Finite element approximation of fractional order elliptic boundary value problems

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Abstract

A finite element numerical method is investigated for fractional order elliptic boundary value problems with homogeneous Dirichlet type boundary conditions. It is pointed out that an appropriate stiffness matrix can be obtained by taking the prescribed fractional power of the stiffness matrix corresponding to the non-fractional elliptic operators. It is proved that this approach, which is also called the matrix transformation or matrix transfer method, delivers optimal rate of convergence in the L_2-norm.

Keywords: fractional order Laplacian, matrix transformation method, finite element method, error estimation

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1. Introduction

A large number of phenomena in the natural sciences is modeled with the conventional diffusion process. In real observations, one can measure the average $\langle |s(t)| \rangle$ of the displacement of particles/individuals on the time interval $[0, t]$. Indeed, $s(t)$ should be considered in discrete model as a random variable. An important characteristics of the diffusive dynamics is that $\langle |s(t)| \rangle$ is linearly proportional with $\sqrt{t}$, which corresponds to the Brownian motion of the individuals. In this case, for the density function $u(t, x)$ of $s(t)$ we have the diffusion equation $\partial_t u(t, x) = D\Delta u(t, x)$, where D corresponds to the linear proportion. At the same time, real observations in the study of groundwater flows [1], motion of predator animals [2], plasma flows [3], wave front propagation [4] and rotating and turbulent flows [5], [6] indicated a proportionality $\langle |s(t)| \rangle \sim \sqrt{t^{\frac{1}{\alpha}}}$. This corresponds to the so-called Lévy flight of the individuals, and the corresponding equation for the density becomes $\partial_t u(t, x) = D\Delta^\alpha u(t, x)$, see [7], [8], which involves fractional order differentiation. The fractional order boundary value problem to be studied in this paper can be considered as the equilibrium state for such a non-standard difusion process. Note that in the background of this dynamics one can

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guess non-local effects. A meaningful model for these phenomena has recently been developed, based on non-local calculus [9]. Also, in a more detailed model, the order α of the differentiation can be time and space-dependent [10].

The need for the simulation called forth also the classical theory of fractional calculus [11]. The finite difference approach based on the Grünwald–Letnikov formula has been considerably developed in the last decade starting with the paper [12]. Nowadays, higher order approximations [13], [14], ADI methods [15] are available for the corresponding numerical simulations. Moreover, recently, several kinds of linear [16], [17] and non-linear problems are studied containing fractional order Laplacian operators [18], [19].

Less results are available for the finite element method even for the related elliptic problem. The favor of this approach is not only that we can deal with complex domains but also that in the corresponding bilinear forms the homogeneous boundary conditions can be included in a natural way. The choice of an appropriate boundary condition - which corresponds to the real-life phenomenon to simulate and leads to a well-posed problem - is not obvious [20], [21] due to the non-local nature [9] of the fractional Laplacian.

The basic theoretical difficulty in the development of the finite element methods is to generalize integration by parts for the fractional Laplacian to obtain appropriate bilinear forms. In [22] the authors successfully deal with this problem by using left and right-sided fractional derivatives and introduce a theoretical framework including error estimation. This work has been generalized in some aspects, *e.g.*, to discontinuous Galerkin discretization [23] and to moving finite element methods [24] but still works only for the spatially one-dimensional problems. To deal with the multi-dimensional case for fractional order elliptic equations, another approach is presented in [25], based on the theoretical results in [26]. Here the fractional order differential operator is recognized as a Dirichlet-to-Neumann operator on a dimensionally extended domain. As a result, in this domain, the convergence in a weighted fractional Sobolev norm has been verified. A related approach is the finite volume method, which has been analyzed and implemented in one-dimensional [27] and two-dimensional cases [28].

A possible natural way to bypass the problem of integration by parts is to define directly the stiffness matrix for the finite element method. Informally, the basic idea of this approach is the following: if we use the stiffness matrix A_h in the finite element discretization of the Laplacian $-\Delta$ with homogeneous Dirichlet boundary conditions, then we should use A_h^α for the discretization of the corresponding operator $(-\Delta)^\alpha$. This is called the *matrix transformation* or *matrix transfer* method and was originally proposed for finite difference approximations of fractional order diffusion problems in [29] and [30]. Its usefulness has been verified experimentally in [31], [32], [17]. Additionally, in these works efficient methods are developed to compute (or approximate) the corresponding matrix powers. At the same time, a general error analysis concerning this approach has not yet been performed. Instead, the authors in [32] refer to [33], which, as we will show, can only be used for smooth analytic solutions.

The aim of this work is to provide a solid and general theoretical basis for the matrix transformation method applied to the finite element discretization of fractional order elliptic problems. We perform an error analysis for this approach in the L_2-norm. We do not assume any extra smoothness for the corresponding standard diffusion problem, or equivalently, we allow reentrant angles in the domain Ω. According to the standard diffusion we can prove superconvergence.

2. Mathematical preliminaries

Fractional power of the Laplacian. The solution operator corresponding to the boundary value problem

$$\begin{cases} -\Delta u = f \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{cases} \tag{1}$$

on a bounded Lipschitz domain Ω with a given $f \in L_2(\Omega)$ is denoted with $(-\Delta_{\mathcal{D}})^{-1}$ such that $(-\Delta_{\mathcal{D}})^{-1} : L_2(\Omega) \to H_0^1(\Omega)$ and $(-\Delta_{\mathcal{D}})^{-1}(f) = u$. With this notation (1) is equivalent with

$$-\Delta_{\mathcal{D}} u = f. \tag{2}$$

Using the compact embedding $H_0^1(\Omega) \hookrightarrow L_2(\Omega)$, the corresponding solution operator $(-\Delta_{\mathcal{D}})^{-1}$ can be recognized as a compact positive self-adjoint operator on $L_2(\Omega)$. The Hilbert–Schmidt theory of compact operators (see, *e.g.*, [34], Section 6.2) implies the existence of the complete system $\{\omega_j\}_{j\in\mathbb{Z}^+}$ of its eigenfunctions with the eigenvalues $\{\lambda_j\}_{j\in\mathbb{Z}^+} \subset \mathbb{R}^+$.

Accordingly, for $\alpha \in [0,1]$ we can define $[(-\Delta_{\mathcal{D}})^{-1}]^\alpha : L_2(\Omega) \to L_2(\Omega)$ for $u = \sum_{j\in\mathbb{Z}^+} u_j\omega_j$ with

$$[(-\Delta_{\mathcal{D}})^{-1}]^\alpha \sum_{j\in\mathbb{Z}^+} u_j\omega_j = \sum_{j\in\mathbb{Z}^+} \lambda_j^\alpha u_j\omega_j.$$

Using these, corresponding to the operator form (2) we investigate here

$$(-\Delta_{\mathcal{D}})^\alpha u = f, \tag{3}$$

which is indeed a boundary value problem involving the homogeneous Dirichlet data.

Functional analysis tools. In the analysis, H denotes a separable Hilbert space and $\mathcal{K}_+(H)$ the set of positive compact self-adjoint operators on H equipped with the usual operator norm. For $s \geq 0$ the norm of the Sobolev space $H^s(\Omega)$ is denoted with $\|\cdot\|_s$. If it is not confusing the same is applied for the corresponding operator norm.

A cornerstone of our analysis is the following statement in [35].

Theorem 1. *Let f denote an operator monotone non-negative real function and $\|\cdot\|_*$ an unitarily equivalent norm on the space of positive semidefinite matrices of dimension N. Then*

$$\|f(B) - f(A)\|_* \leq f(\|(B - A)\|_*).$$

Indeed, we use the following consequence of Theorem 1.

Corollary 1. *If $A, \mathcal{E} \in \mathcal{K}_+(H)$ and $\alpha \in [0, 1]$ then*

$$\|(A + \mathcal{E})^\alpha - A^\alpha\| \leq \|\mathcal{E}\|^\alpha.$$

This has originally been proved in [36], but can also be obtained using Theorem 1. For this, one first has to rewrite Theorem 1 for operators in $\mathcal{K}_+(H)$ and use that the operator norm is unitarily equivalent. Second, one should verify that the power function $A \to A^\alpha$ defined on $\mathcal{K}_+$ is operator monotone for $\alpha \in [0, 1]$.

Finite element discretization. For the finite element discretization of (3) and (1) we use a shape-regular family $\{\mathcal{T}_h\}$ of subdivisions of the computational domain Ω. The finite element subspace $V_h \subset H_0^1(\Omega)$ is identified with $\mathbb{R}^{N_h}$, where $N_h = \dim V_h$ and we introduce the L_2-orthogonal projection $\Pi_{0,h} : L_2(\Omega) \to V_h$.

The discretized variational problem

$$(\nabla u_h, \nabla v_h) = (f, v_h) = (\Pi_{0,h} f, v_h) \quad \forall\, v_h \in V_h \tag{4}$$

corresponding to (1) becomes a linear system with the stiffness matrix $A_h^{N_h \times N_h}$ such that we can give the numerical solution u_h as

$$u_h = A_h^{-1} \Pi_{0,h} f, \tag{5}$$

where we again identified V_h with $\mathbb{R}^{N_h}$.

We also assume that for the solution u of (1) we have $u \in H^{1+s}(\Omega)$ with some $s \in (0, 1]$. For non-convex Lipschitz polyhedra it is satisfied if the maximal reentrant corner is less than $\frac{\pi}{s}$, see [37]. For the convex Lipschitz domains, we have $u \in H^2(\Omega)$. Depending on the exponent s, a higher-order convergence rate can be established in the $L_2(\Omega)$-norm [38]:

$$\|u - u_h\|_0 \lesssim h^{s+1} \|f\|_0, \tag{6}$$

where the relation $\lesssim$ yields that there exists a mesh-independent constant $C \in \mathbb{R}^+$ such that C times the right hand side is larger than the left hand side. This notation will be used also later.

3. Results

According to (5), the proposed numerical method for the finite element solution of (3) is simply the following:

$$u_{h,\alpha} = A_h^{-\alpha}\Pi_{0,h}f. \tag{7}$$

To visualize this approach, we depict the classical finite element method and the one applied to the solution of (3).

$$\begin{tikzcd} L_2(\Omega) \arrow[r, "(-\Delta_{\mathcal{D}})^{-1}"] \arrow[d, "\Pi_{h,0}"'] & H_0^1(\Omega) \arrow[d, "\Pi_{h,1}"] \\ V_h \arrow[r, "{A_h}^{-1}"'] & V_h \end{tikzcd} \qquad \begin{tikzcd} L_2(\Omega) \arrow[r, "(-\Delta_{\mathcal{D}})^{-\alpha}"] \arrow[d, "\Pi_{h,0}"'] & \mathbb{H}^\alpha(\Omega) \arrow[d] \\ V_h \arrow[r, "{A_h}^{-\alpha}"'] & V_h \end{tikzcd} \tag{8}$$

- $\Pi_{h,1}$ denotes the $H^1(\Omega)$-projection to the subspace V_h. The Céa lemma [39] ensures that

$$\|u - u_h\|_{H_1^0(\Omega)} \lesssim \|u - \Pi_h u\|_{H_1^0(\Omega)}.$$

- For the definition and properties of $\mathbb{H}^\alpha(\Omega)$ - including its relation with classical fractional order Sobolev spaces - we refer to [25].

Remark: Whenever we analyze (7), in the practice the right hand side becomes $\Pi_{0,h}f$ only if an $L_2(\Omega)$-orthogonal basis $\{\mathbf{b}_1\}$ is used in V_h. The practical aspects of this problem will be discussed at the end of this section.

Before presenting our convergence proof we give in concrete terms the consequences of the results in [33] which has been referred to in [32]. Using our notations, the following statement is true:

Theorem 2. *Assume that we have the norm estimate $\|A_h\|_0 \leq C_1 h^{-r}$ for some exponent $r \in \mathbb{R}^+$ and for the matrix A_h we have the estimate*

$$\|(A_h^{-1}\Pi_{0,h} - (-\Delta_{\mathcal{D}})^{-1})f\|_0 \leq C_2 h^r \|f\|_0$$

then we also have

$$\|(A_h^{-\alpha}\Pi_{0,h} - (-\Delta_{\mathcal{D}})^{-\alpha})f\|_0 \leq C_3 h^{r\alpha} \|f\|_0.$$

for some positive constants C_1, C_2 and C_3.

The proof of this theorem is given in [33].

Corollary 2. *If we have $u \in H^2(\Omega)$ for the solution of (1), then we have the convergence rate $h^{2\alpha}$ for the solution of (3) using the matrix transformation method.*

Proof: We first note that A_h is symmetric, and therefore, its norm (associated to the Euclidean vector norm) coincides with its largest eigenvalue:

$$\|A_h\|_0 = \lambda_{\max} := \max\{\lambda : \lambda \text{ is an eigenvalue of } A\}.$$

The order of $\lambda_{\max}$ is h^{-2}, see, *e.g.*, [39], pages 390-391. Therefore, the assumption $\|A_h\|_0 \leq C_1 h^{-r}$ in Theorem 2 can satisfied only for $r \geq 2$. In this way, the second assumption in Theorem 2 becomes

$$\|(A_h^{-1}\Pi_{0,h} - (-\Delta_{\mathcal{D}})^{-1})f\|_0 \leq C_2 h^2 \|f\|_0.$$

This, indeed, assumes superconvergence in the $L_2(\Omega)$-norm [38], which is only possible if $u \in H^2(\Omega)$.
Then, in Theorem 2 both assumptions are satisfied with $r = 2$ and therefore, we have

$$\|(A_h^{-\alpha}\Pi_{0,h} - (-\Delta_{\mathcal{D}})^{-\alpha})f\|_0 \leq C_3 h^{2\alpha} \|f\|_0,$$

which has been stated. $\square$

To sharpen this result and extend to analytic solutions $u \notin H^2(\Omega)$, we need an additional statement by completing the functional analysis tools.

Lemma 1. *If $A \in \mathcal{K}_+(H)$ and $\alpha \in [0,1]$ then for all $x \in H$ the following inequality is valid*

$$\|A^\alpha x\| \leq \|Ax\|^\alpha \|x\|^{1-\alpha}.$$

Proof: We first prove the statement for operators $A_n \in \mathcal{K}_+(H)$ with a finite-dimensional range of dimension D_n. Its normed eigenfunctions and eigenvalues - in a decreasing order - are denoted with $\{\omega_{n,j}\}_{j\in\mathbb{Z}^+}$ and $\{\lambda_{n,j}\}_{j\in\mathbb{Z}^+}$, respectively, where the first D_n terms are non-zero. Then for

$$x = \sum_{j\in\mathbb{Z}^+} x_j \omega_{n,j}$$

we have

$$\|A_n x\|^2 = \left\| \sum_{j\in\mathbb{Z}^+} \lambda_{n,j} x_j \omega_j \right\|^2 = \sum_{j=1}^{D_n} \lambda_{n,j}^2 x_j^2 \quad \text{and} \quad \|A_n^\alpha x\|^2 = \sum_{j=1}^{D_n} \lambda_{n,j}^{2\alpha} x_j^2. \tag{9}$$

Since the cases $\alpha = 0, 1$ are trivial, we may assume that $\alpha \in (0,1)$ such that $\frac{1}{\alpha}, \frac{1}{1-\alpha} \in (1, \infty)$. We apply then Hölder's inequality with $p = \frac{1}{\alpha}$ and $q = \frac{1}{1-\alpha}$ for the vectors

$$((\lambda_{n,1}x_1)^{2\alpha}, \ldots, (\lambda_{n,D_n}x_{D_n})^{2\alpha}) \quad \text{and} \quad (x_1^{2-2\alpha}, \ldots, x_{D_n}^{2-2\alpha}),$$

which gives

$$\|A_n^\alpha x\|^2 = \sum_{j=1}^{D_n} (\lambda_{n,j} x_j)^{2\alpha} x_j^{2-2\alpha} \leq \sum_{j=1}^{D_n} \left[\left((\lambda_{n,j} x_j)^{2\alpha} \right)^{\frac{1}{\alpha}} \right]^\alpha \sum_{j=1}^{D_n} \left[\left(x_j^{2-2\alpha} \right)^{\frac{1}{1-\alpha}} \right]^{1-\alpha}$$

$$= \sum_{j=1}^{D_n} (\lambda_{n,j} x_j)^{2\alpha} \sum_{j=1}^{D_n} x_j^{2(1-\alpha)} = \|A_n x\|^{2\alpha} \|x\|^{2-2\alpha}.$$

Taking the square root, this gives the desired inequality for operators with a finite dimensional range.

We use that for all $A \in \mathcal{K}_+(H)$ and there is a sequence (A_n) with $A_n \to A$ in the operator norm, where each operator A_n is of finite dimensional range. Note also that Theorem 1 implies

$$\|A_n^\alpha - A^\alpha\| \leq \|A_n - A\|^\alpha$$

such that $A_n^\alpha \to A^\alpha$ in the operator norm. Therefore,

$$\|A^\alpha x\| = \lim \|A_n^\alpha x\| \leq \lim \|A_n x\|^\alpha \|x\|^{1-\alpha} = \|Ax\|^\alpha \|x\|^{1-\alpha}$$

as stated in the lemma. $\square$

Proposition 1. *For all $f \in L_2(\Omega)$ we have the following estimate:*

$$\|(-\Delta_{\mathcal{D}})^{-1}(f - \Pi_{h,0} f)\|_0 \lesssim h^{s+1} \|f\|_0.$$

Proof: Observe first that the finite element solution of the problems

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} -\Delta u = \Pi_{h,0} f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

in V_h coincide, since both of them are given with the variational form given in (4).

$$(\nabla u_h, \nabla v_h) = (f, v_h) = (\Pi_{h,0} f, v_h) \quad \forall\, v_h \in V_h.$$

Therefore, using (6), which estimates the error of the approximation u_h the above problems, we get

$$\|(-\Delta_{\mathcal{D}})^{-1}(f - \Pi_{h,0} f)\|_0 \leq \|(-\Delta_{\mathcal{D}})^{-1} f - u_h\|_0 + \|u_h - (-\Delta_{\mathcal{D}})^{-1} \Pi_{h,0} f)\|_0$$

$$\lesssim h^{s+1} \|f\|_0 + h^{s+1} \|\Pi_{h,0} f\|_0 \lesssim h^{s+1} \|f\|_0$$

as stated in the proposition. $\square$

Theorem 3. *If the regularity assumption in* (6) *is satisfied then the finite element method given in* (7) *for the numerical solution of* (3) *is quasi-optimal in the* $L_2(\Omega)$*-norm in the sense that*

$$\|u - u_{h,\alpha}\|_0 = \|(-\Delta_{\mathcal{D}})^{-\alpha} f - A_h^{-\alpha}(\Pi_0 f)\|_0 \lesssim h^{\alpha(s+1)}\|f\|_0. \tag{10}$$

Proof: To estimate the computational error, we first rewrite it as

$$(-\Delta_{\mathcal{D}})^{-\alpha} f - A_h^{-\alpha}(\Pi_{h,0} f) = (-\Delta_{\mathcal{D}})^{-\alpha}(f - \Pi_{h,0} f) + ((-\Delta_{\mathcal{D}})^{-\alpha} - A_h^{-\alpha})\Pi_{h,0} f.$$

Consequently,

$$\begin{aligned} &\|(-\Delta_{\mathcal{D}})^{-\alpha} f - A_h^{-\alpha}(\Pi_{h,0} f)\|_0 \\ &\leq \|(-\Delta_{\mathcal{D}})^{-\alpha}(f - \Pi_{h,0} f)\|_0 + \|((-\Delta_{\mathcal{D}})^{-\alpha} - A_h^{-\alpha})\Pi_{h,0} f\|_0. \end{aligned} \tag{11}$$

For the first term, Lemma 1 and Proposition 1 imply

$$\begin{aligned} &\|(-\Delta_{\mathcal{D}})^{-\alpha}(f - \Pi_{h,0} f)\|_0 \leq \|(-\Delta_{\mathcal{D}})^{-1}(f - \Pi_{h,0} f)\|_0^{\alpha} \|f - \Pi_{h,0} f\|_0^{1-\alpha} \\ &\lesssim h^{\alpha(s+1)}\|f\|_0^{\alpha}\|f\|_0^{1-\alpha} = h^{\alpha(s+1)}\|f\|_0. \end{aligned} \tag{12}$$

To estimate the second term in (11) we first extend the discrete solution operator A_h^{-1} to $L_2(\Omega)$ such that it is zero on the orthocomplement of V_h. This does not affect the second term in (11). Then using (5) and the regularity assumption in (6) we have

$$\|((-\Delta_{\mathcal{D}})^{-1} - A_h^{-1})f\|_0 = \|(-\Delta_{\mathcal{D}})^{-1} f + A_h^{-1}\Pi_{h,0} f\|_0 = \|u - u_h\|_0 \lesssim h^{s+1}\|f\|_0$$

such that

$$\|(-\Delta_{\mathcal{D}})^{-1} - A_h^{-1}\|_0 \lesssim h^{s+1}. \tag{13}$$

Using Theorem 1 with (13) we also have

$$\|(-\Delta_{\mathcal{D}})^{-\alpha} - A_h^{-\alpha}\|_0 \leq \|(-\Delta_{\mathcal{D}})^{-1} - A_h^{-1}\|_0^{\alpha} \lesssim h^{\alpha(s+1)},$$

which together with the boundedness of the projection $\Pi_{h,0}$ delivers the following estimate for the second term in (11):

$$\|((-\Delta_{\mathcal{D}})^{-\alpha} - A_h^{-\alpha})\Pi_{h,0} f\|_0 \lesssim h^{\alpha(s+1)}\|\Pi_{h,0} f\|_0 \leq h^{\alpha(s+1)}\|f\|_0. \tag{14}$$

Finally, inserting (12) and (14) into (11) gives

$$\|(-\Delta_{\mathcal{D}})^{-\alpha} f - A_h^{-\alpha}(\Pi_{h,0} f)\|_0 \lesssim h^{\alpha(s+1)}\|f\|_0,$$

which is the desired error estimate. $\square$

Numerical method. In the practice, we do not use $L_2(\Omega)$-orthogonal bases such that we first rewrite the approximation $A_h^{-1}\Pi_{h,0}f \in \mathbb{R}^{N_h}$ as

$$A_h^{-1}\Pi_{h,0}f = S^T(SA_hS^T)^{-1}S\Pi_{h,0}f,$$

where the matrix $S \in \mathbb{R}^{N_h \times N_h}$ gives the basis transformation between the orthogonal one $\{\mathbf{b}_1\}$ and a conventional finite element basis $\{\mathbf{b}_2\}$. Then the right hand side of the discretized variational problem using the basis $\{\mathbf{b}_2\}$ is $S\Pi_{h,0}f$ and the corresponding stiffness matrix is SA_hS^T. These two terms are given in the practice. Unfortunately, we can not apply here the matrix transformation method since in general,

$$A_h^{-\alpha}\Pi_{h,0}f \neq S^T(SA_hS^T)^{-\alpha}S\Pi_{h,0}f,$$

where the left hand side gives the quasi-optimal approximation analyzed above and the right hand side is a straightforward application of the matrix transformation method.

Instead, we use the following algorithm:

- We compute the right hand side f_2, the mass matrix B_2 and the stiffness matrix A_2 (corresponding to the bilinear form $(\nabla b_{2,j}, \nabla b_{2,k})$) for the standard finite element basis $\{\mathbf{b}_2\}$.
- Using Cholesky factorization, we compute the matrix S^{-1} such that $SB_2S^T = I$, or equivalently, $B_2 = S^{-1}(S^T)^{-1}$.
- We compute the modified right hand side $f = S^{-1}f_2$ and the stiffness matrix $A_h = S^{-1}A_2(S^T)^{-1}$.
- We solve the equation $A_h^\alpha x = f$, which gives the quasi optimal approximation in an orthogonal basis.
- $x_2 = (S^T)^{-1}x$ gives the result in the basis $\{\mathbf{b}_2\}$.

Remark: It is not necessary to give the orthogonal basis in concrete terms. Using the matrix S in the Cholesky factorization, we can simply expand any any vector in this basis.

4. Numerical experiments

The usefulness of the matrix transformation (or matrix transfer) method has already been verified in some numerical experiments, see [31], [32]. We focus here to the convergence rate for problems where the inverse of the Laplace operator delivers non-smooth solution.

Model problem. We choose the L-shaped non-convex computational domain $\Omega = (-1,1) \times (-1,1) \setminus [0,1] \times [0,-1]$. λ_1 denotes the smallest eigenvalue of the Laplacian $-\Delta_{\mathcal{D}}$ defined on Ω with a corresponding eigenfunction Φ_1. Then for any $\alpha \in [0,1]$ we investigate the fractional order elliptic boundary value problem

$$\begin{cases} (-\Delta)^\alpha u(\mathbf{x}) = \lambda_1 \Phi_1(\mathbf{x}) \quad \mathbf{x} \in \Omega \\ u(\mathbf{x}) = 0 \quad \mathbf{x} \in \partial\Omega, \end{cases} \tag{15}$$

which, according to (3), in a correct form reads

$$u = (-\Delta_{\mathcal{D}})^{-\alpha}(\lambda_1 \Phi_1).$$

We first note that using the definition of Φ_1 and the fractional Laplacian we have

$$(-\Delta_{\mathcal{D}})^\alpha(\lambda_1^{1-\alpha}\Phi_1) = \lambda_1^{1-\alpha}(-\Delta_{\mathcal{D}})^\alpha \Phi_1 = \lambda_1^{1-\alpha}(\lambda_1^\alpha)^\alpha \Phi_1 = \Phi_1$$

such that the function $u = (\lambda_1)^{1-\alpha}\Phi_1$ is the unique solution of (15).

We first need to approximate the eigenfunction Φ_1 and the corresponding eigenvalue λ_1 on a fine grid. This requires the solution of the generalized eigenvalue problem $A_2 x = \lambda B_2 x$, which is performed using an inverse iteration procedure. Using the sparsity of B_2 this can be computed on a very fine grid leading to an accurate approximation $\Phi_{1,0}$ of Φ_1 and the corresponding eigenvalue $\lambda_{1,0}$.

Finite element discretization and numerical solution. To specify the computational algorithm given at the end of Section 3, we give the details of the computation.

We have used a family uniform square meshes, determined by the edge-length h.

The local finite element basis $\{b_{21}, b_{22}, b_{23}, b_{24}\}$ on $[0,h] \times [0,h]$ is defined with

$$\left\{1 - \frac{x}{h} - \frac{y}{h} + \frac{xy}{h^2}, \frac{x}{h} - \frac{xy}{h^2}, \frac{xy}{h^2}, \frac{y}{h} - \frac{xy}{h^2}\right\},$$

which are shifted to obtain the local basis in an arbitrary square of the mesh. These determine the global basis functions and define the stiffness matrix A_2 and the mass matrix B_2.

To compute the right side f_2 of the corresponding linear system, we used a third-order quadrature with the Gauss points $(\xi_1,\xi_1), (\xi_1,\xi_2), (\xi_2,\xi_1), (\xi_2,\xi_2)$, on $[0,h] \times [0,h]$, where $\xi_1 = \frac{h}{2} - \frac{h}{2}\frac{\sqrt{3}}{3}$ and $\xi_2 = \frac{h}{2} + \frac{h}{2}\frac{\sqrt{3}}{3}$.

The Cholesky decomposition was computed using the built-in command of MATLAB.

These specify the first three steps of the solution algorithm.

To compute the matrix power A_h^α, we applied two approaches. First, we computed it as

$$A_h^\alpha = V D^\alpha V^{-1},$$

where the matrix V consists of the eigenvectors of A_h and the diagonal matrix D the corresponding eigenvalues. Second, for special powers we made use of an efficient algorithm for taking the square root of (positive) matrices. Finally, the linear system $A_h^\alpha x = f$ was solved. These steps were also executed using the built-in MATLAB functions.

Experimental error analysis. We have experimentally invesigated the convergence of the approximation of u in the sense that the difference $u_h - \lambda_{1,0}^{1-\alpha}\Phi_{1,0}$ has been computed so that $\lambda_{1,0}^{1-\alpha}\Phi_{1,0}$ serves as a reference solution.

The numerical results are presented for $\alpha = 0.7$ and $\alpha = 0.75$ in Table 1 and 2. The algorithms were implemented in MATLAB.

Table 1: Computational error and convergence rate in the $L_2(\Omega)$ norm for the numerical solution of (15) with $\alpha = 0.7$ using the matrix transformation technique. Left side: results using a standard basis, right side: results using an orthogonal basis

	orthogonal basis		standard basis	
h^{-1}	$\|u_h - \lambda_{1,0}^{0.3}\Phi_{1,0}\|_0$	convergence rate	$\|u_h - \lambda_{1,0}^{0.3}\Phi_{1,0}\|_0$	convergence rate
2	0.0020	–	0.0016	–
4	$7.16 \cdot 10^{-4}$	1.48	$4.20 \cdot 10^{-4}$	1.92
8	$3.25 \cdot 10^{-4}$	1.14	$1.63 \cdot 10^{-4}$	1.36
16	$1.55 \cdot 10^{-4}$	1.06	$7.32 \cdot 10^{-5}$	1.16
32	$7.59 \cdot 10^{-5}$	1.03	$3.48 \cdot 10^{-5}$	1.07

One can observe that computation using the standard basis results in similar approximation order (even, a slightly better one) compared to the method with an orthogonal basis. This also means that the analysis could still be improved to suppress the assumption on the orthogonality.

Note that according to (6), in the first case we have $\Phi_1 \in H^{\frac{5}{3}}(\Omega)$ such that we expect a convergence rate $\frac{3.5}{3} \approx 1.17$, while in the second case the expected rate is $\frac{3.75}{3} = 1.25$. These rates were approximated in both cases. At the same time, the computation of the matrix power and the reference solution may decrease the accuracy of the matrix transformation method. Moreover, using the algorithm at the end of Section 3 to obtain an orthogonal basis can contain further computational errors which is a possible explanation of the slightly slower convergence in this case.

Computational efficiency. The bottleneck in the practice of the matrix transformation method is the computation of the matrix power. In case of $\alpha = 0.75$ we have compared two different approaches: first we computed the eigenvalues and eigenvectors of the matrix A_h and took the power of the diagonal consisting of the eigenvalues. Surprisingly, this method in MATLAB is less expensive than applying the built-in subroutine `mpower.m`. In the second approach we have applied the subroutine `sqrtm.m` for the matrix power A_h^3. The corresponding results are shown in Table 2. Unfortunately, none of the above subroutines are able to handle sparse matrices, which is a real barrier for performing further spatial refinement steps.

Table 2: Computational error and convergence rate in the $L_2(\Omega)$ norm for the numerical solution of (15) with $\alpha = 0.75$. The matrix power is first computed using the eigenvalue decomposition, and second taking the square root of the third power. An orthogonal basis has been used.

	eigenvector-decomposition		square root	
h^{-1}	$\|u_h - \lambda_{1,0}^{0.25}\Phi_{1,0}\|_0$	convergence rate	$\|u_h - \lambda_{1,0}^{0.25}\Phi_{1,0}\|_0$	convergence rate
2	0.0017	–	0.0014	–
4	$6.05 \cdot 10^{-4}$	1.49	$3.59 \cdot 10^{-4}$	1.96
8	$2.80 \cdot 10^{-4}$	1.11	$1.33 \cdot 10^{-4}$	1.49
16	$1.37 \cdot 10^{-4}$	1.04	$5.84 \cdot 10^{-5}$	1.19
32	$6.76 \cdot 10^{-5}$	1.02	$2.76 \cdot 10^{-5}$	1.08

To have a further comparison of the two approaches above and analyze the algorithm at the end of Section 3, we have also compared the computational time of the consecutive steps. The results are given in Table 3. These results

Table 3: Computational time for the matrix transformation method applied for the numerical solution of (15) with $\alpha = 0.75$ and using the mesh parameter $h = \frac{1}{32}$. Step 1: Cholesky factorization, Step 2: computation of fractional matrix powers, Step 3: solution of the resulting linear system. Method 1: application of eigenvector decomposition, Method 2: application of the subroutine `sqrtm.m`. In both cases an orthogonal basis has been used.

	Percentage of the computational time			Total computational time
	Step 1	Step 2	Step 3	
Method 1	0.03%	72.5%	27.47%	38.69 s
Method 2	0.04%	97.02%	2.94%	253.65 s

also show that the extra costs for using an orthogonal basis is completely negligible. We note that the second method is not necessarily slower since it delivers a more accurate approximation according to Table 2. It is also clear that a straightforward way to make the above computations more efficient is to apply an improved algorithm for calculating the matrix powers. For related results we refer to [31], [40] and [32].

5. Conclusion

A convergence analysis is provided for the matrix transformation method, which is used to the finite element solution of fractional order elliptic boundary value problems. It is proved that the method exhibits an optimal convergence rate with respect to the L_2-norm. The cornerstone of the analysis is an estimate on the non-integer powers of compact operators. Numerical experiments were carried out on a non-smooth domain, which confirm the expected convergence rate. It was also observed that the application of orthogonal bases - corresponding to our theory - does not increase singificantly the computational costs.

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