

Leader and Predecessor Following Robust Controller Synthesis for String Stable Heterogeneous Vehicle Platoons^{*}

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Abstract: Conditions of string (in)stability for look-ahead interconnected vehicle systems are presented based on a compact characterization of the interconnected system. Continuous-time local models describe the temporal evolution of the state-variables of one vehicle. Discrete spatially varying systems describe the spatial evolution of local systems. The problem of string stability analysis of large scale interconnected systems is reduced to the stability analysis of two simple dynamic systems. The evaluation of worst-case spacing errors, as responses to bounded \mathcal{L}_2 leader acceleration, can be upper-bounded by computing the step-response to a simple discrete linear system. Based on the compact characterization, it is also shown that the string stability requirement can be directly converted to a standard \mathcal{H}_∞ control problem. The efficiency of the synthesis method is demonstrated on a numerical example.

Keywords: Interconnected systems, distributed control, robust control design, string stability.

1. INTRODUCTION

In order to increase road capacity and avoid congestions on highways vehicles can be organized in automated platoons, Sheikholeslam and Desoer (1990), Alvarez and Horowitz (1997). Based on local radar measurements and information shared over a communication network, on-board controllers have to keep a prescribed space between the vehicles. Disturbances, nonzero initial conditions, and change in the reference speed induce transients in the spacing errors which are propagated along the platoon. It is important that the spacing errors remain uniformly bounded, so a vehicle may join the platoon without the need of redesign or reanalyzing the platoon. The related term to this property is called *string stability*.

Many definitions of string stability emerged in the literature. (*Asymptotic*) *string stability* in the sense of Lyapunov has been formalized by Swaroop and Hedrick (1996): for finite initial conditions the evaluation of spacing errors must be bounded (or must tend to zero). A generalization to 2D formations, called *mesh stability* (Pant et al. (2002)), added the requirement of non-increasing spacing error bounds. These definitions consider the interconnected systems (IS) as autonomous systems without inputs. The effect of inputs are examined for given finite formations, for example by Tanner et al. (2004), where the formation errors are bounded in terms of both initial conditions and inputs, but the error-propagation is not examined. A generalization of the previous definition to infinite strings, called \mathcal{L}_p *string stability*, is given by Ploeg et al. (2014). The \mathcal{L}_p *string stability* is called *strict*, when the error bounds are non-increasing. Shaw and Hedrick (2007b) demonstrated that the latter condition is rather

restrictive for heterogeneous platoons where the spacing error of a slow vehicle can be larger than the spacing error of its faster predecessor.

A further practical demand against vehicle formations is the permission of *independent control design*, i.e. to grant the different manufacturers as much freedom as possible in constructing their own controllers with their own communication architecture and spacing policy. The ad-hoc organization of heterogeneous formations of such different control strategies should be allowed without the danger of violating safety and performance constraints. In the majority of the papers, string stability of vehicle formations are analyzed or controllers are designed with the assumption of specific and common control laws. Some exceptions in the field of analysis are by Middleton and Braslavsky (2010), Lestas and Vinnicombe (2006), Seiler et al. (2004), Barooah and Hespanha (2005), and in the field of control synthesis by Shaw and Hedrick (2007a).

In the present paper the results by Shaw and Hedrick (2007a) are generalized in a number of ways. General definitions for heterogeneous string stability are formulated in Section 2. The main contribution of the paper is a compact characterization of vehicle strings with *leader and predecessor following* control architecture. It is discussed in Section 4. The main advantages of this characterization may be summarized in two points. 1) It allows simple conditions for string stability and disturbance attenuation and 2) directly indicates a distributed control design method to achieve string stability. This is presented in Section 5. The results have important consequences related to platoon heterogeneity and the necessary specifications for *individual control design*. The effectiveness of the suggested method is demonstrated in Section 6.

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Notations. \mathbb{R} (\mathbb{C}) denotes the real (complex) numbers. For $x \in \mathbb{R}$ or $x \in \mathbb{C}$ $|x|$ denotes the absolute value of x . x_i is the i th element of vector x . The ℓ_p vector norm on \mathbb{R}^n is defined by $\|x\|_p = (\sum_{k=1}^n |x_k|^p)^{1/p}$ for $p \in [1, \infty)$ and $\|x\|_\infty = \max_i |x_i|$ for $p = \infty$. The function space \mathcal{L}_p^n denotes $\{x : \mathbb{R} \mapsto \mathbb{R}^n : x \text{ is measurable and } \|x\|_{p,q} < \infty, p, q \in [1, \infty]\}$, where $\|x\|_{p,q} = (\int_0^\infty \|x(t)\|_q^p dt)^{1/p}$ for $p \in [1, \infty)$ and $\|x\|_{\infty,q} = \sup_{t \geq 0} \|x(t)\|_q$ for $p = \infty$. Note that space \mathcal{L}_p^n is identical for every $q \in [1, \infty]$. If $n = 1$, then q can be dropped from the notation and the \mathcal{L}_p^1 norm of the scalar signal is denoted by $\|x\|_p$. Let $\text{vec}_{i=0}^n \{x_i\} = [x_0^T, x_1^T, \dots, x_n^T]^T$ is a hyper-vector. Let $\mathcal{L}_{r,p}^n$ denote the vector space

$$\mathcal{L}_{r,p}^n = \{X = \text{vec}_{i=0}^\infty \{x_i(t)\} : x_i \in \mathcal{L}_p^n : \|X\|_{r,p,q} < \infty\}$$

with norm defined by $\|X\|_{r,p,q} = (\sum_{i=0}^\infty \|x_i(t)\|_{p,q}^r)^{1/r}$ for $r \in [1, \infty)$ and $\|X\|_{\infty,p,q} = \sup_{i \geq 0} \|x_i(t)\|_{p,q}$ for $r = \infty$. If $n = 1$, then q can be dropped from the notations. The induced norm of the system $\mathcal{G} : \mathcal{L}_{r',p'}^{n_u} \mapsto \mathcal{L}_{r,p}^{n_y}$, with $r, p, q, r', p', q' \in [1, \infty]$, is denoted by

$$\|\mathcal{G}\|_{(r,p,q),(r',p',q')} \triangleq \sup_{0 \neq W \in \mathcal{L}_{r',p'}^{n_u}} \frac{\|E\|_{r,p,q}}{\|W\|_{r',p',q'}}$$

For a signal $x(t)$, $\hat{x}(s) = \mathcal{L}\{x(t)\}$ denotes its Laplace transform.

2. DEFINITION FOR MIXED NORM STRING STABILITY

In this section, string stability is defined for a general class of systems, denoted by $\mathcal{G} : \mathcal{L}_{r',p'}^{n_d} \mapsto \mathcal{L}_{r,p}^{n_e}$,

$$\mathcal{G} : \dot{x}_i(t) = f_i(X(t), W(t)), \quad x_i(0) = x_0, \quad (1)$$

$$e_i(t) = h_i(X(t), W(t)), \quad i = 0, 1, 2, \dots \quad (2)$$

where $X(t) \triangleq \text{vec}_{i=0}^\infty \{x_i(t)\}$, $x_i(t) \in \mathbb{R}^n$, $W(t) \triangleq \text{vec}_{i=0}^\infty \{d_i(t)\}$ and $E(t) \triangleq \text{vec}_{i=0}^\infty \{e_i(t)\}$ denote respectively the sequence of state-variables, disturbances and outputs of the connected subsystems, $f_i : (\mathbb{R}^n \times \dots \times \mathbb{R}^n, \mathbb{R}^{n_d} \times \dots \times \mathbb{R}^{n_d}) \mapsto \mathbb{R}^n$ and $h_i : (\mathbb{R}^n \times \dots \times \mathbb{R}^n, \mathbb{R}^{n_d} \times \dots \times \mathbb{R}^{n_d}) \mapsto \mathbb{R}^{n_e}$. Reconsidering the definitions given by Swaroop and Hedrick (1996), Pant et al. (2002), Ploeg et al. (2014) and Shaw and Hedrick (2007b), the following generalized string stability definitions are proposed.

Definition 1. (Mixed norm string stability). The interconnected system (1)-(2) with $X(0) = 0$ is *mixed norm string stable* with respect to input $W \in \mathcal{L}_{r',p'}^{n_d}$ and output $E \in \mathcal{L}_{r,p}^{n_e}$, if $\|\mathcal{G}\|_{(r,p,q),(r',p',q')} < \infty$, where $r, p, q, r', p', q' \in [1, \infty]$.

Suppose that f_i and h_i belong to some sets of functions \mathcal{F} and \mathcal{H} , respectively.

Definition 2. (Robust mixed norm string stability). The interconnected system (1)-(2) with $X(0) = 0$ is *robustly mixed norm string stable* with $r, p, q, r', p', q' \in [1, \infty]$, if it is mixed norm string stable with any $f_i \in \mathcal{F}$ and $h_i \in \mathcal{H}$, $i \geq 0$.

3. PROBLEM FORMULATION

Consider V_i as the following model for the longitudinal dynamics of the i th vehicle in the platoon (Shaw and Hedrick (2007b)),

$$\hat{a}_i(s) = H_i(s)\hat{u}_i(s) + \hat{d}_i(s), \quad (3)$$

$$\hat{v}_i(s) = \hat{a}_i(s)/s, \quad (4)$$

$$\hat{p}_i(s) = \hat{v}_i(s)/s = V_i(s)\hat{u}_i(s) + \hat{d}_i(s)/s^2, \quad (5)$$

where $H_i(s) = g_i/(\tau_i s + 1)$ represents a first-order actuator dynamics, $V_i(s) = H_i(s)/s^2$ and $\hat{a}_i, \hat{v}_i, \hat{p}_i$ and \hat{d} denote the Laplace-transform of acceleration, speed, position and disturbance, respectively. Vehicle V_0 is the lead vehicle driven by a driver. The follower vehicles ($i > 0$) are driven by LTI controllers, $K_i = [K_{ia}, K_{iy}, K_{ia}^0, K_{iy}^0]$, defined by

$$\hat{u}_1(s) = K_{1a}(s)\hat{a}_0(s) + K_{1y}(s)(\hat{a}_1(s) - \hat{a}_0(s)), \quad (6)$$

$$\hat{u}_i(s) = K_{ia}(s)\hat{a}_{i-1}(s) + K_{iy}(s)(\hat{a}_i(s) - \hat{a}_{i-1}(s)) + K_{ia}^0(s)\hat{a}_0(s) + K_{iy}^0(s)(\hat{a}_i(s) - \hat{a}_0(s)), \quad i > 1 \quad (7)$$

The *predecessor following* controller (6) is in a form of Cooperative Adaptive Cruise Control (CACC), while the other controllers (7) have *leader and predecessor following* architecture. It is known that the above architecture can be designed to be string stable even with constant spacing policy (see Swaroop and Hedrick (1996) for homogeneous platoons). In most of the literature, the controllers have specific structure. The analysis in the following sections is valid for general proper LTI controllers.

The performance of the platoon can be measured in terms of the spacing errors defined as

$$\hat{e}_i(s) \triangleq \hat{p}_i(s) - \hat{p}_{i-1}(s) + \hat{L}_i(s), \quad i > 0, \quad (8)$$

where $L_i(t) = \mathcal{L}^{-1}\{\hat{L}_i(s)\}$ defines the space prescribed between vehicle i and $i - 1$. For analysis and design purposes $L_i(t) = 0$ can be assumed in the case of constant spacing policy.

In a state-space representation of the platoon model (3)-(7), x_i consists of the state-variables of vehicle i and controller i , the disturbance is defined by $W = [u_0, d_0, d_1, \dots]^T$, and the output by $E = [e_1, e_2, \dots]^T$. The notion of mixed norm string stability is related to the evolution of maximum spacing errors (in some norm) along the string as effects of disturbances. The determination of the maximum spacing errors are of practical interest. For example peak value, $\|e_i\|_\infty$, can be used to set constant L_i .

In the platoon problem, $\dim(d_i) = n_d = 1$, $\dim(e_i) = n_e = 1$, so indexes q and q' can be dropped from the notations of norms. It is practical to assume that d_i has a uniform (in spatial variable i) bound for its \mathcal{L}_2^1 or \mathcal{L}_∞^1 norm, i.e. $W \in \mathcal{L}_{\infty,2}^1$ or $W \in \mathcal{L}_{\infty,\infty}^1$. Uniform boundedness of the spacing errors (Shaw and Hedrick (2007b)), or convergence of the spacing errors to zero (Seiler et al. (2004), Barooah and Hespanha (2005)) can be expressed in terms of norms $\|E\|_{\infty,p}$ or $\|E\|_{2,p}$, respectively.

Remark 1. Note that asymptotic stability of the closed-loop system (3)-(7) is not required in the definitions. Indeed, the vehicle platoon is only partially stable due to the two integrators in each vehicle models. It is realistic to assume that $u_0, a_i \in \mathcal{L}_2^1 \cap \mathcal{L}_\infty^1$, but $v_i \in \mathcal{L}_\infty^1$ and p_i are unbounded. On the other hand it can be expected from the control law that $v_i - v_j \in \mathcal{L}_2^1$ and $p_i - p_j \in \mathcal{L}_2^1$ for any pair i, j . State-transformation $p_i \rightarrow e_i, v_i \rightarrow \dot{e}_i$ for $i > 0$ and omitting states v_0 and p_0 would result in a stable closed-loop system. In the subsequent sections we follow a different approach which fits better to controller

synthesis. Instead of state transformation we consider the evaluation of the stable parts of the states (leaving out the integrators), then derive conditions for string stability with respect to outputs a_i . It is shown that, with appropriate controllers, the string stability property holds also with respect to outputs e_i and \dot{e}_i .

Problem 1. Derive conditions for robust mixed norm string stability/instability of the heterogeneous vehicle platoon (3)-(8) with leader and predecessor following architecture and constant spacing policy.

Problem 2. Design robust dynamic controllers that solve Problem 1.

4. A COMPACT CHARACTERIZATION OF PLATOON MODELS

The following two dimensional description of the platoon model is shown to be useful for analysis and control synthesis. In the 2D framework by Knorn and Middleton (2012) and Sebek and Hurak (2011), the temporal and spatial dimensions are related to the state variable $x(i, t) = x_i(t)$. The constructed 2D system depends on the states of the i th vehicle, and also on the states of the neighboring vehicles. Thus the approach is applicable only in the case of limited communication range.

In contrast, in the following description the temporal dimension is related to the states $x_i(t)$ for every $i = 0, 1, \dots$. Continuous time systems describe the dynamics of the i th vehicle locally. In the spatial dimension a discrete dynamic system is defined as a mapping from the space of continuous time systems to the space of continuous time systems, where the coefficients of this discrete dynamic system are also continuous-time dynamic systems. As a result, the stability properties of a discrete dynamic system will characterize the string stability properties of the network and these properties can be converted to specifications of the local controller design.

In order to introduce the approach of the compact description of distributed systems, we examine the transients propagating on a vehicle string. Zero initial conditions and zero disturbances are assumed for simplicity. The only excitation to the system is the leader vehicle acceleration reference $W(t) = u_0(t) \in \mathcal{L}_2^1$. The effect of initial conditions and disturbances can be modeled similarly.

4.1 A frequency-domain local model

Local models are defined as the closed-loop systems mapping the accelerations received from the leader and the preceding vehicle to the acceleration of the i th vehicle,

$$\hat{a}_1(s) = T_{p1}(s)\hat{a}_0(s), \quad (9)$$

$$\hat{a}_i(s) = T_{pi}(s)\hat{a}_{i-1}(s) + T_{li}(s)\hat{a}_0(s), \quad i > 1, \quad (10)$$

where, from (3)-(8),

$$T_{p1}(s) = S_1(s)(K_{1a}(s) - K_{1y}(s)), \quad (11)$$

$$S_1(s) = (1 - H_1(s)K_{1y}(s))^{-1}, \quad (12)$$

$$T_{pi}(s) = S_i(s)(K_{ia}(s) - K_{iy}(s)), \quad i > 1, \quad (13)$$

$$T_{li}(s) = S_i(s)(K_{ia}^0(s) - K_{iy}^0(s)), \quad i > 1, \quad (14)$$

$$S_i(s) = (1 - H_i(s)(K_{iy}(s) + K_{iy}^0(s)))^{-1}, \quad i > 1. \quad (15)$$

Without loss in generality, local models can also be defined in the state-space and are not restricted to LTI systems.

4.2 Spatial dimension: discrete dynamics

Let G_i denote the system with input a_0 and output a_i ,

$$\hat{a}_i(s) \triangleq G_i(s)\hat{a}_0(s). \quad (16)$$

Then $G_1(s) = T_{p1}(s)$. A state-space realization of G_i would involve the state-variables of all vehicles and their controllers between and including V_0 and V_i . Given G_i , G_{i+1} can be computed by using (10) as

$$G_{i+1}(s) = T_{p,i+1}(s)G_i(s) + T_{l,i+1}(s). \quad (17)$$

If, for example, we are interested in $a_{i+1} - a_i$, we can define the system F_i as

$$\hat{a}_{i+1}(s) - \hat{a}_i(s) \triangleq F_i(s)\hat{a}_0(s), \quad (18)$$

where $F_i(s) = G_{i+1}(s) - G_i(s)$. Define the following spatially discrete system

$$\begin{bmatrix} G_{i+1}(s) \\ F_i(s) \end{bmatrix} = \begin{bmatrix} T_{p,i+1}(s) & | & T_{l,i+1}(s) \\ T_{p,i+1}(s) - 1 & | & T_{l,i+1}(s) \end{bmatrix} \begin{bmatrix} G_i(s) \\ U_i(s) \end{bmatrix} \quad (19)$$

with initial conditions $T_{l1}(s) = 0$ and $G_0(s) = 1$, and U_i being a stable continuous time system.

System (19) resembles a discrete state-space system that maps sequence U_i , $i = 0, 1, \dots$ to sequence F_i , $i = 0, 1, \dots$. Its state variable is G_i and its varying coefficient matrix is a continuous-time dynamic system in each step. When all continuous time systems are LTI transfer functions, then for every fixed s , (19) maps a sequence of complex numbers to a sequence of complex numbers, and the coefficient matrices are also complex. Note that the sequence of $\hat{a}_{i+1}(s) - \hat{a}_i(s)$, $i = 0, 1, \dots$, can be obtained as the step response of system (19), i.e. with $U_i(s) = 1$.

4.3 Conditions of robust mixed norm string stability

The following theorem provides conditions for robust mixed norm string stability of the vehicle string (3)-(7), which can equivalently be described by (9)-(15). It is assumed that $T_{pi} \in \mathcal{T}_p$ and $T_{li} \in \mathcal{T}_l$, $i > 1$, for some sets of transfer functions \mathcal{T}_p and \mathcal{T}_l , respectively. Let us introduce the notation $\|\mathcal{T}_p\|_\infty = \sup_{T \in \mathcal{T}_p} \|T\|_\infty$, and $\|\mathcal{T}_l\|_\infty = \sup_{T \in \mathcal{T}_l} \|T\|_\infty$.

Theorem 1. (Robust mixed norm string stability). Let the vehicle string be denoted by \mathcal{G}_a , where the input to \mathcal{G}_a is $W = a_0 \in \mathcal{L}_2^1$ and its output is defined by $E_a = \text{vec}_{i=0}^\infty \{a_{i+1} - a_i\} \in \mathcal{L}_{\infty,2}^1$.

- (1) \mathcal{G}_a is robustly mixed norm string stable with respect to W , E_a , \mathcal{T}_p and \mathcal{T}_l , i.e.

$$\|\mathcal{G}_a\|_{(\infty,2)(2)} = \sup_{W \in \mathcal{L}_2^1, \|W\|_2 \leq 1} \|E_a\|_{\infty,2} < \infty \quad (20)$$

for any \mathcal{G}_a with $T_{pi} \in \mathcal{T}_p$ and $T_{li} \in \mathcal{T}_l$, $i > 1$, if $\|\mathcal{T}_p\|_\infty < 1$, $\|T_{p1}\|_\infty < \infty$ and $\|\mathcal{T}_l\|_\infty < \infty$.

- (2) If $\|\mathcal{T}_p\|_\infty > 1$, then there exist a \mathcal{G}_a with $T_{pi} \in \mathcal{T}_p$ and $T_{li} \in \mathcal{T}_l$, $i > 1$, such that (20) does not hold, i.e. \mathcal{G}_a is not mixed norm string stable.

Proof. (1) Let $\alpha(s), \beta(s) \in \mathbb{R}$ be defined as $\alpha(s) \triangleq \sup_{i>1} |T_{pi}(s)|$ and $\beta(s) \triangleq \sup_{i>1} |T_{li}(s)|$. Then $\alpha(s) < 1$ and $\beta(s) < \infty$ by hypothesis. The step response of the discrete LTI system

$$\begin{bmatrix} \delta_{i+1}(s) \\ \gamma_i(s) \end{bmatrix} = \begin{bmatrix} \alpha(s) & | & \beta(s) \\ \alpha(s) + 1 & | & \beta(s) \end{bmatrix} \begin{bmatrix} \delta_i(s) \\ \nu_i(s) \end{bmatrix} \quad (21)$$

starting from $i = 1$ with initial condition $\delta_1(s) = |T_{p1}(s)|$, provide upper-bounds for $G_i(s)$ and $F_i(s)$ such that $|G_i(s)| \leq \delta_i(s)$ and $|F_i(s)| \leq \gamma_i(s)$. Since $\alpha(s) < 1$ for all $s \in \mathbb{C}$ the step response of (21) is bounded for all $s \in \mathbb{C}$, thus there exist finite constants M_1 and M_2 such that $\|G_i\|_\infty < M_1$ and $\|F_i\|_\infty < M_2$ for all i , which implies that $\|a_i\|_2 < M_1$ and $\|a_{i+1} - a_i\|_2 < M_2$ for all $i \geq 0$ and $\|u_0\|_2 \leq 1$.

(2) If $\|\mathcal{T}_p\|_\infty > 1$, then there exist a $T_{pj} \in \mathcal{T}_p$, such that $\|T_{pj}\|_\infty > 1$. Then, for the homogeneous platoon with $T_{pi} = T_{pj}$ for all $i > 1$, there exists an $s = j\omega$ such that $|G_i(s)|$ increases without bound according to (17). This implies that there exists an excitation $\hat{a}_0(s) = V_0(s)\hat{u}_0(s)$ such that $|a_i(s)|$ increases without bound as i increases, i.e. the platoon is string unstable. ■

Shaw and Hedrick (2007b) calculated an explicit formulae for F_i , with controllers such that $K_{ia} = K_{ia}^0 = 0$ in (6)-(7). In contrast, introducing the compact characterization (9), (10) and (19) allows a much simpler proof and the possibility for generalization of the theorem to other interconnection topologies.

Remark 2. Whenever string stability is established, it can be interesting to have a picture about the size of the worst-case errors along the string. It can be observed from the discrete upper-bound system (21) that there is an exponential decreasing term due to the nonzero initial condition, $G_0 = 1$, and a monotonously increasing convergent term. The example with static controller in Section 6 illustrates this phenomenon, see Figure 1. It is easy to show that for homogeneous platoons (i.e. when $T_{pi} = T_{p2}$ for all $i > 1$) F_i is proportional to $T_{p,2}^{i-2}$, therefore $|F_i(s)|$ and $|a_{i+1} - a_i|$ tend to zero.

Remark 3. Part (2) of Theorem 1 is due to the fact that $\|T_{pi}^j\|_\infty = \|T_{pi}\|_\infty^j$ for all $T_{pi} \in \mathcal{H}_\infty$. For the peak to peak gains, on the other hand, $\|T_{pi}^j\|_1 < \|T_{pi}\|_1^j$ in general, therefore the platoon might be string stable with respect to $W = u_0 \in \mathcal{L}_\infty^1$ and $E_a \in \mathcal{L}_{\infty,\infty}^1$ even if $\|T_{pi}\|_1 > 1$.

Remark 4. An important consequence of Theorem 1 follows from that the condition of string stability $\|T_{pi}\|_\infty < 1$ is a local condition. It has to be satisfied for all i independently. The smaller T_{pi} and T_{li} are, the smaller is the contribution of vehicle i to the error bounds. As a conclusion: *the worst-case spacing errors in ad-hoc organized platoons are smaller if the controllers are optimized independently.* That is, controller K_i should be designed based on the knowledge about vehicle V_i only.

5. CONTROL DESIGN FOR ROBUST STRING STABILITY

Theorem 1 provides conditions for string stability for the stable part of the dynamics, see Remark 1. It is further required that $\dot{e}_i = v_i - v_{i-1} \in \mathcal{L}_2$ and $e_i = p_i - p_{i-1} \in \mathcal{L}_2$ whenever $a_0 \in \mathcal{L}_2$. In the next subsection the predecessor following controller K_1 is designed which ensures the latter two conditions for $i = 1$.

5.1 Design of predecessor following robust controllers with constant spacing policy

Let $\mathcal{I} \triangleq [1, 1/s, 1/s^2]^T$. Condition $\|\mathcal{IF}_1\|_\infty < \infty$ ensures the above requirement, i.e. $\ddot{e}_1, \dot{e}_1, e_1 \in \mathcal{L}_2$ whenever

$a_0 \in \mathcal{L}_2$. System \mathcal{IF}_1 can be augmented by a performance output penalizing the control effort and disturbance and noise inputs which also act on the system. Furthermore, the performance inputs and outputs can be weighted by stable weighting functions to form the generalized plant, $\begin{bmatrix} \hat{z}_1(s) \\ \hat{y}_1(s) \end{bmatrix} = P_1(s) \begin{bmatrix} \hat{w}_1(s) \\ \hat{u}_1(s) \end{bmatrix}$. The perfor-

mance outputs, $z_1 = [z_u, z_a, z_v, z_p]^T$, penalize weighted control effort $\hat{z}_u(s) = W_u(s)\hat{u}_1(s)$, acceleration difference $\hat{z}_a(s) = W_a(s)(\hat{a}_1(s) - \hat{a}_0(s))$, speed difference $\hat{z}_v(s) = W_v(s)(\hat{v}_1(s) - \hat{v}_0(s))$, and spacing error $\hat{z}_p(s) = W_p(s)\hat{e}_1(s)$. $W_r, W_d, W_n, W_u, W_a, W_v$ and W_p denote stable weighting functions which are design parameters. The normalized performance inputs to the plant are $w_1 = [a_0, n_0, d_1, n_1]^T$, where n_0, n_1 denote the measurement noise on a_0 and a_1 , respectively. The measurements are defined by $y_1 = [a_0 + n_0, a_1 + n_1 - a_0 - n_0, v_1 - v_0, e_1]^T$. The control input is formulated by $\hat{u}_1(s) = K_1(s)\hat{y}_1(s)$, where controller K_1 is the solution of the standard \mathcal{H}_∞ control problem

$$\min_{K_1} \|\mathcal{F}_U(P_1, K_1)\|_\infty. \quad (22)$$

5.2 Design of leader and predecessor following robust controllers with constant spacing policy

Suppose that for each vehicle V_i a predecessor following controller $\tilde{K}_i = [\tilde{K}_{ia}, \tilde{K}_{iy}]$ is designed according to Section 5.1. Let \tilde{T}_{pi} denote the corresponding closed-loop transfer function computed as in (11) with (12), i.e.

$$\tilde{T}_{pi}(s) = \tilde{S}_i(s)(\tilde{K}_{ia}(s) - \tilde{K}_{iy}(s)), \quad (23)$$

$$\tilde{S}_i(s) = (1 - H_i(s)\tilde{K}_{iy}(s))^{-1}. \quad (24)$$

The simplest approach to obtain leader and predecessor following controller is to excite controller \tilde{K}_i by the linear combination of the inputs received from the leader and the preceding vehicle, respectively, as proposed by Shaw and Hedrick (2007a)

$$\hat{u}_i(s) = \tilde{K}_i(s) \begin{bmatrix} \hat{a}_{\rho_i}(s) \\ \hat{a}_i(s) - \hat{a}_{\rho_i}(s) \end{bmatrix}. \quad (25)$$

where $a_i = \tilde{T}_{pi}a_{\rho_i}$, $a_{\rho_i} \triangleq \rho_i a_{i-1} + (1 - \rho_i)a_0$, and $0 \leq \rho_i < 1$ is a constant design parameter. Using this choice for u_i , we have

$$T_{pi} = \rho_i \tilde{T}_{pi}, \quad (26)$$

$$T_{li} = (1 - \rho_i) \tilde{T}_{pi}. \quad (27)$$

The following theorem shows that with the above controllers, not only the partially stable state-variables, but the spacing errors are uniformly bounded as well.

Theorem 2. (Uniform boundedness of the spacing errors). Let the vehicle string be denoted by \mathcal{G}_e , where the input to \mathcal{G}_e is $W = a_0 \in \mathcal{L}_2^1$ and its output is defined by $E_e = \text{vec}_{i=1}^\infty \{e_i\} \in \mathcal{L}_{\infty,2}^1$. Let the controllers are chosen according to (25) and $\rho_i < 1$ satisfy also $\rho_i < \|\tilde{T}_{pi}\|_\infty^{-1}$, $i > 0$. Then the vehicle string is robustly mixed norm string stable with respect to W and E_e , i.e. $\|\|\mathcal{G}_e\|\|_{(\infty,2)(2)} < \infty$.

Proof. According to Theorem 1 robust mixed norm string stability with respect to output E_a follows by the choice $\rho_i < \|\tilde{T}_{pi}\|_\infty^{-1}$, $i = 2, 3, \dots$. This implies also that the sequence of bounds, $\|a_i\|_2$, $i \geq 0$, is uniformly bounded, consequently the sequence of bounds, $\|a_{\rho_i}\|_2$, $i > 1$, is

uniformly bounded. Since \tilde{K}_i is designed according to Section 5.1, it follows that systems $(a_{\rho_i} \mapsto \mathcal{I}(a_i - a_{\rho_i})) \in \mathcal{H}_\infty$, which implies that $\|p_i - p_{\rho_i}\|_2 < \beta$, for all $i > 1$, for some constant β , where $p_{\rho_i} = \rho_i p_{i-1} + (1 - \rho_i) p_0$. This is equivalent to $\|(p_i - p_0) - \rho_i(p_{i-1} - p_0)\|_2 < \beta$, $i > 1$. By triangle inequality, we have $\|p_i - p_0\|_2 < \beta + \rho_i \|p_{i-1} - p_0\|_2$, $i > 1$. Since $\|p_1 - p_0\|_2$ is finite by the choice of K_1 , and $\rho_i < 1$, it follows that the above iteration is convergent and $\|p_i - p_0\|_2$, $i > 1$, is uniformly bounded. This implies that $\|p_i - p_{i-1}\|_2$, $i > 0$, is uniformly bounded. ■

6. EXAMPLE

In this section the design method of Section 5 is tested and compared to a verified leader and predecessor following controller by Rödönyi et al. (2014).

6.1 Controller K^{stat}

The experimentally tuned and verified controller presented by Rödönyi et al. (2014) is common for all vehicles. It is denoted by K^{stat} and has the same structure as (6)-(7), where $K_{1a}(s) = 1$, $K_{1y}(s) = -\frac{0.7}{s} - \frac{0.1127}{s^2}$, $K_{ia}(s) = 0.0449$, $K_{iy}(s) = -\frac{0.236}{s} - \frac{0.0564}{s^2}$, $K_{ia}^0(s) = 0.9551$, $K_{iy}^0(s) = -\frac{0.4642}{s} - \frac{0.0564}{s^2}$.

6.2 Controller K^{dyn}

For each vehicle a dynamic controller \tilde{K}_i is designed according to Section 5. The design parameters in the generalized plant have been chosen as $W_r(s) = 1$, $W_d(s) = 0.3 \frac{7s+1}{0.1s+1}$, $W_n(s) = 0.01$, $W_u(s) = 0.01$, $W_a(s) = 1$, $W_v(s) = 1$ and $W_p(s) = 1$. The scaling parameters in (25) are chosen to be $\rho_i = 0.5/\|\tilde{T}_{pi}\|_\infty$. The order of each LTI controller is four.

6.3 Analysis of robust mixed norm string stability

The uncertainty set is defined by a finite set of vehicle parameters. Each vehicle model has time-constant either $\tau_i = 0.6$ or $\tau_i = 0.9$ and gain $g_i = 1$. A specific vehicle platoon indexed by j can be characterized by defining the sequence of parameters, $\pi_j(n) = [\tau_{j_0}, \tau_{j_1}, \tau_{j_2}, \dots, \tau_{j_n}]^T$. For a platoon of $n + 1$ vehicles there are 2^{n+1} possible values of vector $\pi_j(n)$. Let $\Pi(n)$ denote the set of all such combinations, i.e. $\pi_j(n) \in \Pi(n)$. Let the platoon model with parameters $\pi_j(n)$ is denoted by $\mathcal{G}_e(\pi_j(n))$. The input to $\mathcal{G}_e(\pi_j(n))$ is $u_0 \in \mathcal{L}_2^1$ and the output is $E_e = \text{vec}_{i=1}^n \{e_i\} \in \mathcal{L}_{\infty,2}^n$. The worst-case spacing errors subject to bounded input and arbitrary vehicle ordering is defined by

$$\gamma_i \triangleq \sup_{\pi_j(i) \in \Pi(i), \|u_0\|_2 < 1} \|e_i(\pi_j(i))\|_2. \quad (28)$$

An upper index indicates the applied controller as γ_i^{stat} or γ_i^{dyn} . Figure 1 shows the evaluation of worst-case spacing errors (28). The 4th order controllers, K^{dyn} , heavily decrease the error bounds. The convergence of the error-bounds to zero refers to homogeneity of the platoon, which is due to the independently designed controllers, see Remark 3.

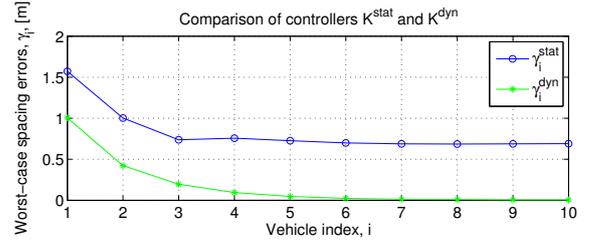


Fig. 1. Worst-case spacing error bounds (28) as functions of vehicle index

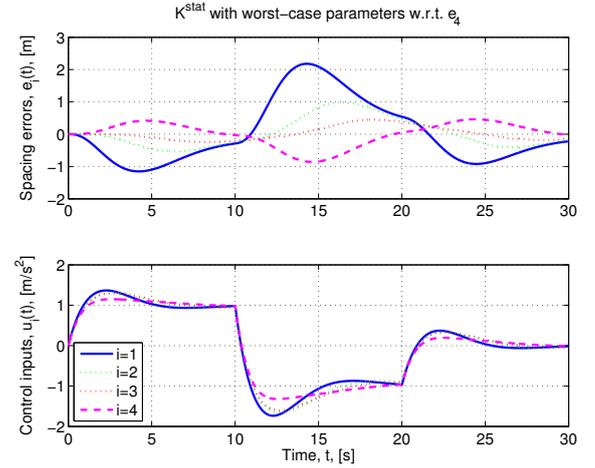


Fig. 2. Simulation with a five-vehicle platoon of $\pi_{WC}^{stat}(4) = [0.6, 0.9, 0.9, 0.9, 0.6]$

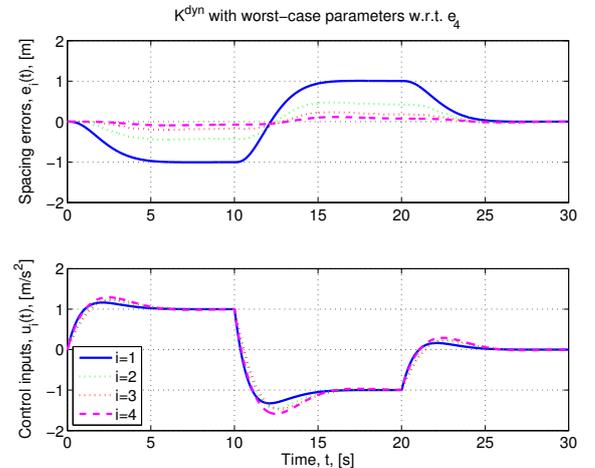


Fig. 3. Simulation with a five-vehicle platoon of $\pi_{WC}^{dyn}(4) = [0.6, 0.6, 0.6, 0.6, 0.9]$

6.4 Simulation results

Though saturation and rate limitations in the control action are not examined in this study, simulation results show that the improvement is achieved along with moderated amplitude and comparable rate limits of the control inputs. Figures 2 and 3 show the spacing error and control input responses of a platoon to reference input $u_0(t) = 1$, if $t \in [0, 10)$, $u_0(t) = -1$ if $t \in [10, 20)$, $u_0(t) = 0$ if $t \in [20, 30]$.

It can be observed that spacing errors are constant for step-like acceleration demand in case of controller K^{dyn} .

Without loss in generality, integrators can be added to the controllers by adding further integrators to the performance weighting functions, as proposed by Hara et al. (1994). The simulations were carried out with the worst-case vehicle combinations with respect to e_4 . Table 1 shows the worst-case platoon configurations. It can be observed that it depends on the controller. In fact, it depends also on the set of vehicle dynamics and the chosen output signal norm $\mathcal{L}_{\infty,2}^1$ or $\mathcal{L}_{\infty,\infty}^1$, so it is not easy to set up a general rule to answer which is the worst-case vehicle combination.

Table 1. Worst-case vehicle combinations

$\pi_{WC}^{stat}(1) =$	[0.6, 0.9]
$\pi_{WC}^{dyn}(1) =$	[0.6, 0.9]
$\pi_{WC}^{stat}(2) =$	[0.6, 0.6, 0.9]
$\pi_{WC}^{dyn}(2) =$	[0.6, 0.6, 0.9]
$\pi_{WC}^{stat}(3) =$	[0.6, 0.9, 0.9, 0.6]
$\pi_{WC}^{dyn}(3) =$	[0.6, 0.6, 0.6, 0.9]
$\pi_{WC}^{stat}(4) =$	[0.6, 0.9, 0.9, 0.9, 0.6]
$\pi_{WC}^{dyn}(4) =$	[0.6, 0.6, 0.6, 0.6, 0.9]
$\pi_{WC}^{stat}(5) =$	[0.6, 0.6, 0.9, 0.9, 0.9, 0.6]
$\pi_{WC}^{dyn}(5) =$	[0.6, 0.6, 0.6, 0.6, 0.6, 0.9]
$\pi_{WC}^{stat}(6) =$	[0.6, 0.6, 0.6, 0.9, 0.9, 0.9, 0.6]
$\pi_{WC}^{dyn}(6) =$	[0.6, 0.9, 0.6, 0.9, 0.6, 0.9, 0.6]
$\pi_{WC}^{stat}(7) =$	[0.6, 0.6, 0.6, 0.6, 0.9, 0.9, 0.9, 0.6]
$\pi_{WC}^{dyn}(7) =$	[0.6, 0.9, 0.9, 0.6, 0.9, 0.6, 0.9, 0.6]
$\pi_{WC}^{stat}(8) =$	[0.6, 0.6, 0.6, 0.6, 0.6, 0.9, 0.9, 0.9, 0.6]
$\pi_{WC}^{dyn}(8) =$	[0.9, 0.9, 0.6, 0.9, 0.9, 0.6, 0.6, 0.9, 0.9]

7. CONCLUSION

It is shown in the paper that vehicle platoons with a leader and predecessor following control architecture and constant spacing policy can be characterized by two simple dynamic systems. The one is an LTI system, (T_{pi}, T_{li}) , a *local description* of the dynamics of a single vehicle and its controller. The other is a *discrete, spatially varying system* (19) which defines an iteration on the systems mapping the leader inputs to the local state-variables. Conditions for robust string stability follow from the stability properties of the second, spatially varying system.

Some properties of the approach are summarized in the following.

- There is no need to compute spacing-error transfer functions that would limit the applicable interconnection topologies, control strategies, the class of model uncertainties and the chosen norms. The method is not bound to frequency-domain descriptions of local models.
- The approach provides a systematic way to design distributed robust controllers. The stability conditions for the spatially discrete dynamics indicate specifications for the control design. The design is carried out based on the local models.
- An upper-bound to the worst-case spacing errors can be computed based on the *discrete, spatially varying system*, but this upper-bound is conservative.

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