2015 European Control Conference (ECC)
July 15-17, 2015. Linz, Austria

$\mathcal{L}_2$-gain analysis of systems with state and input delays

Gábor Rödönyi[†] and Gabriella Varga[†]

Abstract— **Sufficient conditions are presented for a system with input delays having finite $\mathcal{L}_2$-gain. The bounded real lemma conditions are infeasible when the actual and the delayed values of the input act on the system simultaneously. Considering the actual and delayed inputs as two independent inputs is shown to lead to very high upper-bound of the true $\mathcal{L}_2$-gain. Several approaches are presented involving a scaling method which is based on upper-bounds of the delayed signal norms; a method which transforms the input delays to state delays with the help of introducing additional dynamics; and a method based on integral quadratic constraints. Both time-invariant and time-varying delays are considered. The methods are evaluated on an example of interconnected vehicles.**

I. Introduction

Robust stability and performance in the presence of uncertain time-delays in the state-variables has been extensively studied recently [1], [2], [3], [4], [5], [6]. In contrast, time delay in the input has avoided the attention of the control community so far. Only very few papers consider both state and input delay in the input-output analysis of linear systems [7]. Indeed, in many cases the problem of input delay can be resolved. A single input with a constant time-delay $w(t-h) \in \mathcal{L}_2[0,\infty]$ does not influence the $\mathcal{L}_2$-gain of the system. Time-varying delay on the other hand modifies the norm of the signal, but introducing a new input $\tilde{w}(t) = w(t-h(t))$ might be sufficient. The problem emerges when the input acts on the system with and without delay ($w(t-h)$ and $w(t)$, respectively). Considering them as independent inputs and disregarding information about the relation between them [7] will result in overestimation of the gain.

In the literature, the treatment of input delay in the $\mathcal{L}_2$-gain analysis is confined to delays in the control input [8], not disturbance inputs, thus closing the loop with the controller leads to a problem with pure state-delay only.

The problem of input delays may arise in the analysis of distributed systems and large scale interconnected systems. In these applications, conclusions on stability/performance of the overall system can be drawn based on analysis of local behavior [9], [10], [11]. The inputs to the local system are transmitted through the network, possibly through multiple channels, and may therefore contain different delays.

In this paper several approaches are presented for the computation of induced $\mathcal{L}_2$-gain of systems with input delays. The results are compared with the condition proposed in [7]. Cases for both time-invariant and time-varying delays are considered. The problem with input delays is illustrated in Section II. The practical approaches are presented in Section III. The efficiency of the methods are demonstrated on an interconnected vehicle model in Section IV.

Notations. $\mathbb{R}$ and $\mathbb{C}$ denote the real and the complex fields, respectively. Matrix inequality $M > 0$ ($M \geq 0$) denotes that M is symmetric and positive (semi-) definite, i.e. all of its eigenvalues are positive (or zero). Negative (semi-) definiteness is denoted by $M < 0$ ($M \leq 0$). The transpose and conjugate transpose of a matrix M is denoted by M^T and M^*, respectively. $\bar{\sigma}(M)$ denotes the maximum singular value of matrix M. $\mathrm{col}\{x, y\}$ denotes the column vector built from vectors x and y. $\mathcal{L}_2^n$ denotes the space of square integrable signals with norm defined by $\|x\|_2 = \left(\int_0^\infty \|x(t)\|^2 dt\right)^{1/2}$, where $\|x(t)\|$ denotes the Euclidean norm on $\mathbb{R}^n$. Space $\mathcal{H}_\infty$ consists of all complex valued functions which are analytic in $\Re s > 0$ and for which $\|G\|_\infty \triangleq \sup_{\Re s>0} \bar{\sigma}(G(s)) = \sup_\omega \bar{\sigma}(G(j\omega)) < \infty$. If $G \in \mathcal{H}_\infty$ is an asymptotically stable system with zero initial conditions, then $\|G\|_\infty = \sup_{u\in\mathcal{L}_2^n} \frac{\|Gu\|_2}{\|u\|_2}$ is the induced $\mathcal{L}_2$-gain of G.

II. Problem formulation and preliminaries

Sufficient and necessary conditions for a system having finite $\mathcal{L}_2$-gain is briefly summarised. Consider an LTI system G described by the equations

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

$$y(t) = Cx(t) \tag{2}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^{n_u}$ and $y(t) \in \mathbb{R}^{n_y}$.

Definition 1 (Dissipativity [12]): System G with *supply rate* $s(u(t), y(t))$ is said to be *strictly dissipative* if there exists a non-negative function $V : \mathbb{R}^n \mapsto \mathbb{R}$ such that

$$V(x(t)) - V(x(0)) - \int_0^t s(t)dt < 0 \tag{3}$$

for all $t > 0$ and all trajectories (x, u, y) which satisfy (1)-(2). Function V is called *storage function*.

Lemma 1 (Bounded Real Lemma (BRL), [13]): Suppose that system G is controllable. Let

$$s(t) = \gamma^2 u(t)^T u(t) - y(t)^T y(t) \tag{4}$$

be the supply rate. Then the following statements are equivalent

1) System G is strictly dissipative with $s(t)$.
2) System G is asymptotically stable and $\|G\|_\infty < \gamma$.

This paper was supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences

[†] Systems and Control Laboratory, Computer and Automation Research Institute of Hungarian Academy of Sciences. E-mail: varga.gabriella@sztaki.mta.hu, rodonyi@sztaki.hu

3) There exists a $P > 0$, $P \in \mathbb{R}^{n\times n}$ such that

$$\begin{bmatrix} A^T P + PA + C^T C & PB \\ B^T P & -\gamma^2 I \end{bmatrix} < 0. \quad (5)$$

Moreover, $V(x) = x^T P x$ defines a quadratic storage function if and only if $P > 0$ satisfies (5).
Linear matrix inequality (LMI) condition (5) is derived with assuming $x(0) = 0$ and differentiating (3), which results in $(d/dt)V(x(t)) - s(t) =$

$$\xi(t)^T M_V \xi(t) + \xi(t)^T M_S \xi(t) < 0 \quad (6)$$

where $\xi(t) = \mathrm{col}\{x(t), u(t)\}$ and

$$M_V = \begin{bmatrix} PA + A^T P & PB \\ B^T P & 0 \end{bmatrix}, M_S = \left[\begin{array}{c|c} C^T C & 0 \\ \hline 0 & -\gamma^2 I \end{array}\right]. \quad (7)$$

A. Problem with input delays

Let $B = [B_0,\ B_1]$ and $u(t) = \mathrm{col}\{w(t), w(t-h)\}$ for some $h > 0$ constant time delay. The only input of system G is $w(t)$. *The delay is part of* G. With supply rate as in (4) the matrices in (7) reveal

$$M_V = \begin{bmatrix} PA + A^T P & PB_0 & PB_1 \\ B_0^T P & 0 & 0 \\ B_1^T P & 0 & 0 \end{bmatrix} \quad (8)$$

$$M_S = \left[\begin{array}{c|cc} C^T C & 0 & 0 \\ \hline 0 & -\gamma^2 I & 0 \\ 0 & 0 & 0 \end{array}\right] \quad (9)$$

and $\xi(t) = \mathrm{col}\{x(t), w(t), w(t-h)\}$. By Schur complement argument, a necessary condition for a matrix being negative definite is that all of the diagonal entries are negative definite. Since the $(3, 3)$ block of matrix $M_V + M_S$ is zero, the BRL condition will fail in proving asymptotic stability and computing the $\mathcal{L}_2$-gain of the system.

B. Problem formulation

In the paper the following linear time-delay system, denoted by Σ, is considered

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + A_1 x(t-h) + B_0 w(t) + B_1 w(t-h) \\ y(t) &= Cx(t) \end{aligned} \quad (10)$$

where $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^{n_w}$ and $y(t) \in \mathbb{R}^{n_y}$. The goal of the paper is to develop and evaluate some modified BRL conditions for computing upper-bounds on the system's $\mathcal{L}_2$-gain performance. The following cases will be examined.

- The time-delay is uncertain and time-varying, $h = h(t)$ and $h(t) \in [h_1, h_2]$, $h_1 \geq 0$. Define $h_{12} \triangleq h_2 - h_1$.
- Time-delay h is constant and known, $h_1 = h_2 = h$.

In the $\mathcal{L}_2$-gain analysis, the initial conditions of the system are assumed to be zero, $x(t) = 0$ for $t \leq 0$ and $w(t) = 0$ for $t < 0$.

Since (10) is a retarded functional differential equation, the storage-function in Definition 1 can be replaced by a Lyapunov-Krasovskii functional (LKF). Two candidate LKFs are presented in Section II-C.

C. Lyapunov-Krasovskii functionals (LKF)

In case of constant and known time-delays the complete Lyapunov-Krasovskii functional

$$\begin{aligned} V_{TI}(x_t) &= x^T(t)Px(t) + 2x^T(t)\int_{-h}^{0} Q(\xi)x(t+\xi)d\xi \\ &+ \int_{-h}^{0}\int_{-h}^{0} x^T(t+\xi)R(\xi,\eta)x(t+\eta)d\eta d\xi \\ &+ \int_{-h}^{0} x^T(t+\xi)S(\xi)x(t+\xi)d\xi \end{aligned} \quad (11)$$

provides sufficient and necessary condition for uniform asymptotic stability of Σ. Matrix functions Q, R and S can be approximated as piecewise linear functions. For more details, see [14].

Due to its advantageous parametrisation, the LKF candidate

$$V_{TV}(t) = \sum_{i=1}^{9} V_i(t) \quad (12)$$

developed recently by Lee et al., [15], will be applied for the case of time-varying delays where

$$\begin{aligned} V_1(t) &= \zeta^T(t)P\zeta(t), \\ V_2(t) &= \int_{t-h_1}^{t} x^T(s)Q_1 x(s)ds, \\ V_3(t) &= \int_{t-h_2}^{t-h_1} x^T(s)Q_2 x(s)ds \\ V_4(t) &= h_1\int_{-h_1}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)R_1\dot{x}(s)dsd\theta, \\ V_5(t) &= h_{12}\int_{-h_2}^{-h_1}\int_{t+\theta}^{t} \dot{x}^T(s)R_2\dot{x}(s)dsd\theta, \\ V_6(t) &= \int_{-h_1}^{0}\int_{\eta}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)Z_1\dot{x}(s)dsd\theta d\eta, \\ V_7(t) &= \int_{-h_1}^{0}\int_{-h_1}^{\eta}\int_{t+\theta}^{t} \dot{x}^T(s)Z_2\dot{x}(s)dsd\theta d\eta, \\ V_8(t) &= \int_{-h_2}^{-h_1}\int_{\eta}^{-h_1}\int_{t+\theta}^{t} \dot{x}^T(s)Z_3\dot{x}(s)dsd\theta d\eta, \\ V_9(t) &= \int_{-h_2}^{-h_1}\int_{-h_2}^{\eta}\int_{t+\theta}^{t} \dot{x}^T(s)Z_4\dot{x}(s)dsd\theta d\eta \end{aligned}$$

and $\zeta(t) = \mathrm{col}\{x(t), \int_{t-h_1}^{t} x(s)ds, \int_{t-h_2}^{t-h_1} x(s)ds\}$. Given the choice of storage function, M_V, $\xi(t)$ and the upper-left block of M_S, denoted by $M_{S,11}$, must be modified accordingly.

D. A sampled-data network model

Time-varying delays can drastically modify the norm of the signal. It is worth considering application specific properties of the delay. The class of time-varying delays in consideration is specialised to those present in sampled-data networks. It is assumed that input $w(t) \in \mathcal{L}_2^{n_w}$ affects the system both directly and through the network (see the motivating example in Section IV). The following model will serve as the basis for deriving upper bounds of $\|w(t-h(t))\|_2$ in terms of $\|w(t)\|_2$ in Section III-A.

The input to the network is realised through a Zero Order Hold (ZOH) device. Let $s_k = kT_s,\ k = 0, 1, ...$ denote the measurement instants, while

$$0 \leq t_0 < t_1 < \ldots < t_k < \ldots, \lim_{k\to\infty} t_k = \infty \tag{13}$$

denote the time instants when the new data arrives at the receiver. Define the time of transmission η_k as $t_k = s_k + \eta_k$. Assume that there exist known constants h_1 and h_2 such that $0 \leq h_1 \leq \eta_k \leq h_2,\ k = 0, 1, 2, ...$ Define

$$h(t) = t - s_k, \quad t_k \leq t < t_{k+1}. \tag{14}$$

i.e. $w(t - h(t)) = w(s_k)$ for $t_k \leq t < t_{k+1}$.

For more details on the model and a stability analysis of systems with control over sampled-data networks, see [16].

III. $\mathcal{L}_2$-GAIN UPPER-BOUNDS FOR SYSTEMS WITH INPUT DELAYS

In this section the input-delay problem in the BRL is relaxed in several ways. It is assumed in the case of time-varying delays that the delay is due to a sampled data network. Some of the presented methods will require bounds on the norms of the transmitted signal $w(t - h(t))$ in terms of $\|w(t)\|_2$. The first subsection is devoted to this issue.

A. Bounds on $\|w(t - h(t))\|_2$ in case of sampled data networks

Assume that the input $w(t)$ to be transmitted through the network is band limited such that the following integral can be well approximated by

$$W(t) \triangleq \int_0^t \|w(\tau)\|^2 d\tau \approx \sum_{k=0}^{N} \|w(s_k)\|^2 T_s \tag{15}$$

where $N = t/T_s$. With the network model presented in Section II-D, we have

$$\begin{aligned} W_h(t) &\triangleq \int_0^t \|w(\tau - h(\tau))\|^2 d\tau \\ &= \sum_{k=0}^{N} \|w(s_k)\|^2 (t_{k+1} - t_k) \end{aligned} \tag{16}$$

for the transmitted input. Define $\bar{h} = T_s + h_2$ as the maximum time span between time s_k and the next update time t_{k+1}. Let $0 \leq \alpha_k, \alpha_{k-1} \leq h_2 - h_1 = h_{12}$ be constants such that

$$t_{k+1} - s_k + \alpha_k = \bar{h} \tag{17}$$
$$t_k - s_{k-1} + \alpha_{k-1} = \bar{h} \tag{18}$$

Subtracting (18) from (17) gives $t_{k+1} - t_k = T_s - \alpha_k + \alpha_{k-1}$, which is in the interval $[T_s - h_{12}, T_s + h_{12}]$. Combining this result with (15) and (16) it follows that

$$\left(1 - \frac{h_{12}}{T_s}\right) W(t) \leq W_h(t) \leq \left(1 + \frac{h_{12}}{T_s}\right) W(t). \tag{19}$$

for all $t \geq 0$. The bounds for the $\mathcal{L}_2$-norm of $w(t - h(t))$ is obtained by the limit $\|w\|_2^2 = \lim_{t\to\infty} W(t)$.

B. Exact gain of LTI systems (EG-TI)

In case of time-invariant delays, the induced $\mathcal{L}_2$-gain of Σ is given by $\|\Sigma\|_\infty = \max_\omega \bar{\sigma}\left(C(j\omega I - A_0 - A_1 e^{-j\omega h})^{-1}(B_0 + B_1 e^{-j\omega h})\right)$, which will be used as reference in the evaluation of BRL-based results.

C. Independent inputs (II-TI and II-TV)

Cheng [7] proposed to incorporate $w(t - h)$ in the supply rate as $s(t) = \gamma^2(u_1(t)^T u_1(t) + u_2(t)^T u_2(t)) - y(t)^T y(t)$, where

$$u_1(t) = w(t), \qquad u_2(t) = w(t - h) \tag{20}$$

With this modification, the BRL condition (6) with $M_S = \text{diag}\{M_{S,11}, -\gamma_{II}^2 I, -\gamma_{II}^2 I\}$ is solvable for some γ_{II} if the system is asymptotically stable.[1] In this approach the two inputs, u_1 and u_2, are considered as independent inputs, correlation (20) is not exploited, and consequently, the result is necessarily conservative. On the other hand, the resulted γ_{II} is not the gain of the original system, whose only input is $w(t) \in \mathcal{L}_2^{n_w}$. With initial condition $w(t) = 0$ for $t < 0$, we have $\|u\|_2 = \left(\|u_1\|_2^2 + \|u_2\|_2^2\right)^{1/2} \leq \sqrt{2}\|w\|_2$ for constant delays and $\|u\|_2 \leq \sqrt{2 + h_{12}/T_s}\|w\|_2$ for time-varying delays. With this approach it can be proven that

$$\|\Sigma\|_\infty < \gamma_{II}\sqrt{2} \text{ or} \tag{21}$$
$$\|\Sigma\|_\infty < \gamma_{II}\sqrt{2 + h_{12}/T_s} \tag{22}$$

depending on the class of delay. This BRL condition will be referred as the method of independent inputs (II) since input $w(t - h)$ could be replaced by any other independent input $u_2(t)$ for which $\|u_2\|_2 \leq \|w\|_2$.

D. Scaled inputs with time-invariant delays (SI-TI)

A slight improvement can be achieved with respect to method II-TI due to the following observations. Assume that $w(t) = 0$ for $t < 0$. Then, time-invariant delays do not change the $\mathcal{L}_2$-norm of a signal, $\|w(t)\|_2 = \|w(t - h)\|_2 = \alpha\|w(t)\|_2 + (1 - \alpha)\|w(t - h)\|_2$ for any $\alpha \in \mathbb{R}$; On finite interval with $h(\tau) = h$, however, we have

$$W_h(t) = W(t - h) \leq W(t). \tag{23}$$

Let the supply rate be defined by (4) with $u(t) = w(t)$. An upper-bound on the left-hand side of the dissipation inequality (3) is derived by replacing

$$S(t) \triangleq \int_0^t s(\tau) d\tau$$

with

$$S_\alpha(t) \triangleq \int_0^t s_\alpha(\tau) d\tau \leq S(t)$$

where

$$s_\alpha(t) \triangleq \gamma^2 \left(\alpha\|w(t)\|^2 + (1 - \alpha)\|w(t - h)\|^2\right) - \|y(t)\|^2$$

[1] M_V and $M_{S,11}$ depend on the chosen storage function.

with $\alpha \in \mathbb{R}$. Then

$$V(t) - V(0) - S(t) \leq V(t) - V(0) - S_\alpha(t) \tag{24}$$

The derivative of the right-hand side being negative definite provides a sufficient condition for strict dissipativity of Σ with supply rate $s(t)$. Moreover, in the BRL condition $M_S = \mathrm{diag}\{M_{S,11}, -\alpha\gamma^2 I, -(1-\alpha)\gamma^2 I\}$. For any fixed $\alpha \in (0,1)$ an LMI problem have to be solved which minimises γ^2. A line-search in the outer loop can be performed to find the optimal α. It can be shown that for $\alpha = 0.5$ the BRL conditions SI-TI and II-TI are equivalent.

E. Scaled inputs with time-varying delays (SI-TV)

The same idea as before applies also to the case of time-varying delays, except that now the bound (19) must be considered. A straightforward derivation results in $M_S = \mathrm{diag}\{M_{S,11}, -\alpha\gamma^2 I, -\frac{1-\alpha}{1+h_{12}/T_s}\gamma^2 I\}$. Again, $\alpha = 0.5$ coincides with the corresponding method of independent inputs.

F. Additional dynamics (AD-TI and AD-TV)

Assume that the input $w(t)$, which is to be transmitted through the network, is band limited. Then, a sufficiently high-bandwidth low-pass filter, $W_d(s)$, with $\|W_d\|_\infty = 1$ and state-space realisation

$$\begin{aligned} \dot{x}_d(t) &= A_d x_d(t) + B_d w(t), \\ w_d(t) &= C_d x_d(t) \end{aligned} \tag{25}$$

has negligible effect on $w(t)$. Assuming zero initial conditions, approximation $w_d(t) \approx w(t)$ holds. System (10) can be replaced by (25), (26) and (27)

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + B_0 w(t) + B_1 w_d(t-h) \tag{26}$$

$$y(t) = Cx(t). \tag{27}$$

This model transforms the input-delay to state-delay and can be used for both time-invariant and time-varying delays. For general inputs, the high-frequency components of w are filtered out, but the system gain is, usually, also small at high frequencies, therefore this effect has low impact on the system gain. On the other hand, the phase lag of the additional dynamics is added to the phase lag caused by the delay.

G. Integral quadratic constraints (IQC-TI)

Time-delay operators can be described by IQCs in both the frequency- and time-domain [17]. Let $\Pi : j\mathbb{R} \mapsto \mathbb{C}^{2n_w \times 2n_w}$ be a bounded and Hermitian-valued function called multiplier. A bounded, causal operator Δ defined on the extended space $\mathcal{L}_{2e}$ satisfies the IQC defined by Π, denoted by $\Delta \in \mathrm{IQC}(\Pi)$, if

$$\int_{-\infty}^{\infty} \begin{bmatrix} w(j\omega) \\ z(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} w(j\omega) \\ z(j\omega) \end{bmatrix} d\omega \geq 0 \tag{28}$$

where $z = \Delta(w)$. Let D_h denote the delay operator. In [17], the operator $S_h \triangleq D_h - 1$ is embedded into a set of operators which satisfy IQC (28) with multiplier

$$\Pi(j\omega) = \lambda_1 \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix} + \lambda_2 T(j\omega) \begin{bmatrix} |\phi(j\omega)|^2 & 0 \\ 0 & -1 \end{bmatrix} \tag{29}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ are arbitrary non-negative parameters, $T(j\omega) = |\frac{j\omega+1}{10j\omega+1}|^2$ and

$$\phi(j\omega) = 2\frac{(j\omega h)^2 + 3.5(j\omega h) + 10^{-6}}{(j\omega h)^2 + 4.5(j\omega h) + 7.1}.$$

When the input-delay is described by the above IQC, system (10) can be embedded into the following model

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + A_1 x(t-h) + B_1 z(t) + (B_0 + B_1) w(t) \\ z(t) &= \Delta(w(t)), \qquad \Delta \in \mathrm{IQC}(\Pi) \\ y(t) &= Cx(t) \end{aligned} \tag{30}$$

Dissipation inequality (3) with LKF (11) or (12) and IQC (28)-(29) can be combined by S-procedure to obtain a sufficient LMI condition for the $\mathcal{L}_2$-gain of the system being less than γ.

IV. EXAMPLE: CONNECTED VEHICLES

Consider a string of vehicles, where each follower vehicle maintains a constant distance from the preceding vehicle by using locally available radar information (relative position and relative velocity) and measurements transmitted through the network: acceleration of the preceding vehicle, acceleration, position and velocity of the lead vehicle. Disturbances act on every vehicle. Through the acceleration measurements, disturbances are transmitted through the network. Depending on the modeling approach, disturbances act with and without delay on the vehicle string. In the following analysis $\mathcal{L}_2$-gains of the system from disturbances to spacing errors are computed with the presented methods.

A. Vehicle platoon model

Let the ith vehicle be described by the following third-order continuous-time state-space model

$$\dot{p}_i(t) = v_i(t), \tag{31a}$$

$$\dot{v}_i(t) = q_i(t) + d_i(t), \tag{31b}$$

$$\dot{q}_i(t) = -\frac{1}{\tau_i} q_i(t) + \frac{g_i}{\tau_i} u_i(t), \tag{31c}$$

where p_i, v_i denote position and velocity, d_i is a disturbance representing both outer effects and modelling error, q_i is an internal state such that the acceleration of the vehicle is $a_i(t) = q_i(t) + d_i(t)$. Gain calculations will be carried out on a homogeneous platoon with parameters $\tau_i = 0.7$ and $g_i = 1$. The leader vehicle is driven by a human driver, u_0 denotes the acceleration demand computed from the pedal signals. u_i, $i = 1, 2, ..., n$ for the follower vehicles denote the acceleration demand generated by the controllers.

Let the distributed platoon controller be given and designed for a constant spacing policy, see [18]. Taking inter-vehicle communication delays into account, the controllers can be described by the following equations

$$u_1(t) = -k_1\delta_1(t) - k_2 e_1(t) + \hat{a}_0(t) \tag{32a}$$

$$\begin{aligned} u_i(t) = &- k_{1\beta}\delta_i(t) - k_{2\beta} e_i(t) + k_{a0}\hat{a}_0(t) + k_{a1}\hat{a}_{i-1}(t) \\ &- k_{1\alpha}(\hat{v}_i(t) - \hat{v}_0(t)) - k_{2\alpha}(\hat{p}_i(t) - \hat{p}_0(t)), \\ &i = 2, ..., n \end{aligned} \tag{32b}$$

where $\delta_i \triangleq v_i - v_{i-1}$ and $e_i \triangleq p_i - p_{i-1} + L_i$ are the relative speed and spacing error, respectively. The prescribed spacing L_i can be set to zero in the analysis without loss of generality. $k_1 = 0.7, k_2 = 0.1127, k_{1\alpha} = 0.4642, k_{2\alpha} = 0.0564, k_{1\beta} = 0.2358, k_{2\beta} = 0.0564, k_{a1} = 0.0449, k_{a0} = 0.9551$ are constant parameters. Symbol ˆ denotes the effect of the network as follows. In case the network is modelled by a constant delay, $\hat{x}(t) = x(t-h)$. In case the network is a sampled data network, $\hat{x}(t) = x(s_k)$ for $t_k \le t < t_{k+1}$.

The closed-loop platoon model described by (31) and (32) with $i = 0, 1, 2, ..., n$ is excited by both the actual disturbance inputs and their delayed values. For example, the system $(d_0 \mapsto e_1)$ with state vector defined by $x = [e_1, \delta_1, q_1]^T$ is described by the closed-loop matrices $A_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.16 & -1 & -1.43 \end{bmatrix}$, $B_0 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, $B_1 = \begin{bmatrix} 0 \\ 0 \\ 1.43 \end{bmatrix}$, $A_1 = 0$, $C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$. Since $A_1 = 0$, the $\mathcal{L}_2$-gain is computed by applying a quadratic storage function, $V(x) = x^T P x$.

System $(d_1 \mapsto e_2)$ with $x = [e_1, \delta_1, q_1, e_2, \delta_2, q_2]^T$ is described by the following state-space matrices

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -0.16 & -1 & -1.43 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -0.08 & -0.34 & -1.43 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.08 & -0.66 & 0.06 & -0.08 & -0.66 & 0 \end{bmatrix},$$

$B_0 = [0, 1, 0, 0, -1, 0]^T$, $B_1 = [0, 0, 0, 0, 0, 0.06]^T$, $C = [0, 0, 0, 1, 0, 0]$. The $\mathcal{L}_2$-gain is computed by applying LKF storage functions V_{TI} and V_{TV} in case of time-invariant and time-varying delays, respectively.

B. $\mathcal{L}_2$-gain analysis results

Tables I - IV show the $\mathcal{L}_2$-gains of systems $(d_0 \mapsto e_1)$ and $(d_1 \mapsto e_2)$ for time-invariant and time-varying delays. For time-invariant delays the accurate gain (row EG) can be computed.

In the method of additional dynamics (AD), $W_d(s) = \frac{1}{\tau_d s+1}$ has been introduced. It can be observed that higher time-constant τ_d increases the gain, as the phase lag of the additional dynamics is added to the lag caused by the time-delay. On the other hand, too small τ_d results in numerical problems during the solution of the LMIs of the BRL, see e.g. the outlier 8.87 and the NaN entry in row $\tau_d = 0.05$, Table II. Method AD with small τ_d provided the smallest upper-bounds for the gains.

For time-varying delays the range of delay is chosen such that the results can be compared to the case of constant delays. The centre of the interval is $0.5(h_1 + h_2) = 0.05$ and $0.5(h_1 + h_2) = 0.25$, respectively, in the two parts of Table II.

When the input delay is characterised by IQCs, the upper-bound is a bit higher as compared to method AD. Pfifer and Seiler [17] generalised the approach to time-varying delays, but they assume norm bounds on the rate of the time-delay, which cannot be applied for sampled data networks where $h(t)$ has jumps.

Both the method of independent inputs and scaled inputs are very conservative as compared to the other methods. SI is a bit less conservative than II, when the optimal value for α differs from 0.5.

TABLE I

$\mathcal{L}_2$-GAIN OF SYSTEM $(d_0 \mapsto e_1)$ WITH TIME-INVARIANT DELAY.

h [s]	0.05	0.1	0.25	0.5	0.8
EG-TI	1.27	1.35	1.6	2.02	2.51
AD-TI($\tau_d = 0.5$)	2.07	2.15	2.4	2.8	3.26
AD-TI($\tau_d = 0.2$)	1.6	1.69	1.94	2.35	2.83
AD-TI($\tau_d = 0.1$)	1.44	1.52	1.77	2.19	2.68
AD-TI($\tau_d = 0.05$)	1.35	1.44	1.69	2.1	2.6
IQC-TI	3.55	3.55	3.56	3.58	3.63
SI-TI	17.75	17.75	17.75	17.75	17.75
II-TI	17.75	17.75	17.75	17.75	17.75

TABLE II

$\mathcal{L}_2$-GAIN OF SYSTEM $(d_0 \mapsto e_1)$ WITH TIME-VARYING DELAY.

h_1	0.04	0	0.24	0.2	0.15
h_2	0.06	0.1	0.26	0.3	0.35
AD-TV($\tau_d = 0.5$)	2.09	2.16	2.42	2.48	2.57
AD-TV($\tau_d = 0.2$)	1.62	2.22	1.95	2.25	4.44
AD-TV($\tau_d = 0.1$)	1.48	4.44	1.84	4.44	8.87
AD-TV($\tau_d = 0.05$)	1.77	8.87	2.15	8.87	NaN
SI-TV	24.24	38.3	24.24	38.3	49.54
II-TV	25.1	43.47	25.1	43.47	58.86

V. Conclusions

Four methods (scaling of the inputs, independent inputs, integral quadratic constraints and incorporation of additional dynamics) are presented for solving the $\mathcal{L}_2$-gain computation problem with input delays. It can be concluded the method of additional fast dynamics is the most advantageous. It converts the input delay to state delay, and very tight upper-bounds can be calculated for the gains, in case of both time-invariant and time-varying delays.

References

[1] C. Jeong, P. Park, and S. H. Kim, "Improved approach to robust stability and $\mathcal{H}_\infty$ performance analysis for systems with an interval time-varying delay," *Applied Mathematics and Computation*, vol. 218, no. 21, pp. 10533–10541, 2012.

[2] J.-H. Kim, "Note on stability of linear systems with time-varying delay," *Automatica*, vol. 47, no. 9, pp. 2118–2121, 2011.

[3] E. Fridman, "New lyapunov-krasovskii functionals for stability of linear retarded and neutral type systems," *Systems & Control Letters*, vol. 43, no. 4, pp. 309–319, 2001.

[4] Z.-G. Wu, J. H. Park, H. Su, and J. Chu, "Dissipativity analysis for singular systems with time-varying delays," *Applied Mathematics and Computation*, vol. 218, no. 8, pp. 4605–4613, 2011.

[5] U. Shaked, I. Yaesh, and C. De Souza, "Bounded real criteria for linear time-delay systems," *Automatic Control, IEEE Transactions on*, vol. 43, no. 7, pp. 1016–1022, 1998.

[6] K. Liu and E. Fridman, "Wirtinger's inequality and lyapunov-based sampled-data stabilization," *Automatica*, vol. 48, no. 1, pp. 102–108, 2012.

[7] G. Cheng, Z. Ding, and J. Fang, "Dissipativity analysis of linear state/input delay systems," in *Abstract and Applied Analysis*, vol. 2012. Hindawi Publishing Corporation, 2012.

TABLE III

$\mathcal{L}_2$-GAIN OF SYSTEM $(d_1 \mapsto e_2)$ WITH TIME-INVARIANT DELAY.

h [s]	0.05	0.8
EG-TI	4.44	4.44
AD-TI($\tau_d = 0.5$)	4.44	4.44
AD-TI($\tau_d = 0.2$)	4.44	4.44
AD-TI($\tau_d = 0.1$)	4.44	4.44
AD-TI($\tau_d = 0.01$)	4.44	4.44
IQC-TI	4.52	4.52
SI-TI	5.23	5.23
II-TI	6.86	6.86

TABLE IV

$\mathcal{L}_2$-GAIN OF SYSTEM $(d_1 \mapsto e_2)$ WITH TIME-VARYING DELAY.

h_1	0.04	0	0.24	0.2	0.15
h_2	0.06	0.1	0.26	0.3	0.35
AD-TV($\tau_d = 0.1$)	4.45	4.45	4.45	4.45	4.45
SI-TV	5.28	5.41	5.28	5.4	5.54
II-TV	13.75	23.81	13.75	23.8	32.24

[8] E. Fridman and U. Shaked, “Input delay approach to robust sampled-data $\mathcal{H}_\infty$ control,” in *Decision and Control, 2004. CDC. 43rd IEEE Conference on*, vol. 2. IEEE, 2004, pp. 1950–1951.

[9] D. Swaroop, “String stability of interconnected systems: An application to platooning in automated highway systems,” *PhD dissertation*, 1994.

[10] A. Pant, P. Seiler, and K. Hedrick, “Mesh stability of look-ahead interconnected systems,” *Automatic Control, IEEE Transactions on*, vol. 47, no. 2, pp. 403–407, Feb 2002.

[11] S. Öncü, N. van de Wouw, and H. Nijmeijer, “Cooperative adaptive cruise control: Tradeoffs between control and network specifications,” in *14th International IEEE Conference on Intelligent Transportation Systems (ITSC)*, 2011, pp. 2051 –2056.

[12] J. C. Willems, “Dissipative dynamical systems part ii: Linear systems with quadratic supply rates,” *Archive for Rational Mechanics and Analysis*, vol. 45, no. 5, pp. 352–393, 1972.

[13] C. Scherer and S. Weiland, “Lecture notes disc course on linear matrix inequalities in control, version 2.0,” 1999.

[14] K. Gu, V. Kharitonov, and J. Chen, *Stability of time-delay systems*. Birkhäuser, Boston, 2003.

[15] W. I. Lee, S. Y. Lee, and P. Park, “Improved criteria on robust stability and $\mathcal{H}_\infty$ performance for linear systems with interval time-varying delays via new triple integral functionals,” *Applied Mathematics and Computation*, vol. 243, pp. 570–577, 2014.

[16] E. Gyurkovics and D. Eszes, “Sufficient conditions for stability and stabilization of networked control systems with uncertainties and non-linearities,” *International Journal of Robust and Nonlinear Control*, 2014.

[17] H. Pfifer and P. Seiler, “Integral quadratic constraints for delayed non-linear and parameter-varying systems,” 2014, submitted to Automatica.

[18] D. Swaroop and J. K. Hedrick, “Constant spacing strategies for platooning in automated highway systems,” *ASME Journal of Dynamic Systems, Measurement and Control*, vol. 121, pp. 462–470, 1999.