a finite subset of \mathbb{Z}^d , we define the capacity of K by the formula

$$\text{cap}(K) = \sum_{x \in K} P_x[\tilde{H}_K = +\infty],$$

where P_x is the law of simple random walk X on \mathbb{Z}^d started from $x \in \mathbb{Z}^d$ and $\tilde{H}_K = \min\{n \geq 1 : X_n \in K\}$ is the hitting time of the set K .

First, we formulate the following result about correlations of functions supported on disjoint finite sets, which is in the spirit of the basic correlation inequality for random interacements, see [12, (2.15)].

Proposition 1.1. *Let $d \geq 3$. There exist constants $0 < c_d \leq C_d < +\infty$ that depend only on d such that if K_1, K_2 are disjoint finite subsets of \mathbb{Z}^d and*

$$\text{dist}(K_1, K_2) \geq \max\{\text{diam}(K_1), \text{diam}(K_2)\}, \quad (1.2)$$

then

$$c_d \frac{(\text{cap}(K_1)\text{cap}(K_2))^{1/2}}{\text{dist}(K_1, K_2)^{d-2}} \leq \sup_{f_1, f_2} \text{Cov}(f_1(\varphi), f_2(\varphi)) \leq C_d \frac{(\text{cap}(K_1)\text{cap}(K_2))^{1/2}}{\text{dist}(K_1, K_2)^{d-2}}, \quad (1.3)$$

where the supremum is taken over $[0, 1]$ -valued functions $f_1, f_2 : \mathbb{R}^{\mathbb{Z}^d} \rightarrow [0, 1]$, and where f_1 is supported on K_1 and f_2 is supported on K_2 .

Let us remark that the assumption (1.2) is only used in the proof the lower bound of (1.3).

While the above result does indeed give the right order of decay of correlations, it is not always the right tool one looks for. The reason is that the covariance decreases polynomially in distance, which makes renormalization arguments more difficult. One can circumvent this problem by using the method of *sprinkling*, which has been effectively applied to produce powerful decoupling inequalities for the excursion sets of the GFF (see [10, Proposition 2.2] and [2, Lemma 2.6]) and the vacant set of random interlacements (see [13, Section 2]).

To explain this approach, we need more definitions. Write $\eta \leq \eta'$ if $\eta_x \leq \eta'_x$ for all $x \in \mathbb{Z}^d$. A function $f : \mathbb{R}^{\mathbb{Z}^d} \rightarrow \mathbb{R}$ is called *increasing* if $\eta \leq \eta'$ implies $f(\eta) \leq f(\eta')$, and *decreasing* if $(-f)$ is increasing. For $\eta \in \mathbb{R}^{\mathbb{Z}^d}$ and $a \in \mathbb{R}$ we use the shorthand $\eta + a$ for the configuration defined by $(\eta + a)_x = \eta_x + a$, $x \in \mathbb{Z}^d$.

Let us fix two disjoint sets $K_1, K_2 \subset \mathbb{Z}^d$, such that K_1 is finite. It is known (see [14, Proposition 2.3] and [10, Lemma 1.2]) that \mathbb{P} -a.s. there exists a decomposition

$$\varphi = \tilde{\varphi} + h \tag{1.4}$$

into a sum of independent Gaussian fields, where $\tilde{\varphi}$ is a centered field such that $\tilde{\varphi}_{K_1} \equiv 0$, and

$$h_x = \sum_{y \in K_1} P_x[H_{K_1} < \infty, X_{H_{K_1}} = y] \cdot \varphi_y, \quad \mathbb{P}\text{-a.s.}, \tag{1.5}$$

where $H_{K_1} = \min\{n \geq 0 : X_n \in K_1\}$ is the entrance time of the random walk to the set K_1 . Note that h is measurable with respect to the sigma-algebra generated by φ_{K_1} and that one has $h_{K_1} \equiv \varphi_{K_1}$.

For $\delta > 0$, define the $\sigma(\varphi_{K_1})$ -measurable event

$$G_\delta = \left\{ \sup_{x \in K_2} |h_x| \leq \frac{\delta}{2} \right\}. \tag{1.6}$$

Our main result is about the *conditional* decoupling inequalities:

Theorem 1.2. *Assume that $f_2 : \mathbb{R}^{\mathbb{Z}^d} \rightarrow [0, 1]$ is increasing and supported on K_2 . For all $\delta > 0$ it \mathbb{P} -a.s. holds that*

$$(\mathbb{E}(f_2(\varphi - \delta)) - \mathbb{P}[G_\delta^c]) \mathbf{1}_{G_\delta} \leq \mathbb{E}(f_2(\varphi) \mid \varphi_{K_1}) \mathbf{1}_{G_\delta} \leq (\mathbb{E}(f_2(\varphi + \delta)) + \mathbb{P}[G_\delta^c]) \mathbf{1}_{G_\delta}. \tag{1.7}$$

An (almost) immediate consequence of this result is the following

Corollary 1.3. *Assume that $f_2 : \mathbb{R}^{\mathbb{Z}^d} \rightarrow [0, 1]$ is increasing and supported on K_2 , and $f_1 : \mathbb{R}^{\mathbb{Z}^d} \rightarrow [0, 1]$ is any function supported on K_1 . Then for all $\delta > 0$ we have*

$$\mathbb{E}f_1(\varphi)\mathbb{E}f_2(\varphi - \delta) - 2\mathbb{P}[G_\delta^c] \leq \mathbb{E}f_1(\varphi)f_2(\varphi) \leq \mathbb{E}f_1(\varphi)\mathbb{E}f_2(\varphi + \delta) + 2\mathbb{P}[G_\delta^c]. \quad (1.8)$$

It is straightforward to see that the corresponding results (with obvious changes) also hold if f_2 is a *decreasing* function.

We refer to the quantity δ in (1.7)–(1.8) as the amount of sprinkling. In Proposition 1.4 below we shall see that the term $\mathbb{P}[G_\delta^c]$ decreases quite fast as $\text{dist}(K_1, K_2)$ increases; so one can decrease the correlation term of Proposition 1.1 at the cost of “changing the level” of the field for the monotone function f_2 .

The next proposition tells us how to choose the amount of sprinkling δ if we want a useful upper bound for $\mathbb{P}[G_\delta^c]$. Let us denote by $|K|$ the cardinality of $K \subset \mathbb{Z}^d$. For any $K \subseteq \mathbb{Z}^d$ we define

$$\begin{aligned} K^{(\geq s)} &= \{y \in \mathbb{Z}^d : \text{dist}(y, K) \geq s\} \\ K^{(=s)} &= \{y \in \mathbb{Z}^d : s \leq \text{dist}(y, K) < s + 1\}. \end{aligned}$$

Note that a nearest-neighbor walk from K to $K^{(\geq s)}$ must pass through $K^{(=s)}$.

Having fixed the disjoint subsets K_1 and K_2 of \mathbb{Z}^d , let us define $s = \text{dist}(K_1, K_2) > 0$ and the auxiliary sets $H_1, H_2 \subseteq \mathbb{Z}^d$ in the following way:

- if $\text{diam}(K_1) \leq \text{diam}(K_2)$, define $H_2 = K_1^{(\geq s)}$ and $H_1 = H_2^{(\geq s)}$,
- if $\text{diam}(K_1) > \text{diam}(K_2)$, define $H_1 = K_2^{(\geq s)}$ and $H_2 = H_1^{(\geq s)}$.

With these definitions we have

$$K_i \subseteq H_i, \quad H_{3-i} = H_i^{(\geq s)}, \quad |H_i^{(=s)}| < \infty, \quad i \in \{1, 2\}. \quad (1.9)$$

Let us define

$$g_s = \sup_{y: \|y\| \geq s} g(0, y).$$

Proposition 1.4. *Denote by $s = \text{dist}(K_1, K_2) > 0$. Then*

$$\mathbb{P}[G_\delta^c] \leq 2|H_1^{(=s)}| \exp\left(-\frac{\delta^2}{8g_s}\right). \quad (1.10)$$

Remark 1.5. *Let us compare the inequality obtained from the combination of (1.8) and (1.10) with the main decoupling result (Theorem 2.1) of [8]. Observing that $g_s = \mathcal{O}(s^{2-d})$ by (1.1), one can note the similarity of the expression in the exponent in the right-hand side of (1.10) with that in the error term in [8, Theorem 2.1]. Also, let r be the minimum of the Euclidean diameters of K_1 and K_2 . Then, one can (very crudely) bound $|H_1^{(=s)}|$ by $\text{const} \times (r + s)^d$ and again observe the similarity with the error term in [8, Theorem 2.1].*

Before we prove the results stated above in Section 3, we give an application of our decoupling inequality in Section 2.

2 Connectivity decay for percolation of excursion sets

For any $h \in \mathbb{R}$, we define the *excursion set* above level h as

$$E_\varphi^{\geq h} = \{x \in \mathbb{Z}^d : \varphi_x \geq h\}.$$

We view $E_\varphi^{\geq h}$ as a random subgraph of \mathbb{Z}^d , and, naturally, one may be interested in studying its percolation properties. Let us write $\{x \overset{\geq h}{\longleftrightarrow} y\}$ for the event when $x, y \in \mathbb{Z}^d$ are connected in $E_\varphi^{\geq h}$. As an application of Corollary 1.3, in this section we establish a result on the decay of the connectivity function of the excursion set $E_\varphi^{\geq h}$.

In [1] (for $d = 3$) and in [10] (for all $d \geq 3$) it was shown that there exists $h_* = h_*(d) \in [0, +\infty)$ such that $E_\varphi^{\geq h}$ percolates for $h < h_*$ and does not percolate for $h > h_*$; moreover, it was also proved that $h_*(d) > 0$ for all sufficiently large d . Further developments regarding percolation of $E_\varphi^{\geq h}$ and its connection to interlacement percolation (based on the isomorphism theorem [15]) can be found in [7, 11].

In [10, (0.6)] the authors define another critical parameter $h_{**} \geq h_*$ as the threshold above which one has at least polynomial decay of the probabilities of certain crossing events:

$$h_{**}(d) = \inf\{h \in \mathbb{R} ; \text{for some } \alpha > 0, \lim_{L \rightarrow \infty} L^\alpha \mathbb{P}[B(0, L) \overset{\geq h}{\longleftrightarrow} S(0, 2L)] = 0\},$$

where the event $\{B(0, L) \overset{\geq h}{\longleftrightarrow} S(0, 2L)\}$ refers to the existence of a nearest-neighbour path in $E_\varphi^{\geq h}$ connecting $B(0, L)$, the ball of radius L around 0 in the ℓ^∞ -norm, to $S(0, 2L)$, the ℓ^∞ -sphere of radius $2L$ around 0.

In [10, Theorem 2.6] they prove that $h_{**}(d) < \infty$ for all $d \geq 3$, and also give a stretched exponential upper bound for the connectivity function $\mathbb{P}[0 \overset{\geq h}{\longleftrightarrow} x]$ as $\|x\| \rightarrow \infty$ for $h > h_{**}$. In the next theorem, we further weaken the definition of h_{**} and, more importantly, we improve on the stretched exponential bound for values of h above h_{**} .

Theorem 2.1. *For $d \geq 4$, given $h > h_{**}(d)$, there exist positive constants $\gamma_1 = \gamma_1(d, h)$ and $\gamma_2 = \gamma_2(d, h)$ such that*

$$\mathbb{P}[0 \overset{\geq h}{\longleftrightarrow} x] \leq \gamma_1 \exp\{-\gamma_2 \|x\|\}, \text{ for every } x \in \mathbb{Z}^d. \quad (2.1)$$

If $d = 3$ and $h > h_{**}(3)$, then for any $b > 1$ there exist $\gamma'_1 = \gamma'_1(h, b)$ and $\gamma'_2 = \gamma'_2(h, b)$ such that

$$\mathbb{P}[0 \overset{\geq h}{\longleftrightarrow} x] \leq \gamma'_1 \exp\left\{-\gamma'_2 \frac{\|x\|}{\log^{3b} \|x\|}\right\}, \text{ for every } x \in \mathbb{Z}^d. \quad (2.2)$$

Moreover, we show that the quantity h_{**} can be defined as

$$h_{**} = \inf \left\{ h > 0; \liminf_{L \rightarrow \infty} \mathbb{P}[[0, L]^d \overset{\geq h}{\longleftrightarrow} \partial[-L, 2L]^d] < \frac{7}{2d \cdot 21^d} \right\}. \quad (2.3)$$

Proof of Theorem 2.1. The reader may have noticed that the above result is a copy of the statement of [8, Theorem 3.1], with obvious notational changes. Indeed, as observed in [8, Remark 3.4], the proof of that theorem can be adapted to *any* percolation model which satisfies certain monotonicity and decoupling properties, that the excursion sets of the GFF do possess.

More specifically, let us denote by \mathbb{P}_h the law of the excursion set $E_\varphi^{\geq h}$. Then (1.8) and (1.10) imply that for any increasing events A_1, A_2 that depend on disjoint boxes of size r within distance at least s from each other, we have

$$\mathbb{P}_h[A_1 \cap A_2] \leq \mathbb{P}_{h-\delta}[A_1] \mathbb{P}_{h-\delta}[A_2] + C(r+s)^d \exp(-C'\delta^2 s^{d-2}). \quad (2.4)$$

This decoupling inequality is a special case of the one in [8, Remark 3.4], the proof of this result is also practically a copy of the proof of [8, Theorem 3.1]. \square

It is important to observe that (2.4) may be seen as a partial replacement of the BK inequality, which is very useful for proving exponential decay of crossing probabilities in the subcritical phase of classical (Bernoulli) percolation, see e.g. Section 5.2 of [4]. While the “pure” BK inequality generally does not hold in the dependent percolation models we mentioned here, inequalities similar to (2.4) are still very useful, even though they usually involve

sprinkling and the additive “error” term. It turns out that, for renormalization arguments, this additive term is pivotal, in the sense that the smaller it is, the better results one obtains for the decay of the probabilities of crossing events. In fact one can even achieve exponential decay (if $d \geq 4$) in polynomially correlated percolation models without the BK inequality.

Another important observation is that the question whether $h_* = h_{**}$ is still open (as well as the corresponding question for random interacements). In our opinion, the conditional decoupling result of our Theorem 1.2 might help in proving it; in fact, it did help in the proof of the fact that $h_*(d)/h_{**}(d) \rightarrow 1$ as $d \rightarrow \infty$, see [3].

3 Proofs of the decoupling results

We start by deducing the correlation bounds with no sprinkling from the corresponding general results of [5, Chapter 10].

Proof of Proposition 1.1. First note that a function $f : \mathbb{R}^{\mathbb{Z}^d} \rightarrow \mathbb{R}$ is supported on K if and only if f is $\sigma(\varphi_K)$ -measurable.

Denote by \mathbf{G} the Gaussian Hilbert space that arises as the closure of vector space of linear combinations of φ_x , $x \in \mathbb{Z}^d$ under the norm $\|X\| = \sqrt{\langle X, X \rangle}$ given by the inner product $\langle X, Y \rangle = \text{Cov}(X, Y)$.

Denote by \mathbf{H} and \mathbf{K} the subspaces of \mathbf{G} spanned by linear combinations of φ_x , $x \in K_1$ and φ_x , $x \in K_2$, respectively. Let us introduce the sigma-algebras $\mathcal{F} = \sigma(\mathbf{H}) = \sigma(\varphi_{K_1})$ and $\mathcal{G} = \sigma(\mathbf{K}) = \sigma(\varphi_{K_2})$.

Recall from [5, Definitions 10.5, 10.6] the notion of the strong mixing coefficient $\alpha(\cdot, \cdot)$ and the maximal correlation coefficient $\rho(\cdot, \cdot)$:

$$\alpha(\mathbf{H}, \mathbf{K}) = \alpha(\mathcal{F}, \mathcal{G}) = \sup_{A \in \mathcal{F}, B \in \mathcal{G}} |\mathbb{P}[A \cap B] - \mathbb{P}[A]\mathbb{P}[B]|,$$

$$\rho(\mathbf{H}, \mathbf{K}) = \rho(\mathcal{F}, \mathcal{G}) = \sup_{X \in L^2(\mathcal{F}), Y \in L^2(\mathcal{G})} \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}.$$

With this notation we have

$$\alpha(\mathbf{H}, \mathbf{K}) \stackrel{(*)}{\leq} \sup_{f_1, f_2} \text{Cov}(f_1(\varphi), f_2(\varphi)) \stackrel{(**)}{\leq} \rho(\mathbf{H}, \mathbf{K}), \quad (3.1)$$

where the supremum is taken over $[0, 1]$ -valued functions $f_1, f_2 : \mathbb{R}^{\mathbb{Z}^d} \rightarrow [0, 1]$, and where f_1 is supported on K_1 and f_2 is supported on K_2 . Indeed, $(*)$ follows if we choose $f_1 = \mathbb{1}_A$ and $f_2 = \mathbb{1}_B$ (or $f_2 = \mathbb{1}_{B^c}$ if $\text{Cov}(\mathbb{1}_A, \mathbb{1}_B) < 0$) and $(**)$ follows because $\text{Var}(f_1), \text{Var}(f_2) \leq 1$.

Denote by $P_{\mathbf{K}} : \mathbf{G} \rightarrow \mathbf{K}$ the orthogonal projection to the subspace \mathbf{K} and by $P_{\mathbf{HK}} : \mathbf{H} \rightarrow \mathbf{K}$ the restriction of $P_{\mathbf{K}}$ to \mathbf{H} . Similarly, let $P_{\mathbf{H}}$ denote the projection to \mathbf{H} and $P_{\mathbf{KH}}$ its restriction to \mathbf{K} . Denote by $\|\cdot\|$ the operator norm of linear operators on (subspaces of) \mathbf{G} .

[5, Theorem 10.11] states that $\rho(\mathbf{H}, \mathbf{K}) = \|P_{\mathbf{HK}}\|$ and [5, Remark 10.1(ii), Theorem 10.13] imply that $\frac{1}{2\pi}\|P_{\mathbf{HK}}\| \leq \alpha(\mathbf{H}, \mathbf{K})$. Combining these results with (3.1) we see that in order to prove Proposition 1.1 we only need to show that there exist constants $0 < c'_d \leq C'_d < +\infty$ that depend only on d such that if K_1, K_2 satisfy (1.2) then we have

$$c'_d \frac{(\text{cap}(K_1)\text{cap}(K_2))^{1/2}}{\text{dist}(K_1, K_2)^{d-2}} \leq \|P_{\mathbf{HK}}\| \leq C'_d \frac{(\text{cap}(K_1)\text{cap}(K_2))^{1/2}}{\text{dist}(K_1, K_2)^{d-2}}. \quad (3.2)$$

First note that the adjoint of $P_{\mathbf{HK}}$ is $P_{\mathbf{KH}}$ (see [5, Remark 10.1]), thus if we define $A : \mathbf{H} \rightarrow \mathbf{H}$ by $A = P_{\mathbf{KH}}P_{\mathbf{HK}}$, then A is self-adjoint and we have (see, for example, [5, Appendix H])

$$\|P_{\mathbf{KH}}\| = \|P_{\mathbf{HK}}\| = \sqrt{\|A\|}. \quad (3.3)$$

Note that we have

$$\|P_{\mathbf{HK}}\| = \sup_{X \in \mathbf{H}, Y \in \mathbf{K}} \frac{\langle P_{\mathbf{K}}(X), Y \rangle}{\|X\|\|Y\|} = \sup_{X \in \mathbf{H}, Y \in \mathbf{K}} \frac{\langle X, Y \rangle}{\|X\|\|Y\|}. \quad (3.4)$$

Now $X \in \mathbf{H}$ if and only if $X = \sum_{x \in K_1} \alpha_x \varphi_x$ for some $\alpha \in \mathbb{R}^{K_1}$ and $Y \in \mathbf{K}$ if and only if $Y = \sum_{y \in K_2} \beta_y \varphi_y$ for some $\beta \in \mathbb{R}^{K_2}$, thus we can use this coordinatization and $\text{Cov}[\varphi_x \varphi_y] = g(x, y)$ to write

$$\langle X, Y \rangle = \text{Cov}(X, Y) = \sum_{x \in K_1, y \in K_2} \alpha_x \beta_y g(x, y), \quad (3.5)$$

$$\|X\|^2 = \text{Var}(X) = \sum_{x \in K_1, y \in K_1} \alpha_x \alpha_y g(x, y). \quad (3.6)$$

Also note that $P_{\mathbf{H}}(\varphi_x) \stackrel{(1.5)}{=} h_x$, $x \in \mathbb{Z}^d$ and that an analogous formula holds for $P_{\mathbf{K}}(\varphi_x)$. In particular, the entries of the matrices of $P_{\mathbf{KH}}$, $P_{\mathbf{HK}}$ and $A = P_{\mathbf{KH}}P_{\mathbf{HK}}$ (expressed in the basis φ_x , $x \in \mathbb{Z}^d$) are all non-negative, thus we can use the Perron-Frobenius theorem to infer that the self-adjoint matrix A has an eigenvector $X_* \in \mathbf{H}$, $\|X_*\| = 1$ such that $\langle X_*, AX_* \rangle = \|A\|$ and X_* has non-negative coordinates in the basis φ_x , $x \in K_1$. We claim that if we define $Y_* = P_{\mathbf{HK}}X_*$, then the pair (X_*, Y_*) maximizes the correlation functional on the right-hand side of (3.4):

$$\|Y_*\| = \sqrt{\langle P_{\mathbf{HK}}X_*, P_{\mathbf{HK}}X_* \rangle} = \sqrt{\langle X_*, P_{\mathbf{KH}}P_{\mathbf{HK}}X_* \rangle} = \sqrt{\|A\|} \stackrel{(3.3)}{=} \|P_{\mathbf{HK}}\|,$$

$$\frac{\langle X_*, Y_* \rangle}{\|X_*\| \|Y_*\|} = \frac{\langle X_*, P_{\mathbf{HK}} X_* \rangle}{\|P_{\mathbf{HK}}\|} = \frac{\langle P_{\mathbf{HK}} X_*, P_{\mathbf{HK}} X_* \rangle}{\|P_{\mathbf{HK}}\|} = \frac{\|A\|}{\|P_{\mathbf{HK}}\|} \stackrel{(3.3)}{=} \|P_{\mathbf{HK}}\|.$$

We can thus infer that the maximum in (3.4) remains unchanged if we assume

$$\alpha_x \geq 0, x \in K_1, \quad \beta_y \geq 0, y \in K_2, \quad \sum_{x \in K_1} \alpha_x = 1 \quad \text{and} \quad \sum_{y \in K_2} \beta_y = 1. \quad (3.7)$$

Using these assumptions we can bound

$$\begin{aligned} c'_d \text{dist}(K_1, K_2)^{2-d} &\stackrel{(1.1), (1.2)}{\leq} \min_{x \in K_1, y \in K_2} g(x, y) \stackrel{(3.5), (3.7)}{\leq} \langle X, Y \rangle \\ &\stackrel{(3.5), (3.7)}{\leq} \max_{x \in K_1, y \in K_2} g(x, y) \stackrel{(1.1)}{\leq} C'_d \text{dist}(K_1, K_2)^{2-d}. \end{aligned} \quad (3.8)$$

A combination of (3.6) and the variational characterization [9, Lemma 2.3] of capacity gives

$$\sup_{\alpha} \frac{1}{\|X\|^2} = \text{cap}(K_1), \quad \sup_{\beta} \frac{1}{\|Y\|^2} = \text{cap}(K_2), \quad (3.9)$$

where the maximum is taken over all $\alpha \in \mathbb{R}^{K_1}$ and $\beta \in \mathbb{R}^{K_2}$ satisfying (3.7). Putting together (3.4), (3.8) and (3.9) we arrive at (3.2). The proof of Proposition 1.1 is complete. \square

Now we prove the conditional decoupling result:

Proof of Theorem 1.2. Recall the decomposition $\varphi = \tilde{\varphi} + h$ from (1.4). Let us assume without loss of generality that our probability space is rich enough to carry an independent copy \hat{h} of the field h . Denote

$$\hat{\varphi} = \tilde{\varphi} + \hat{h}.$$

Clearly, φ and $\hat{\varphi}$ have the same law. Let \hat{G}_δ be the event defined as in (1.6), but with \hat{h} replacing h . Then, write

$$\begin{aligned} \mathbb{E}(f_2(\varphi) \mid \varphi_{K_1}) \mathbf{1}_{G_\delta} &= \mathbb{E}(f_2(\tilde{\varphi} + h) \mid \varphi_{K_1}) \mathbf{1}_{G_\delta} \\ &= \mathbb{E}(f_2(\hat{\varphi} + h - \hat{h}) \mid \varphi_{K_1}) \mathbf{1}_{G_\delta} \\ &= \mathbb{E}(f_2(\hat{\varphi} + h - \hat{h}) \mathbf{1}_{G_\delta \cap \hat{G}_\delta} \mid \varphi_{K_1}) \\ &\quad + \mathbb{E}(f_2(\hat{\varphi} + h - \hat{h}) \mathbf{1}_{G_\delta \cap \hat{G}_\delta^c} \mid \varphi_{K_1}) \\ &=: T_1 + T_2. \end{aligned} \quad (3.10)$$

Clearly, we have

$$0 \leq T_2 \leq \mathbb{P}[\widehat{G}_\delta^c] \mathbf{1}_{G_\delta}. \quad (3.11)$$

Since, by construction, $\widehat{\varphi}$ is independent of φ_{K_1} , moreover $|h - \widehat{h}| \leq \delta$ on $G_\delta \cap \widehat{G}_\delta$ and f_2 is increasing, we can write

$$\begin{aligned} T_1 &\leq \mathbb{E}(f_2(\widehat{\varphi} + \delta) \mathbf{1}_{G_\delta \cap \widehat{G}_\delta} \mid \varphi_{K_1}) \\ &\leq \mathbb{E}(f_2(\widehat{\varphi} + \delta) \mid \varphi_{K_1}) \mathbf{1}_{G_\delta} \\ &= \mathbb{E}(f_2(\widehat{\varphi} + \delta)) \mathbf{1}_{G_\delta}. \end{aligned} \quad (3.12)$$

Also, we have

$$\begin{aligned} T_1 &\geq \mathbb{E}(f_2(\widehat{\varphi} - \delta) \mathbf{1}_{\widehat{G}_\delta} \mid \varphi_{K_1}) \mathbf{1}_{G_\delta} \\ &= \mathbb{E}(f_2(\widehat{\varphi} - \delta)(1 - \mathbf{1}_{\widehat{G}_\delta^c})) \mathbf{1}_{G_\delta} \\ &\geq \mathbb{E}(f_2(\widehat{\varphi} - \delta)) \mathbf{1}_{G_\delta} - \mathbb{P}[\widehat{G}_\delta^c] \mathbf{1}_{G_\delta}. \end{aligned} \quad (3.13)$$

Inserting (3.11)–(3.13) into (3.10) and using the fact that $\widehat{\varphi}$ and φ are equally distributed and $\mathbb{P}[\widehat{G}_\delta^c] = \mathbb{P}[G_\delta^c]$, we conclude the proof of Theorem 1.2. \square

Proof of Corollary 1.3. Now, let $f_1 : \mathbb{R}^{\mathbb{Z}^d} \rightarrow [0, 1]$ be a function supported on K_1 . Since

$$\mathbb{E}f_1(\varphi) - \mathbb{P}[G_\delta^c] \leq \mathbb{E}(f_1(\varphi) \mathbf{1}_{G_\delta}) \leq \mathbb{E}f_1(\varphi),$$

it is then straightforward to obtain (1.8) by multiplying (1.7) by $f_1(\varphi)$ and integrating. \square

Proof of Proposition 1.4. Define the events

$$\Lambda_{\delta, x} = \{|h_x| \leq \delta/2\}, \quad x \in H_1^{(=s)}.$$

Clearly, h_x is a centered Gaussian random variable, thus we can use (1.5) and $\mathbb{E}[\varphi_x \varphi_y] = g(x, y)$ to bound the variance of h_x , $x \in H_1^{(=s)}$:

$$\begin{aligned} \text{Var } h_x &= \sum_{y \in K_1} P_x[H_{K_1} < \infty, X_{H_{K_1}} = y] \sum_{z \in K_1} P_x[H_{K_1} < \infty, X_{H_{K_1}} = z] g(z, y) \\ &\stackrel{(*)}{=} \sum_{y \in K_1} P_x[H_{K_1} < \infty, X_{H_{K_1}} = y] g(x, y) \leq \sup_{y \in K_1} g(x, y) \stackrel{(1.9)}{\leq} g_s, \end{aligned}$$

where (*) holds by the strong Markov property of simple random walk:

$$g(x, y) = E_x \left[\sum_{n=0}^{\infty} \mathbb{1}_{[X_n=y]} \right] = E_x \left[\sum_{n=H_{K_1}}^{\infty} \mathbb{1}_{[X_n=y]} \right] =$$

$$E_x \left(g(X_{H_{K_1}}, y); H_{K_1} < \infty \right) = \sum_{z \in K_1} P_x[H_{K_1} < \infty, X_{H_{K_1}} = z] g(z, y).$$

Thus we can use the exponential Chebyshev's inequality to bound

$$\mathbb{P}[\Lambda_{\delta,x}^c] \leq 2 \exp \left(- \frac{\delta^2}{8g_s} \right).$$

Observe that by (1.9) any nearest-neighbor walk from K_2 to K_1 must pass through $H_1^{(=s)}$, so by the strong Markov property of the simple random walk on \mathbb{Z}^d and (1.5), for any $y \in K_2$, the value of h_y is a weighted sum of the values $(h_x, x \in H_1^{(=s)})$, with total weight at most 1. In particular, we have

$$G_{\delta}^c \subseteq \bigcup_{x \in H_1^{(=s)}} \Lambda_{\delta,x}^c.$$

Using the union bound we conclude the proof of Proposition 1.4. \square

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