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A short proof of the phase transition for the vacant set of random interlacements*

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Abstract

The vacant set of random interlacements at level $u > 0$, introduced in [8], is a percolation model on $\mathbb{Z}^d$, $d \geq 3$ which arises as the set of sites avoided by a Poissonian cloud of doubly infinite trajectories, where u is a parameter controlling the density of the cloud. It was proved in [6, 8] that for any $d \geq 3$ there exists a positive and finite threshold u_* such that if $u < u_*$ then the vacant set percolates and if $u > u_*$ then the vacant set does not percolate. We give an elementary proof of these facts. Our method also gives simple upper and lower bounds on the value of u_* for any $d \geq 3$.

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1 Introduction

The model of random interlacements was introduced in [8]. The interlacement $\mathcal{I}^u$ at level $u > 0$ is a random subset of $\mathbb{Z}^d$, $d \geq 3$ that arises as the local limit as $N \to \infty$ of the range of the first $\lfloor uN^d \rfloor$ steps of a simple random walk on the discrete torus $(\mathbb{Z}/N\mathbb{Z})^d$, $d \geq 3$, see [14]. The law of $\mathcal{I}^u$ is characterized by

$$\mathbb{P}[\mathcal{I}^u \cap K = \emptyset] = e^{-u \cdot \mathrm{cap}(K)}, \quad \text{for any finite } K \subseteq \mathbb{Z}^d, \tag{1.1}$$

where $\mathrm{cap}(K)$ denotes the discrete capacity of K, see (2.5). The vacant set of random interlacements $\mathcal{V}^u$ at level u is defined as the complement of $\mathcal{I}^u$ at level u:

$$\mathcal{V}^u = \mathbb{Z}^d \setminus \mathcal{I}^u, \quad u > 0. \tag{1.2}$$

By [8, (1.68)] the correlations of $\mathcal{V}^u$ decay polynomially for any $u > 0$:

$$\mathbb{P}[x, y \in \mathcal{V}^u] - \mathbb{P}[x \in \mathcal{V}^u] \cdot \mathbb{P}[y \in \mathcal{V}^u] \asymp (|x-y| \vee 1)^{2-d}, \qquad x, y \in \mathbb{Z}^d. \tag{1.3}$$

One is interested in the connectivity properties of the subgraphs of the nearest-neighbour lattice $\mathbb{Z}^d$ spanned by the above random sets. For any $u > 0$, $\mathcal{I}^u$ is a $\mathbb{P}$-a.s. connected random subset of $\mathbb{Z}^d$ (see [8, (2.21)]), but $\mathcal{V}^u$ exhibits a percolation phase transition: there exists $u_* \in (0, \infty)$ such that

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(i) for any $u > u_*$, $\mathbb{P}$-a.s. all connected components of $\mathcal{V}^u$ are finite, and

(ii) for any $u < u_*$, $\mathbb{P}$-a.s. $\mathcal{V}^u$ contains an infinite connected component.

The fact that $u_* < \infty$ was proved in [8, Section 3], and the positivity of u_* was established in [8, Section 4] when $d \geq 7$, and later in [6] for all $d \geq 3$.

There is no reason to believe that an exact formula for the value of the critical threshold $u_* = u_*(d)$ exists. However, it is proved in [9, 10] that

$$\lim_{d \to \infty} \frac{u_*(d)}{\ln(d)} = 1, \tag{1.4}$$

in agreement with the principal asymptotic behaviour of the critical threshold of random interlacements on $2d$-regular trees, which is explicitly computed in [12, Proposition 5.2].

The aim of this paper is to give a short proof of the non-triviality of phase transition of $\mathcal{V}^u$ and to provide simple explicit upper and lower bounds on the value of $u_* = u_*(d), d \geq 3$.

For any $d \geq 3$ let us denote by $0 < c_g = c_g(d)$ and $C_g = C_g(d) < +\infty$ the best constants such that the inequalities

$$c_g \cdot (|x-y| \vee 1)^{2-d} \leq g(x,y) \leq C_g \cdot (|x-y| \vee 1)^{2-d}, \qquad x, y \in \mathbb{Z}^d \tag{1.5}$$

hold, where $|\cdot|$ is the ℓ^∞-norm on $\mathbb{Z}^d$ and $g(\cdot,\cdot)$ is the Green function of simple random walk on $\mathbb{Z}^d$, see (2.3). The positivity of c_g and $C_g < +\infty$ follow from [4, Theorem 1.5.4].

Theorem 1.1. *For any $d \geq 3$, we have*

$$\frac{c_g}{L_0} \frac{1}{\mathcal{C}_2} 2^{-(d+5)} \leq u_* \leq \frac{5}{2} C_g \ln(\mathcal{C}_d), \tag{1.6}$$

where

$$\mathcal{C}_d = (13^d - 11^d)(25^d - 23^d), \quad d \geq 2, \tag{1.7}$$

and

$$L_0 = \begin{cases} \left\lceil \exp\left(48 \frac{C_g}{c_g} \mathcal{C}_2\right) \right\rceil & \textit{if} \quad d = 3, \\ \left\lceil \left(48 \frac{C_g}{c_g} \mathcal{C}_2\right)^{\frac{1}{d-3}} \right\rceil & \textit{if} \quad d \geq 4. \end{cases} \tag{1.8}$$

The bounds (1.6) are not at all sharp, especially if we compare them with (1.4) as $d \to \infty$. This shortcoming of Theorem 1.1 is counterbalanced by the fact that its proof is very simple. In particular, our self-contained proof does not use the *"sprinkling" technique* and *decoupling inequalities* usually applied in order to overcome the long-range correlations (1.3) present in the model. The proof of $u_*(d) > 0$ for $d \geq 7$ in [8, Section 4] does not use "sprinkling", but the proof of $u_*(d) < +\infty$ for any $d \geq 3$ in [8, Section 3] and the proof of $u_*(d) > 0$ for $3 \leq d \leq 7$ in [6] does. Various forms of decoupling inequalities have been subsequently developed to study the connectivity properties of $\mathcal{V}^u$ in the subcritical [5, 7, 11] and supercritical [2, 13] phases. These techniques are very useful once they are available, but the elementary method of our paper seems to be easier to adapt to other percolation models with long-range correlations, e.g., *branching interlacements* [1].

Let us briefly describe the idea of the proof of Theorem 1.1. We employ multi-scale renormalization. In order to prove $u_* < +\infty$ we show that if $\mathcal{V}^u$ crosses an annulus at scale $L_n = 6^n$ then this vacant crossing contains a set $\mathcal{X}_\mathcal{T}$ of 2^n well-separated vertices which arises as the image of leaves under an embedding $\mathcal{T}$ of the dyadic tree of depth n (this method already appears in [11]). By construction, the number of possible embeddings is less than $\mathcal{C}_d^{2^n}$ (c.f. (1.7)), so we only need to show that $\mathrm{cap}(\mathcal{X}_\mathcal{T}) \asymp 2^n$ if we want to use (1.1) to to show that crossing of the annulus by $\mathcal{V}^u$ is unlikely when u is big enough. This is indeed the case, because by construction the embedding $\mathcal{T}$ is "spread-out on all scales", thus the cardinality and the capacity of $\mathcal{X}_\mathcal{T}$ are comparable.

In order to prove $u_* > 0$, we restrict our attention to a plane inside $\mathbb{Z}^d$. By planar duality we only need to show that a $*$-connected crossing of a planar annulus at scale $L_n = L_0 \cdot 6^n$ by $\mathcal{I}^u$ is unlikely. We show that such a crossing must intersect 2^n "frames", where each frame is the union of four "sticks" of length $2L_0 - 1$. Such a collection of frames again arises from a spread-out embedding of the dyadic tree of depth n. We use that $\mathcal{I}^u$ can be written as the union of the ranges of a Poissonian cloud of independent random walks and the fact that random walks tend to avoid sticks if L_0 is large enough (c.f. (1.8)) to arrive at a large deviation estimate on the probability that the number of frames that intersect $\mathcal{I}^u$ is 2^n which is strong enough to beat the combinatorial complexity term $\mathcal{C}_2^{2^n}$. This stick-based approach to $u_* > 0$ is already present in [6, Section 3] and our large deviation estimate resembles the one in the proof of [8, Theorem 2.4].

The rest of this paper is organized as follows.

In Section 2 we introduce further notation and recall some useful facts related to the notion of capacity and random interlacements. In Section 3 we define the notion of a *proper embedding* of a dyadic tree into $\mathbb{Z}^d$ and derive some facts about such embeddings. In Sections 4 and 5 we prove the upper and lower bounds on u_* stated in Theorem 1.1.

2 Preliminaries

For a set K, we denote by $|K|$ its cardinality. We denote by $K \subset\subset \mathbb{Z}^d$ the fact that K is a finite subset of $\mathbb{Z}^d$. We denote by $|x|$ the ℓ^∞-norm of $x \in \mathbb{Z}^d$ and by $S(x,R)$ the ℓ^∞-sphere of radius R about x in $\mathbb{Z}^d$:

$$S(x,R) = \{y \in \mathbb{Z}^d \,:\, |y-x| = R\}. \tag{2.1}$$

For $x \in \mathbb{Z}^d$, denote by P_x the law of simple random walk $(X_n)_{n=0}^\infty$ on $\mathbb{Z}^d$ starting at $X_0 = x$. If m is a probability measure on $\mathbb{Z}^d$, we denote by

$$P_m = \sum_{x \in \mathbb{Z}^d} m(x) P_x \tag{2.2}$$

the law of simple random walk with initial distribution m and by E_m the corresponding expectation. The Green function of simple random walk on $\mathbb{Z}^d$ is defined by

$$g(x,y) = \sum_{n=0}^\infty P_x[X_n = y], \quad x, y \in \mathbb{Z}^d. \tag{2.3}$$

Let us denote by $\{X\} \subseteq \mathbb{Z}^d$ the range of the random walk:

$$\{X\} = \cup_{n=0}^\infty \{X_n\} \tag{2.4}$$

2.1 Potential theory

If $K \subset\subset \mathbb{Z}^d$, we define the equilibrium measure $e_K(\cdot)$ of K by

$$e_K(x) = P_x[\, X_n \notin K \text{ for any } n \geq 1\,], \qquad x \in K.$$

The total mass of the equilibrium measure is called the capacity of K:

$$\mathrm{cap}(K) = \sum_{x \in K} e_K(x). \tag{2.5}$$

One defines the normalized equilibrium measure $\widetilde{e}_K(\cdot)$ of K by

$$\widetilde{e}_K(x) = \frac{e_K(x)}{\mathrm{cap}(K)}. \tag{2.6}$$

Let us now collect some facts about capacity that we will use in the sequel. The proofs of the properties (2.7)-(2.10) below can be found in, e.g., [3, Section 1.3].

For any $x \in \mathbb{Z}^d$ and any $K \subset\subset \mathbb{Z}^d$,

$$P_x[\{X\} \cap K \neq \emptyset] = \sum_{y \in K} g(x,y) e_K(y) \overset{(2.5)}{\leq} \mathrm{cap}(K) \max_{y \in K} g(x,y). \tag{2.7}$$

For any $K_1, K_2 \subset\subset \mathbb{Z}^d$,

$$\mathrm{cap}(K_1 \cup K_2) \leq \mathrm{cap}(K_1) + \mathrm{cap}(K_2). \tag{2.8}$$

For any $K \subseteq K' \subset\subset \mathbb{Z}^d$,

$$\mathrm{cap}(K) \leq \mathrm{cap}(K'). \tag{2.9}$$

For any $K \subset\subset \mathbb{Z}^d$,

$$\frac{|K|}{\max_{x \in K} \sum_{y \in K} g(x,y)} \leq \mathrm{cap}(K) \leq \frac{|K|}{\min_{x \in K} \sum_{y \in K} g(x,y)}. \tag{2.10}$$

Let us denote by F the plane

$$F = \mathbb{Z}^2 \times \{0\}^{d-2} \subseteq \mathbb{Z}^d. \tag{2.11}$$

For any $y \in F$ and $L \geq 1$ let us define the frame $\square_y^L \subseteq F$ by

$$\square_y^L \overset{(2.1)}{=} S(y, L-1) \cap F. \tag{2.12}$$

The next lemma gives an explicit upper bound on the capacity of a frame. The bounds of (2.13) are actually sharp up to a dimension-dependent constant factor, but we will only use the upper bounds. The stronger bound for $d = 3$ is crucial to showing that random walks tend to avoid frames in $\mathbb{Z}^3$. The extra $\ln(L_0)$ makes the parameter p defined in (5.6) small, which is necessary for our proof of $u_*(3) > 0$. Recall the notion of c_g from (1.5).

Lemma 2.1. *For any $L \geq 1$ we have*

$$\mathrm{cap}\left(\square_y^L\right) \leq \begin{cases} 8\frac{L}{c_g} & \textit{if} \quad d \geq 4, \\ 8\frac{L}{c_g \cdot (1+\ln(L))} & \textit{if} \quad d = 3. \end{cases} \tag{2.13}$$

Proof. Denote by $\mathcal{S}_\ell = \{1, \dots, \ell\} \times \{0\}^{d-1} \subseteq \mathbb{Z}^d$ the stick of length ℓ. We will use (2.10) to bound $\mathrm{cap}(\mathcal{S}_\ell)$. If $x \in \mathcal{S}_\ell$ then $x = \{i\} \times \{0\}^{d-1}$ for some $1 \leq i \leq \ell$ and

$$\sum_{y \in \mathcal{S}_\ell} g(x,y) \overset{(1.5)}{\geq} \sum_{j=1}^{\ell} c_g \cdot (|j-i| \vee 1)^{2-d} \geq \sum_{j=1}^{\ell} c_g \cdot (|j-1| \vee 1)^{2-d} =$$

$$c_g \cdot \left(1 + \sum_{k=1}^{\ell-1} k^{2-d}\right) \geq \begin{cases} c_g & \text{if} \quad d \geq 4, \\ c_g \cdot \left(1 + \int_1^\ell \frac{1}{s}\, \mathrm{d}s\right) = c_g \cdot (1 + \ln(\ell)) & \text{if} \quad d = 3. \end{cases}$$

Using these bounds, (2.10) and $|\mathcal{S}_\ell| = \ell$ we obtain that $\mathrm{cap}\,(\mathcal{S}_\ell) \leq \ell / c_g$ if $d \geq 4$ and $\mathrm{cap}\,(\mathcal{S}_\ell) \leq \ell / (c_g \cdot (1 + \ln(\ell)))$ if $d = 3$. Now the frame $\square_y^L$ is the union of four sticks of length $2L - 1$, thus (2.13) follows from the above bounds and (2.8), (2.9).

□

2.2 Constructive definition of random interlacements

The definition of the interlacement $\mathcal{I}^u$ at level u by the formula (1.1) is short, but it is not constructive. The construction of [8, Section 1] involves a Poisson point process with intensity measure $u \cdot \nu$, where ν is a sigma-finite measure on the space of equivalence classes of doubly infinite trajectories modulo time-shift. The union of the ranges of trajectories which are contained in the support of this Poisson point process is denoted by $\mathcal{I}^u$, and this random subset of $\mathbb{Z}^d$ indeed satisfies (1.1).

We will not use the full definition of random interlacements, only a corollary of it, which allows one to construct a set with the same law as $\mathcal{I}^u \cap K$ for any $K \subset\subset \mathbb{Z}^d$.

Recall the notion of P_m from (2.2), $\{X\}$ from (2.4) and $\widetilde{e}_K(\cdot)$ from (2.6).

Claim 2.2. *Let $d \geq 3$, $K \subset\subset \mathbb{Z}^d$, N_K be a Poisson random variable with parameter $u \cdot \mathrm{cap}(K)$, and $(X^j)_{j \geq 1}$ i.i.d. simple random walks with distribution $P_{\widetilde{e}_K}$ and independent from N_K. Then $K \cap \cup_{j=1}^{N_K} \{X^j\}$ has the same distribution as $\mathcal{I}^u \cap K$.*

This explicit "local representation" of $\mathcal{I}^u$ follows from the very construction of the sigma-finite measure ν, which is obtained by patching together certain explicit measures Q_K, $K \subset\subset \mathbb{Z}^d$ in a consistent manner in [8, Theorem 1.1]. The above representation of $\mathcal{I}^u \cap K$ is obtained from the Poisson point process with intensity measure uQ_K.

3 Renormalization

For $n \geq 0$, let $T_{(n)} = \{1, 2\}^n$ (in particular, $T_{(0)} = \emptyset$). Denote by

$$T_n = \bigcup_{k=0}^{n} T_{(k)}$$

the dyadic tree of depth n. For $0 \leq k < n$ and $m \in T_{(k)}$, $m = (\xi_1, \ldots, \xi_k)$, we denote by

$$m_1 = (\xi_1, \ldots, \xi_k, 1) \qquad \text{and} \qquad m_2 = (\xi_1, \ldots, \xi_k, 2) \tag{3.1}$$

the two children of m in $T_{(k+1)}$. Given some $L_0 \geq 1$ we define the sequence of scales

$$L_n := L_0 \cdot 6^n, \quad n \geq 0. \tag{3.2}$$

For $n \geq 0$, we denote by $\mathcal{L}_n = L_n \mathbb{Z}^d$ the lattice $\mathbb{Z}^d$ renormalized by L_n.

Definition 3.1. *$\mathcal{T} : T_n \to \mathbb{Z}^d$ is a proper embedding of T_n with root at $x \in \mathcal{L}_n$ if*

1. *$\mathcal{T}(\emptyset) = x$;*
2. *for all $0 \leq k \leq n$ and $m \in T_{(k)}$ we have $\mathcal{T}(m) \in \mathcal{L}_{n-k}$;*
3. *for all $0 \leq k < n$ and $m \in T_{(k)}$ we have*

$$|\mathcal{T}(m_1) - \mathcal{T}(m)| = L_{n-k}, \qquad |\mathcal{T}(m_2) - \mathcal{T}(m)| = 2L_{n-k}. \tag{3.3}$$

We denote by $\Lambda_{n,x}$ the set of proper embeddings of T_n into $\mathbb{Z}^d$ with root at x.

Lemma 3.2. *For any $L_0 \geq 1$, $n \geq 0$ and $x \in \mathcal{L}_n$ the number of proper embeddings of T_n into $\mathbb{Z}^d$ with root at x is equal to*

$$|\Lambda_{n,x}| \overset{(1.7)}{=} \mathcal{C}_d^{2^n - 1}. \tag{3.4}$$

Proof. The claim is trivially true for $n = 0$. If $n \geq 1$, $x \in \mathcal{L}_n$ and $\mathcal{T} \in \Lambda_{n,x}$, we denote by $\mathcal{T}_1$ and $\mathcal{T}_2$ the two embeddings of T_{n-1} which arise from $\mathcal{T}$ as the embeddings of the descendants of the two children of the root, i.e., for any $0 \leq k \leq n-1$ and $m = (\xi_1, \dots, \xi_k) \in T_{(k)}$ let $\mathcal{T}_\xi(m) = \mathcal{T}(\xi, \xi_1, \xi_2, \dots, \xi_k)$ for $\xi \in \{1, 2\}$. By Definition 3.1 we have $\mathcal{T}_\xi \in \Lambda_{n-1,\mathcal{T}(\xi)}$ for $\xi \in \{1, 2\}$, thus we obtain (3.4) by induction on n:

$$|\Lambda_{n,x}| \overset{(3.3)}{=} |S(x, L_n) \cap \mathcal{L}_{n-1}| \cdot |S(x, 2L_n) \cap \mathcal{L}_{n-1}| \cdot |\Lambda_{n-1,\mathcal{T}(1)}| \cdot |\Lambda_{n-1,\mathcal{T}(2)}| \overset{(3.2)}{=}$$
$$|S(0,6)| \cdot |S(0,12)| \cdot |\Lambda_{n-1,\mathcal{T}(1)}| \cdot |\Lambda_{n-1,\mathcal{T}(2)}| \overset{(*)}{=} \mathcal{C}_d \cdot \mathcal{C}_d^{2^{n-1}-1} \cdot \mathcal{C}_d^{2^{n-1}-1} = \mathcal{C}_d^{2^n-1},$$

where in $(*)$ we used the induction hypothesis. □

We say that $\gamma : \{0, \dots, l\} \to \mathbb{Z}^d$ is a $*$-connected path if $|\gamma(i) - \gamma(i-1)| = 1$ for any $1 \leq i \leq l$. For such a path we denote by $\{\gamma\} = \{\gamma(1), \dots, \gamma(l)\}$ the range of γ.

Recall the notion of $S(x, R)$ from (2.1) and note that $S(x, 0) = \{x\}$.

Lemma 3.3. *If γ is a $*$-connected path in $\mathbb{Z}^d$, $d \geq 2$ and $x \in \mathcal{L}_n$ such that*

$$\{\gamma\} \cap S(x, L_n - 1) \neq \emptyset \quad \textit{and} \quad \{\gamma\} \cap S(x, 2L_n) \neq \emptyset \tag{3.5}$$

then there exists $\mathcal{T} \in \Lambda_{n,x}$ such that

$$\{\gamma\} \cap S(\mathcal{T}(m), L_0 - 1) \neq \emptyset \quad \textit{for all} \quad m \in T_{(n)}. \tag{3.6}$$

Proof. We will prove that (3.5) implies that there exists $\mathcal{T} \in \Lambda_{n,x}$ such that for all $0 \leq k \leq n$ we have

$$\begin{matrix} \{\gamma\} \cap S(\mathcal{T}(m), L_{n-k} - 1) \neq \emptyset \\ \{\gamma\} \cap S(\mathcal{T}(m), 2L_{n-k}) \neq \emptyset \end{matrix} \quad \text{for all} \quad m \in T_{(k)}. \tag{3.7}$$

We will construct such a $\mathcal{T} \in \Lambda_{n,x}$ by induction on k. By $\mathcal{T}(\emptyset) = x$ we see that the case $k = 0$ of (3.7) is just (3.5). Assuming that (3.7) holds for some $0 \leq k \leq n-1$ we now show that it also holds for $k+1$. If $m \in T_{(k)}$ then our induction hypothesis (3.7) and the fact that γ is a $*$-connected path imply

$$\{\gamma\} \cap S(\mathcal{T}(m), L_{n-k} + L_{n-k-1} - 1) \neq \emptyset,$$
$$\{\gamma\} \cap S(\mathcal{T}(m), 2L_{n-k} - L_{n-k-1} + 1) \neq \emptyset.$$

We also have

$$S(\mathcal{T}(m), L_{n-k} + L_{n-k-1} - 1) \subseteq \bigcup_{y \in S(\mathcal{T}(m), L_{n-k}) \cap \mathcal{L}_{n-k-1}} S(y, L_{n-k-1} - 1),$$
$$S(\mathcal{T}(m), 2L_{n-k} - L_{n-k-1} + 1) \subseteq \bigcup_{z \in S(\mathcal{T}(m), 2L_{n-k}) \cap \mathcal{L}_{n-k-1}} S(z, L_{n-k-1} - 1),$$

thus we can choose

$$\mathcal{T}(m_1) \in S(\mathcal{T}(m), L_{n-k}) \cap \mathcal{L}_{n-k-1} \quad \text{and} \quad \mathcal{T}(m_2) \in S(\mathcal{T}(m), 2L_{n-k}) \cap \mathcal{L}_{n-k-1}$$

such that

$$\{\gamma\} \cap S(\mathcal{T}(m_1), L_{n-(k+1)} - 1) \neq \emptyset, \qquad \{\gamma\} \cap S(\mathcal{T}(m_2), L_{n-(k+1)} - 1) \neq \emptyset.$$

It follows from this, $|\mathcal{T}(m_1) - \mathcal{T}(m_2)| \geq L_{n-k} = 6L_{n-(k+1)}$ and the fact that γ is a $*$-connected path that we also have

$$\{\gamma\} \cap S(\mathcal{T}(m_1), 2L_{n-(k+1)}) \neq \emptyset, \qquad \{\gamma\} \cap S(\mathcal{T}(m_2), 2L_{n-(k+1)}) \neq \emptyset.$$

We have thus constructed the embedding $\mathcal{T}$ up to depth $k+1$ so that Definition 3.1 is satisfied up to depth $k+1$ and (3.7) also holds for $k+1$. Therefore by induction we have constructed $\mathcal{T} \in \Lambda_{n,x}$ such that (3.7) holds for all $0 \leq k \leq n$, which implies (3.6). The proof of Lemma 3.3 is complete. □

For $0 \le k \le n$ and $m = (\xi_1, \dots, \xi_n) \in T_{(n)}$ we denote $m|_k = (\xi_1, \dots, \xi_k) \in T_{(k)}$. Let us denote the lexicographic distance of $m, m' \in T_{(n)}$ by

$$\rho(m, m') = \min\{k \ge 0 \ : \ m|_{n-k} = m'|_{n-k}\}.$$

For any $m \in T_{(n)}$ and $0 \le k \le n$ we define

$$T_{(n)}^{m,k} = \{m' \in T_{(n)} \ : \ \rho(m, m') = k\}, \tag{3.8}$$

see Figure 1 for an illustration. Note that

$$|T_{(n)}^{m,k}| = 2^{k-1}, \qquad 1 \le k \le n. \tag{3.9}$$

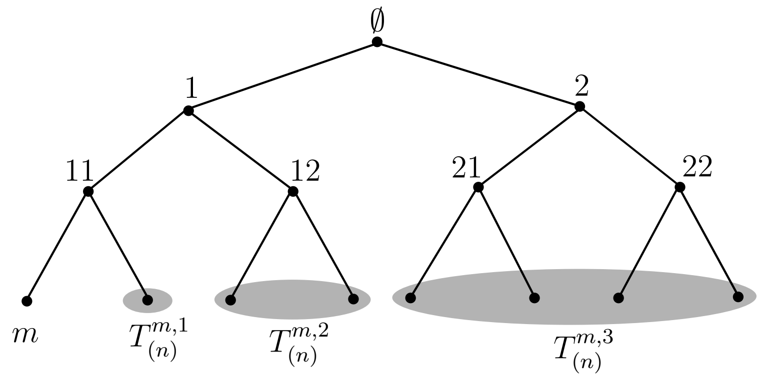

Figure 1: An illustration of the subsets $T_{(n)}^{m,k}$ of leaves of T_n defined in (3.8). The dyadic tree on the picture is of depth $n = 3$ and the leaf denoted by m is $111 \in T_{(n)}$.

The next lemma shows that a proper embedding is “spread-out on all scales."

Lemma 3.4.

$$\begin{gathered} \forall\, n \ge 1,\ x \in \mathcal{L}_n,\ \mathcal{T} \in \Lambda_{n,x},\ m \in T_{(n)},\ k \ge 1, \\ \forall\, m' \in T_{(n)}^{m,k},\ y \in S(\mathcal{T}(m), L_0 - 1),\ z \in S(\mathcal{T}(m'), L_0 - 1): \\ |y - z| \ge L_{k-1}. \end{gathered} \tag{3.10}$$

Proof. Let $m'' = m|_{n-k} = m'|_{n-k} \in T_{(n-k)}$. Recalling (3.1) we may assume w.l.o.g. that $m|_{n-k+1} = m''_1 \in T_{(n-k+1)}$ and $m'|_{n-k+1} = m''_2 \in T_{(n-k+1)}$. We have

$$|\mathcal{T}(m''_1) - \mathcal{T}(m''_2)| \overset{(3.3)}{\ge} L_k \overset{(3.2)}{=} 6L_{k-1},$$

moreover

$$|\mathcal{T}(m''_1) - y| \le |\mathcal{T}(m) - y| + \sum_{j=1}^{k-1} \left| \mathcal{T}(m|_{n-j}) - \mathcal{T}(m|_{n-j+1}) \right| \overset{(3.3)}{\le} \\ L_0 - 1 + \sum_{j=1}^{k-1} 2L_j \overset{(3.2)}{\le} 2L_{k-1} \sum_{i=0}^{\infty} 6^{-i} = \frac{12}{5} L_{k-1},$$

and similarly $|\mathcal{T}(m''_2) - z| \le \frac{12}{5} L_{k-1}$. Putting these bounds together we obtain (3.10). □

4 Upper bound on u_*

Let us choose $L_0 = 1$ in (3.2). For $n \geq 1$ let us denote by A_n^u the event

$$A_n^u = \left\{ \begin{array}{c} \text{there exists a nearest-neighbour path in } \mathcal{V}^u \\ \text{that connects } S(0, L_n - 1) \text{ to } S(0, 2L_n) \end{array} \right\}.$$

Recall the definitions of C_g from (1.5) and $\mathcal{C}_d$ from (1.7).

Proposition 4.1. *For any $d \geq 3$ and*

$$u > \frac{5}{2} C_g \ln(\mathcal{C}_d) \tag{4.1}$$

there exists $q = q(d, u) \in (0, 1)$ such that for any $n \geq 1$ we have

$$\mathbb{P}[A_n^u] \leq q^{2^n}. \tag{4.2}$$

Corollary 4.2. *Proposition 4.1 implies the upper bound of Theorem 1.1, as we now explain. Let us denote by $\widetilde{A}_n^u$ the event that there exists a nearest-neighbour path in $\mathcal{V}^u$ that connects $S(0, L_n - 1)$ to infinity and by $\widetilde{A}_\infty^u$ the event that $\mathcal{V}^u$ has an infinite connected component. If* (4.1) *holds, then*

$$\mathbb{P}[\widetilde{A}_\infty^u] \overset{(*)}{=} \lim_{n\to\infty} \mathbb{P}[\widetilde{A}_n^u] \leq \lim_{n\to\infty} \mathbb{P}[A_n^u] \overset{(4.2)}{=} 0,$$

where $(*)$ holds by monotone convergence. Therefore we have $u_* \leq \frac{5}{2} C_g \ln(\mathcal{C}_d)$.

Proof of Proposition 4.1. For any $n \geq 1$ and $\mathcal{T} \in \Lambda_{n,0}$ we denote $\mathcal{X}_\mathcal{T} = \bigcup_{m \in T_{(n)}} \mathcal{T}(m)$. Noting that $S(\mathcal{T}(m), L_0 - 1) = S(\mathcal{T}(m), 0) = \{\mathcal{T}(m)\}$ for any $m \in T_{(n)}$ and that every nearest-neighbour path is also a $*$-connected path we can apply Lemma 3.3 to infer

$$\mathbb{P}\left[A_n^u\right] \overset{(3.6)}{\leq} \mathbb{P}\left[\bigcup_{\mathcal{T} \in \Lambda_{n,0}} \{\mathcal{X}_\mathcal{T} \subseteq \mathcal{V}^u\}\right] \overset{(1.1),(1.2)}{\leq}$$

$$\sum_{\mathcal{T} \in \Lambda_{n,0}} \exp\left(-u \cdot \mathrm{cap}(\mathcal{X}_\mathcal{T})\right) \overset{(3.4)}{\leq} \mathcal{C}_d^{2^n} \cdot \max_{\mathcal{T} \in \Lambda_{n,0}} \exp\left(-u \cdot \mathrm{cap}(\mathcal{X}_\mathcal{T})\right). \tag{4.3}$$

In order to finish the proof of Proposition 4.1 we only need to show that for any $\mathcal{T} \in \Lambda_{n,0}$ we have

$$\mathrm{cap}(\mathcal{X}_\mathcal{T}) \geq \frac{2}{5} \frac{1}{C_g} 2^n, \tag{4.4}$$

because then we indeed obtain

$$\mathbb{P}\left[A_n^u\right] \overset{(4.3),(4.4)}{\leq} \mathcal{C}_d^{2^n} \exp\left(-u \frac{2}{5} \frac{1}{C_g} 2^n\right) = \left(\mathcal{C}_d \exp\left(-u \frac{2}{5} \frac{1}{C_g}\right)\right)^{2^n} = q^{2^n}, \qquad q \overset{(4.1)}{<} 1.$$

We will show (4.4) using (2.10). For any $\mathcal{T} \in \Lambda_{n,0}$ and any $m \in T_{(n)}$ we have

$$\sum_{m' \in T_{(n)}} g(\mathcal{T}(m), \mathcal{T}(m')) \overset{(3.8)}{=} \sum_{k=0}^{n} \sum_{m' \in T_{(n)}^{m,k}} g(\mathcal{T}(m), \mathcal{T}(m')) \overset{(1.5),(3.10)}{\leq}$$

$$C_g + \sum_{k=1}^{n} C_g L_{k-1}^{2-d} \left| T_{(n)}^{m,k} \right| \overset{(3.2),(3.9)}{=} C_g \cdot \left(1 + \sum_{k=1}^{n} 6^{(k-1)(2-d)} 2^{(k-1)}\right) \overset{d\geq 3}{\leq}$$

$$C_g \cdot \left(1 + \sum_{k=1}^{\infty} 3^{1-k}\right) = \frac{5}{2} C_g. \tag{4.5}$$

Now (4.4) follows from (2.10), (4.5) and the fact that $|\mathcal{X}_\mathcal{T}| = 2^n$. The proof of Proposition 4.1 is complete. ☐

5 Lower bound on u_*

Let us choose L_0 according to (1.8) in (3.2). Recall the notion of the plane F from (2.11). For $n \geq 1$ and $x \in \mathcal{L}_n \cap F$ let us denote by $B^u_{n,x}$ the event

$$B^u_{n,x} = \left\{ \begin{array}{c} \text{there exists a } *\text{-connected path in } \mathcal{I}^u \cap F \\ \text{that connects } S(x, L_n - 1) \text{ to } S(x, 2L_n) \end{array} \right\}.$$

Recall the definitions of c_g, C_g from (1.5) and $\mathcal{C}_d$ from (1.7).

Proposition 5.1. *For any $d \geq 3$ and*

$$u < \frac{c_g}{L_0} \frac{1}{\mathcal{C}_2} 2^{-(d+5)}, \tag{5.1}$$

for any $n \geq 1$ and $x \in \mathcal{L}_n \cap F$ we have

$$\mathbb{P}[B^u_{n,x}] \leq \left(\frac{3}{4}\right)^{2^n}. \tag{5.2}$$

Corollary 5.2. *Proposition 5.1 implies the lower bound of Theorem 1.1, as we now explain. Let us denote by $\widehat{A}^u_n$ the event that there exists a nearest-neighbour path in $\mathcal{V}^u \cap F$ that connects $S(0, L_n)$ to infinity and by $\widehat{A}^u_\infty$ the event that $\mathcal{V}^u \cap F$ has an infinite connected component. By planar duality the event $(\widehat{A}^u_n)^c$ is equal to the event that there exists a $*$-connected path in $\mathcal{I}^u \cap F$ that surrounds $S(0, L_n - 1)$, thus if* (5.1) *holds, then*

$$\mathbb{P}[\widehat{A}^u_n] \geq 1 - \mathbb{P}\left[\bigcup_{k=n}^{\infty} \bigcup_{x \in \mathcal{L}_k,\, |x| \leq 2L_{k+1}} B^u_{k,x}\right] \overset{(3.2),(5.2)}{\geq} 1 - \sum_{k=n}^{\infty} 25^d \cdot \left(\frac{3}{4}\right)^{2^k},$$

which in turn implies $\mathbb{P}[\widehat{A}^u_\infty] = \lim_{n\to\infty} \mathbb{P}[\widehat{A}^u_n] = 1$. Therefore we have $u_* \geq \frac{c_g}{L_0} \frac{1}{\mathcal{C}_2} 2^{-(d+5)}$.

Proof of Proposition 5.1. We say that $\mathcal{T} : T_n \to F$ is a proper embedding of the dyadic tree T_n with root at $x \in \mathcal{L}_n \cap F$ into F if $\mathcal{T} \in \Lambda_{n,x}$ (see Definition 3.1). We denote by $\Lambda^F_{n,x}$ the set of proper embeddings of T_n into F.

For any $y \in \mathcal{L}_0 \cap F$ let us define the frame $\square_y \subseteq F$ by

$$\square_y \overset{(2.12)}{=} \square^{L_0}_y = S(y, L_0 - 1) \cap F.$$

For any $n \geq 1$, $x \in \mathcal{L}_n \cap F$ and $\mathcal{T} \in \Lambda^F_{n,x}$ let us denote by

$$\mathcal{X}^\square_\mathcal{T} = \bigcup_{m \in T_{(n)}} \square_{\mathcal{T}(m)}. \tag{5.3}$$

We start the proof of Proposition 5.1 by an application of Lemma 3.3 with $d = 2$:

$$\begin{aligned}\mathbb{P}\left[B^u_{n,x}\right] &\overset{(3.6)}{\leq} \mathbb{P}\left[\bigcup_{\mathcal{T} \in \Lambda^F_{n,x}} \bigcap_{m \in T_{(n)}} \{\square_{\mathcal{T}(m)} \cap \mathcal{I}^u \neq \emptyset\}\right] \overset{(*)}{\leq} \\ &\mathcal{C}_2^{2^n} \cdot \max_{\mathcal{T} \in \Lambda^F_{n,x}} \mathbb{P}\left[\bigcap_{m \in T_{(n)}} \{\square_{\mathcal{T}(m)} \cap \mathcal{I}^u \neq \emptyset\}\right],\end{aligned} \tag{5.4}$$

where in $(*)$ we used Lemma 3.2 to infer $|\Lambda^F_{n,x}| \leq \mathcal{C}_2^{2^n}$.

In order to bound the probability on the right-hand side of (5.4) let us fix some $\mathcal{T} \in \Lambda^F_{n,x}$, recall the constructive definition of random interlacements from Claim 2.2 and denote the probability underlying the random objects (i.e., N_K and $(X^j)_{j\geq 1}$) introduced in that claim by P when $K = \mathcal{X}_{\mathcal{T}}^{\square}$. For a simple random walk X let us denote by

$$\mathcal{N}(X) = \sum_{m\in T_{(n)}} \mathbb{1}[\{X\} \cap \square_{\mathcal{T}(m)} \neq \emptyset]$$

the number of frames of form $\square_{\mathcal{T}(m)}$, $m \in T_{(n)}$ that X visits. We can bound

$$\mathbb{P}\left[\bigcap_{m\in T_{(n)}} \{\square_{\mathcal{T}(m)} \cap \mathcal{I}^u \neq \emptyset\}\right] \leq \mathrm{P}\left[\sum_{j=1}^{N_K} \mathcal{N}(X^j) \geq 2^n\right]. \tag{5.5}$$

Our next goal is to stochastically bound $\mathcal{N}(X)$. Recall the definitions of c_g, C_g from (1.5) and L_0 from (1.8). Let us define

$$p = \begin{cases} 12C_g/c_g \cdot L_0^{3-d} & \text{if} \quad d \geq 4, \\ 12C_g/c_g \cdot \frac{1}{1+\ln(L_0)} & \text{if} \quad d = 3. \end{cases} \tag{5.6}$$

For any $m \in T_{(n)}$, $y \in \square_{\mathcal{T}(m)}$ we have

$$P_y[\{X\} \cap \mathcal{X}_{\mathcal{T}}^{\square} \setminus \square_{\mathcal{T}(m)} \neq \emptyset] \overset{(3.8),(5.3)}{\leq} \sum_{k=1}^{n} \sum_{m'\in T_{(n)}^{m,k}} P_y[\{X\} \cap \square_{\mathcal{T}(m')} \neq \emptyset] \overset{(1.5),(2.7),(3.10)}{\leq}$$
$$\sum_{k=1}^{n} \sum_{m'\in T_{(n)}^{m,k}} C_g L_{k-1}^{2-d} \mathrm{cap}(\square_{\mathcal{T}(m')}) \overset{(3.2),(3.9)}{=} \sum_{k=1}^{n} 2^{k-1} C_g L_0^{2-d} 6^{(k-1)(2-d)} \mathrm{cap}(\square_0) \overset{d\geq 3}{\leq}$$
$$C_g L_0^{2-d} \mathrm{cap}(\square_0) \sum_{k=1}^{\infty} 3^{1-k} \overset{(2.13),(5.6)}{\leq} p. \tag{5.7}$$

The bound (5.7) together with the strong Markov property of simple random walk imply that $P_{\widetilde{e}_K}[\mathcal{N}(X) \geq k] \leq p^{k-1}$ for any $k \geq 1$. In other words, $\mathcal{N}(X)$ is stochastically dominated by a geometric random variable with parameter $1-p$, which implies $E_{\widetilde{e}_K}\left[z^{\mathcal{N}(X)}\right] \leq \frac{(1-p)z}{1-pz}$ for any $1 \leq z < \frac{1}{p}$. Recalling from Claim 2.2 that N_K is Poisson with parameter $u \cdot \mathrm{cap}(K) = u \cdot \mathrm{cap}(\mathcal{X}_{\mathcal{T}}^{\square})$, for any $1 \leq z < \frac{1}{p}$ we obtain

$$\mathrm{E}\left[z^{\sum_{j=1}^{N_K} \mathcal{N}(X^j)}\right] = \exp\left(u \cdot \mathrm{cap}(\mathcal{X}_{\mathcal{T}}^{\square}) \left(E_{\widetilde{e}_K}\left[z^{\mathcal{N}(X)}\right] - 1\right)\right) \leq$$
$$\exp\left(u \cdot \mathrm{cap}(\mathcal{X}_{\mathcal{T}}^{\square}) \left(\frac{z-1}{1-pz}\right)\right).$$

We can thus apply the exponential Chebyshev inequality with $z = \frac{1}{2p}$ to bound

$$\mathbb{P}\left[B_{n,x}^u\right] \overset{(5.4),(5.5)}{\leq} \mathcal{C}_2^{2^n} \mathrm{E}\left[\left(\frac{1}{2p}\right)^{\sum_{j=1}^{N_K} \mathcal{N}(X^j)}\right] (2p)^{2^n} \leq$$
$$\exp\left(u \cdot \mathrm{cap}(\mathcal{X}_{\mathcal{T}}^{\square}) \left(\frac{\frac{1}{2p} - 1}{1/2}\right)\right) (2p\mathcal{C}_2)^{2^n} \overset{(2.8)}{\leq} \exp\left(u \cdot \frac{\mathrm{cap}(\square_0)}{p}\right)^{2^n} (2p\mathcal{C}_2)^{2^n} \overset{(1.8),(5.6)}{\leq}$$
$$\exp\left(u \cdot \mathrm{cap}(\square_0) 2^d \mathcal{C}_2\right)^{2^n} 2^{-2^n} \overset{(2.13)}{\leq} \exp\left(u \frac{L_0}{c_g} 2^{d+3} \mathcal{C}_2\right)^{2^n} 2^{-2^n} \overset{(5.1)}{\leq} \left(\frac{3}{4}\right)^{2^n}.$$

This completes the proof of Proposition 5.1. □

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