The tail of the crossing probability in near-critical percolation — an appendix to Ahlberg & Steif [arXiv:1405.7144]

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Abstract

We answer a question of Ahlberg and Steif (2014) by finding the tail behaviour of the crossing probability in near-critical planar percolation. Interestingly, this superexponentially small behaviour is different from the case of dynamical percolation, where the analogous tail probability was proved to be at least exponential and at most superpolynomial by Hammond, Mossel and Pete (2012). The proof is simple, given the scale covariance established by Garban, Pete and Schramm (2013).

1 Introduction

Consider site percolation on the triangular lattice $\mathbb{T}$, around the critical density $p_c = 1/2$. See [Gri99, Wer07] for background. Let $\mathsf{LR}_\mathcal{Q}$ denote the left-to-right crossing event in a nice quad $\mathcal{Q}$, by which we mean the image of the square $[0,1]^2$ under a smooth injective map into $\mathbb{C}$. If we magnify $\mathcal{Q}$ by a factor of ρ, the new quad will be denoted by $\rho\mathcal{Q}$, the center of magnification being irrelevant. Furthermore, let $\alpha_4(n)$ denote the critical alternating four-arm probability from a given site to graph distance n, and let $r(n) := 1/\big(n^2\alpha_4(n)\big)$.

It was proved in [GPS13, GPS15+] that, for any $\lambda \in \mathbb{R}$, the limit

$$f(\lambda, \mathcal{Q}) := \lim_{n\to\infty} \mathbf{P}_{p_c+\lambda r(n)}\big[\mathsf{LR}_{n\mathcal{Q}}\big] \tag{1.1}$$

exists, and is conformally covariant. See [GPS15+, Theorems 1.5, 9.5, 10.3]. Instead of defining exactly what conformal covariance means, let us just give a special case that we will use:

$$f(\rho\lambda, \mathcal{Q}) = f(\lambda, \rho^{4/3}\mathcal{Q})\,, \tag{1.2}$$

for any scaling factor $\rho > 0$. Furthermore, we know already from Kesten's work [Kes87, Nol08] (for any subsequential limit, at that time) that $f(\lambda, \mathcal{Q}) \in (0,1)$, and

$$\lim_{\lambda\to-\infty} f(\lambda, \mathcal{Q}) = 0\,, \quad \text{and} \quad \lim_{\lambda\to\infty} f(\lambda, \mathcal{Q}) = 1\,. \tag{1.3}$$

Now, one may naturally wonder about the tail behaviour of $f(\lambda, \mathcal{Q})$ as $\lambda \to \pm\infty$, for any fixed nice quad $\mathcal{Q}$. This question was explicitly asked in [AhS14+], where the possible scaling limits of the threshold window of monotone events were studied. We will give the answer here. For simplicity, let us take $\mathcal{Q} = [0,1]^2$, and notice that $f(-\lambda, [0,1]^2) = 1 - f(\lambda, [0,1]^2)$ by duality, hence it is enough to answer the question for $\lambda \to -\infty$.

Theorem 1.1. *As $\lambda \to -\infty$, we have the superexponential decay*

$$f(\lambda, [0,1]^2) = \exp\Big(-\Theta\big(|\lambda|^{4/3}\big)\Big)\,.$$

Besides [AhS14+], another motivation is [HMP12], where the analogous tail behaviour was studied for the scaling limit of dynamical percolation. Namely, if we start with critical percolation, then resample each site at rate $r(n)$, keeping the configuration stationary, then we may look at

$$g(t, \mathcal{Q}) := \lim_{n\to\infty} \mathbf{P}\big[\mathsf{LR}_{n\mathcal{Q}} \text{ does not hold at any moment in } [0,t]\big]\,. \tag{1.4}$$

Again, this limit exists and is conformally covariant by [GPS13, GPS15+]. Then, regarding the tail behaviour, it was proved in [HMP12] using general Markov chain arguments such as spectral computations and a dynamical (space-time) FKG-inequality, that there exists an absolute constant $c > 0$, and for every $K > 0$ some $c_K > 0$, such that

$$\exp(-ct) \le g(t,[0,1]^2) \le c_K t^{-K}\,, \tag{1.5}$$

for all $t > 0$. Furthermore, the present author was speculating in [Pet12], using non-rigorous renormalization ideas (motivated by [LL94, Lan05, SSmG11]) and a very strong universality hypothesis, that the true behaviour could be $\exp(-t^{2/3+o(1)})$. Several people in the community agreed that the lower bound in this speculation looked quite solid even if non-rigorous, while the upper bound was more questionable. And, as typical for these planar percolation scaling limits, that argument seemed to be working equally well for the symmetric (dynamical) and asymmetric (near-critical) versions. However, our present Theorem 1.1 violates not only this bold prediction (in the near-critical case), but even the rigorous exponential lower bound of (1.5), hence this tail probability question turns out to be an instance where the asymmetric versus symmetric dynamical versions of critical percolation show drastically different behaviour. Regarding the true decay in the symmetric dynamical version, our simulations suggest a subexponential decay, but are far from being conclusive, and are even further from giving a prediction for the exponent. See Figure 1.1.

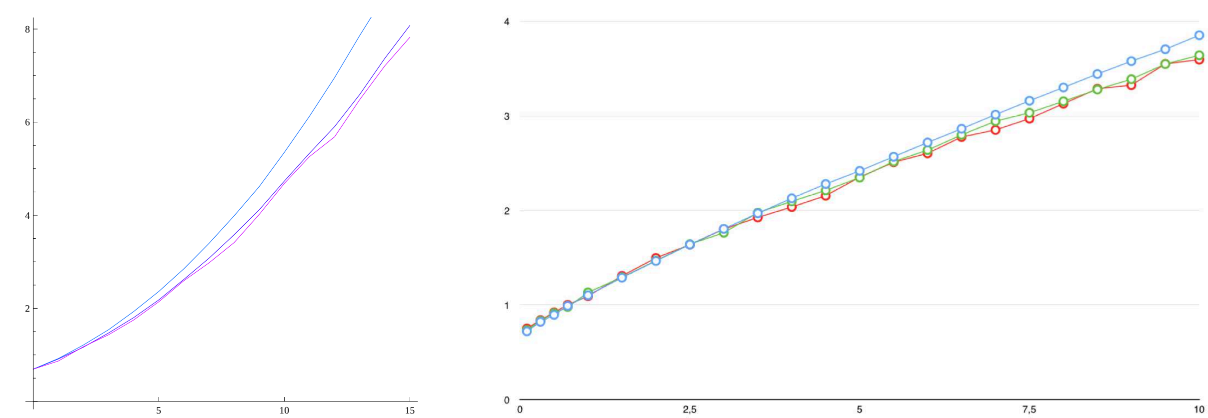

Figure 1.1: On the left, simulation results are shown for $-\log f(\lambda,[0,1]^2)$, with the near-critical percolation parameter varying from $\lambda = 0$ to 1.5, board sizes $n = 10, 100, 500$. On the right, simulation results are shown for $-\log g(t,[0,1]^2)$ in dynamical percolation for scaled time going from $t = 0$ to 10, on board sizes $n = 10, 100, 200$. In both cases, the values are lower and have more fluctuations as n increases, since fewer simulation runs were feasible. The superexponential decay for $f(\lambda,[0,1]^2)$ is apparent, the subexponential decay for $g(t,[0,1]^2)$ is less so.

In the next section, we give the proof of Theorem 1.1, which will be rather simple, given the results of [GPS13, GPS15+] cited above. In fact, only one additional ingredient is needed, which was also one of the key ideas in [DC13], where Hugo Duminil-Copin showed, again building on [GPS13, GPS15+], that the percolation Wulff shape is asymptotically circular, as the density p approaches p_c.

2 Proof

By the scaling covariance (1.2), we need to show that

$$f\big(-1,[0,\lambda^{4/3}]^2\big)=\exp\Big(-\Theta\big(\lambda^{4/3}\big)\Big)\,, \tag{2.1}$$

as $\lambda\to\infty$. For this, the main step is to prove in the scaling limit measure $\mathbf{P}_{\lambda=-1}$ that there exists some $L>0$ such that, for any $\underline{x}\in\mathbb{Z}^2$,

$$\mathbf{P}_{\lambda=-1}\Big[B_L(\underline{0})\longleftrightarrow B_L(L\underline{x})\Big]=\exp\big(-\Theta(\|\underline{x}\|)\big)\,, \tag{2.2}$$

where $\|\underline{x}\|$ is the Euclidean norm, $B_L(\underline{x})$ is the $L\times L$ box centered at $\underline{x}\in\mathbb{Z}^2$, and $\longleftrightarrow$ means being connected to each other.

Indeed, to prove the upper bound in (2.1) from (2.2), cover the left and right sides of the square $[0,\lambda^{4/3}]^2$ by order $\lambda^{4/3}/L$ segments of length L, consider each possible pair with one length-L-segment from the left side and another from the right side, and note that $\mathsf{LR}_{[0,\lambda^{4/3}]^2}$ implies that the event of (2.2) occurs for one of these pairs, with $\|\underline{x}\|\geq\lambda^{4/3}/L$. See the first picture on Figure 2.1. Thus, taking a union bound over these λ-polynomially many possibilities, we get the desired upper bound.

To get the lower bound in (2.1), it is enough to consider a single pair of length-L-segments forming a rectangle, and to note that for any $L>0$ there is some $c=c_L>0$ such that

$$f\big(-1,[0,L]\times[0,Lk]\big)\geq\exp(-c\,k)\,, \tag{2.3}$$

which is clear from the usual RSW and FKG gluing technology along consecutive $L\times L$ boxes; see the second picture on Figure 2.1.

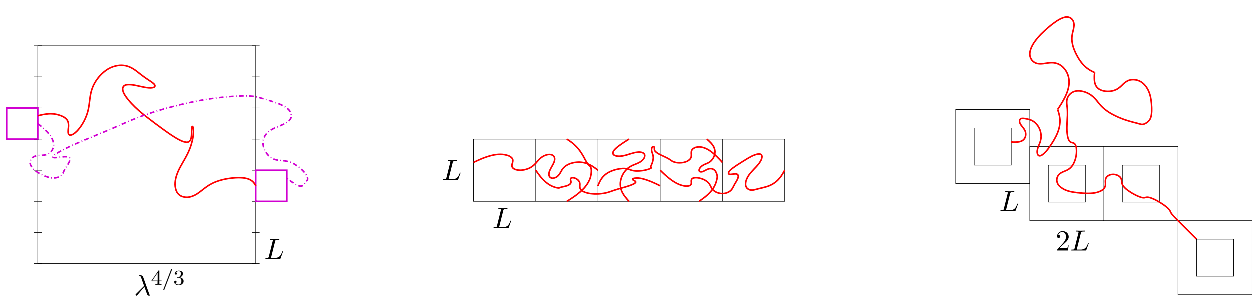

Figure 2.1: Proving (2.1) and (2.2) for the near-critical crossing probability.

The argument proving (2.2) is similar to one in [DC13]. Note that the $\lambda\to-\infty$ case of (1.3), together with the scaling covariance (1.2), implies that for large enough $L>0$, the $\mathbf{P}_{\lambda=-1}$-probability of a radial crossing in the annulus $A_{L,2L}(\underline{0})=B_{2L}(\underline{0})\setminus B_L(\underline{0})$ is at most $1/14$. This will be the L used in (2.2). Consider all the annuli A_i of the form $A_{L,2L}(L\underline{x})$. If $B_L(\underline{0})\longleftrightarrow B_L(L\underline{x})$, then there is a sequence of annuli A_i that are all radially crossed and whose consecutive elements have inner $L\times L$ boxes sharing a side. This path of $L\times L$ boxes may have repetitions, but by taking shortcuts, we can extract a self-avoiding path of disjoint annuli A_i whose consecutive elements have outer $2L \times 2L$ boxes that share some part of their boundaries (of length 0, L or $2L$); see the third picture in Figure 2.1. The number of such annulus paths of length k is at most $16 \cdot 13^{k-1}$, and definitely zero for $k < \|\underline{x}\|/3$. Along any such path, the annulus crossings are independent of each other, and hence the probability of such an annulus path is at most $(1/14)^k$ for some $\delta > 0$. Thus, a union bound over all annulus paths of length at least $\|\underline{x}\|/3$ gives the required exponential decay. An exponential lower bound follows from (2.3). □

3 What about high dimensions?

It is proved in [AhS14+] that for any probability distribution function $F(\cdot)$ there exists a sequence of monotone Boolean functions f_n and some parameters p_n, b_n such that $\mathbf{P}_{p_n+\lambda b_n}\big[f_n = 1\big] \to F(\lambda)$ as $n \to \infty$, for all $\lambda \in \mathbb{R}$. However, all "natural" examples of limit distributions $F(\cdot)$ they have found have exponential or superexponential tails. We are suggesting here that crossing events in near-critical percolation on $\mathbb{Z}^d$, where d is high enough so that already mean field behaviour takes place, could have subexponential tails.

One of the usual definitions of the correlation length $\xi(p)$ is via two-point connectivity:

$$\mathbf{P}_p\big[\underline{x} \longleftrightarrow \underline{y} \not\longleftrightarrow \infty\big] = \exp\Big(- \Theta\big(\|\underline{x} - \underline{y}\|/\xi(p)\big)\Big)\,,$$

which makes sense both for $p < p_c$ and $p > p_c$. For mean field percolation on $\mathbb{Z}^d$, the correlation length exponent $\xi(p) = |p - p_c|^{\nu+o(1)}$ is $\nu = 1/2$, proved in [Har90]. In analogy with the 2-dimensional case, this suggests that the critical window for left-to-right crossing in a box of side-length n is of size $n^{-2+o(1)}$, and the tail behaviour could be $\exp(-|\lambda|^{1/2})$ here. However, since the RSW gluing technology is completely missing, any of this seems far from being provable at this point; to our mind, a small step is the proven equivalence of different possible definitions of the Incipient Infinite Cluster [vdHJ04, HvdHH14].

One might prefer to deal with transitive Boolean functions only. As it was pointed out in [Aiz97], in high-dimensional percolation the finite size boundary effects become important, and the cluster structure in a torus is different from the cluster structure in a box. In particular, for percolation on the torus, already Erdős-Rényi random graph asymptotics take place: the largest critical cluster has size of order $n^{2d/3}$, and the critical window should be $n^{-d/3}$. (Nevertheless, large clusters are still "four-dimensional", similarly to the box case; in particular, their diameter, when pulled back to the universal cover of the torus, is $n^{d/6}$.) Consequently, one expects a tail behaviour $\exp(-|\lambda|^{3/d})$ for the existence of a cluster of pulled-back-diameter $n^{d/6}$, or for the appearance of a cluster of size $n^{2d/3}$. In this torus case, a large part of the conjectured basic near-critical picture has already been proved [B++05a, B++05b, HvdH07, HvdH11, vdHS14], hence it is more hopeful that a subexponential tail can be proved. At first sight, one might also hope to find some event with subexponential tail in the near-critical Erdős-Rényi random graph — however, since the Euclidean structure is not there anymore, the scaling works differently, and there do not seem to be good examples here.

For a recent (in fact, still evolving) survey of high-dimensional percolation and random graphs, see [HvdH15+].

4 Acknowledgments

I am grateful to Daniel Ahlberg for asking me this question, and to Balázs Ráth for an eye-opening discussion and comments on the manuscript. I am also grateful to Christophe Garban and Alan Hammond for many discussions on near-critical and dynamical percolation, and to Remco van der Hofstad for remarks on the high-dimensional case.

This research was partially supported by the Hungarian National Science Fund OTKA grant K109684, and by the MTA Rényi Institute "Lendület" Limits of Structures Research Group.

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