Approximate Hermite–Hadamard inequality

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Abstract

The main results of this paper offer sufficient conditions in order that an approximate lower Hermite–Hadamard type inequality imply an approximate Jensen convexity property. The key for the proof of the main result is a Korovkin type theorem.

1. Introduction

Throughout this paper $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{N}$ and $\mathbb{Z}$ denote the sets of real, nonnegative real, natural and integer numbers respectively. Let X be a real linear space and $D \subset X$ be a convex set.

One can easily see that, for any constant $\varepsilon \geq 0$, the ε-convexity of f (cf. [12]), i.e., the validity of

$$f\big(tx + (1-t)y\big) \leq tf(x) + (1-t)f(y) + \varepsilon \qquad (x, y \in D,\, t \in [0,1]),$$

implies the following lower and upper ε-Hermite–Hadamard inequalities

$$f\Big(\frac{x+y}{2}\Big) \leq \int_0^1 f\big(tx + (1-t)y\big)dt + \varepsilon \qquad (x, y \in D), \tag{1}$$

and

$$\int_0^1 f\big(tx + (1-t)y\big)dt \leq \frac{f(x)+f(y)}{2} + \varepsilon \qquad (x, y \in D). \tag{2}$$

The above implication was discovered if $\varepsilon = 0$ by Hadamard [5] in 1893. (See also [21], [14], and [25] for a historical account). For $\varepsilon = 0$, the converse is also known to be true (cf. [24], [25]), i.e., if a function $f : D \to \mathbb{R}$ which is continuous over the segments of D satisfies (1) or (2) with $\varepsilon = 0$, then it is also convex. Concerning the reversed implication for the case $\varepsilon > 0$, Nikodem, Riedel, and Sahoo in [26] have recently shown that the ε-Hermite–Hadamard inequalities (1) and (2) do not imply the $c\varepsilon$-convexity of f (with any $c > 0$). Thus, in order to obtain results that establish implications between the approximate Hermite–Hadamard inequalities and the approximate Jensen inequality, one has to consider these inequalities with nonconstant error terms.

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In order to describe the old and new results about the connection of an approximate Jensen convexity inequality and the approximate Hermite–Hadamard inequality with variable error terms, we need to introduce the following terminology.

For a function $f : D \to \mathbb{R}$, we say that f is *hemi-P*, if, for all $x, y \in D$, the mapping

$$t \mapsto f((1-t)x + ty) \qquad (t \in [0,1]) \tag{3}$$

has property P. For example f is hemiintegrable, if for all $x, y \in D$ the mapping defined by (3) is integrable. Analogously, we say that a function $h : (D - D) \to \mathbb{R}$ is *radially-P*, if for all $u \in D - D$, the mapping

$$t \mapsto h(tu) \qquad (t \in [0,1])$$

has property P on $[0,1]$. Thus in this paper, we are searching connections beetwen the approximate upper Hermite–Hadamard inequality

$$\int\limits_{[0,1]} f(tx + (1-t)y)d\mu(t) \le \lambda f(x) + (1-\lambda)f(y) + \alpha_H(x-y). \tag{4}$$

the approximate Jensen inequality

$$f\Big(\frac{x+y}{2}\Big) \le \frac{f(x)+f(y)}{2} + \alpha_J(x-y) \qquad (x, y \in D). \tag{5}$$

where $f : D \to \mathbb{R}$, $\alpha_H, \alpha_J : D^* \to \mathbb{R}$ are given even functions, $\lambda \in \mathbb{R}$ and μ is a Borel probability measure on $[0,1]$. In [17] the authors established the connections between an upper Hermite–Hadamard type inequality and a Jensen type inequality, which were stated in the following theorem.

Theorem 1. *Let $\alpha_H : (D - D) \to \mathbb{R}$ be even and radially upper semicontinuous, $\rho : [0,1] \to \mathbb{R}_+$ be integrable with $\int_0^1 \rho = 1$ and there exist $c \ge 0$ and $p > 0$ such that*

$$\rho(t) \le c(-\ln|1-2t|)^{p-1} \qquad (t \in]0, \tfrac{1}{2}[\cup]\tfrac{1}{2}, 1[),$$

and $\lambda \in [0,1]$. Then every $f : D \to \mathbb{R}$ lower hemicontinuous function satisfying the approximate upper Hermite–Hadamard inequality

$$\int_0^1 f\big(tx + (1-t)y\big)\rho(t)dt \le \lambda f(x) + (1-\lambda)f(y) + \alpha_H(x-y) \qquad (x, y \in D),$$

fulfills the approximate Jensen inequality (15)*, provided that $\alpha_J : (D - D) \to \mathbb{R}$ is a radially lower semicontinuous solution of the functional inequality*

$$\alpha_J(u) \ge \int_0^1 \alpha_J(|1-2t|u)\rho(t)dt + \alpha_H(u) \qquad (u \in (D-D))$$

and $\alpha_J(0) \ge \alpha_H(0)$.

In [11], the authors established a connection between a lower Hermite–Hadamard type inequality and a Jensen type inequality by proving the following result.

Theorem 2. *Let $\alpha_H : D^* \to \mathbb{R}_+$ be a nonnegative even function. Assume that $f : D \to \mathbb{R}$ is an upper hemicontinuous function satisfying the approximate lower Hermite–Hadamard inequality*

$$f\left(\frac{x+y}{2}\right) \le \int\limits_0^1 f\big(tx + (1-t)y\big)dt + \alpha_H(x-y) \qquad (x, y \in D), \tag{6}$$

Then f satisfies the approximate Jensen inequality (15) *where $\alpha_J : 2(D-D) \to \mathbb{R}_+$ is a nonnegative radially lower semicontinuous, radially increasing solution of the functional inequality*

$$\alpha_J(u) \geq \int_0^1 \alpha_J(2tu)dt + \alpha_H(u) \qquad (u \in D - D). \tag{7}$$

In [18] using a Korokvkin type theorem the authors prove the following theorem.

Theorem 3. *Let μ be a Borel probability measure on $[0,1]$ with a non-singleton support. Let $\varepsilon : D^2 \to \mathbb{R}$ such that $\varepsilon(x,x) = 0$ for all $x \in D$ and $\varepsilon^* : D^2 \times [0,1] \to \mathbb{R}$ be a function such that, for all $x, y \in D$, $\varepsilon^*(x,y,0) = \varepsilon^*(x,y,1) = 0$ and*

$$\varepsilon^*(x,y,s) \geq \begin{cases} \int\limits_{[0,1]} \varepsilon^*\big(x,y,\frac{st}{\mu_1}\big)d\mu(t) + \varepsilon\big(x, \frac{\mu_1 - s}{\mu_1}x + \frac{s}{\mu_1}y\big) & s \in [0,\mu_1], \\ \int\limits_{[0,1]} \varepsilon^*\big(x,y,\frac{t+s-st-\mu_1}{1-\mu_1}\big)d\mu(t) + \varepsilon\big(\frac{1-s}{1-\mu_1}x + \frac{s-\mu_1}{1-\mu_1}y, y\big) & s \in [\mu_1, 1]. \end{cases}$$

Then every $f : D \to \mathbb{R}$ upper hemi-continuous solution of the following lower Hermite–Hadamard type functional inequality

$$f(\mu_1 x + (1-\mu_1)y) \leq \int\limits_{[0,1]} f(tx + (1-t)y)d\mu(t) + \varepsilon(x,y) \qquad (x, y \in D)$$

also fulfills

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon^*(x,y,t) \qquad (x, y \in D,\ t \in [0,1]).$$

In this paper we examine the implication from an upper Hermite–Hadamard type inequality to a Jensen type inequality. In Theorem 5 below, we generalize Theorem 1 replacing the Lebesgue–Stieltjes integral by an integral with respect to an arbitrary Borel probability measure. This allows to view an approximate Jensen inequality as a particular approximate Hermite–Hadamard inequality.

Throughout this paper, the notation δ_t stands for the Dirac measure concentrated at the point $t \in [0,1]$.

First certain Korovkin type theorems ([13], [1]) will be proved, which will play an important role in the proof of the main result Theorem 5. The subsequent results are Korovkin type theorems. In the sequel, denote by $C([0,1])$ and $B([0,1])$ the space of continuous and bounded Borel measurable real valued functions defined on the interval $[0,1]$ equipped with the usual supremum norm. Denote by $p_i : [0,1] \to \mathbb{R}$ the following polinomyals:

$$p_i(u) := u^i, \qquad (i = 0, 1, 2)$$

Theorem 4. *Let $\mathcal{T}_n : B([0,1]) \to B([0,1])$ $(n \in \mathbb{N})$ be a sequence of positive linear operators such that*

$$\lim_{n\to\infty} (\mathcal{T}_n p_0) = p_0. \tag{8}$$

Suppose that there exists a function $g \in C([0,1])$ with $g(\frac{1}{2}) = 0$ and $g > 0$ on $[0,1] \setminus \{\frac{1}{2}\}$ such that $\lim_{n\to\infty}(\mathcal{T}_n g) = 0p_0$. Then, for all bounded lower semicontinuous function $h : [0,1] \to \mathbb{R}$,

$$\liminf_{n\to\infty} \mathcal{T}_n h \geq h\big(\frac{1}{2}\big)p_0 \tag{9}$$

Remark 1. *It easily follows from the above theorem that, if f is continuous, then* (9) *holds with equality and the "liminf" can be replaced by "lim".*

In what follows, we construct a large family of positive linear operators on $B([a,b])$ which satisfies the assumptions of the previous results and will be instrumental in the investigation of approximate convexity. Let μ be a Borel probability measure on $[0,1]$ and define a sequence of linear operators $\mathcal{T}_n^\mu : B([a,b]) \to B([a,b])$ by the following formula:

$$(\mathcal{T}_n^\mu h)(u) := \int\limits_{[0,1]} \dots \int\limits_{[0,1]} h\big(\tfrac{1}{2} + \tfrac{1}{2}(2t_1 - 1) * \dots * (2t_n - 1)\big) d\mu(t_1) \dots d\mu(t_n) p_0(u). \quad (10)$$

Proposition 1. *Assume that μ is a Borel probability measure on $[0,1]$ and define $\mathcal{T}_n^\mu$ by (10). Then, for all $n \in \mathbb{N}$, $\mathcal{T}_n^\mu : B([a,b]) \to B([a,b])$ is a bounded positive linear operator with*

$$\|\mathcal{T}_n^\mu\| \leq 1. \quad (11)$$

In addition, $\mathcal{T}_n^\mu$ has the following property: For all $h \in B([0,1])$,

$$\mathcal{T}_n^\mu p_0 = p_0 \quad (12)$$

Proposition 2. *Assume that μ is a Borel probability measure on $[0,1]$, such that $\mu \notin \{\alpha\delta_0 + (1-\alpha)\delta_1 \mid \alpha \in [0,1]\}$ and for all $n \in \mathbb{N}$ define $\mathcal{T}_n^\mu$ by (10). Then, for all lower semicontinuous $h \in B([0,1])$,*

$$h\big(\tfrac{1}{2}\big) \leq \liminf_{n\to\infty} (\mathcal{T}_n^\mu h)(u) \qquad (u \in [0,1]). \quad (13)$$

The next theorem gives a connection between an approximate upper Hermite–Hadamard type inequality and a Jensen type inequality. In what follows, let X be a real linear space, $D \subseteq X$ be a convex set and denote by D^* the set $D - D$.

Theorem 5. *Assume that μ is a Borel probability measure on $[0,1]$, such that $\mu \notin \{\alpha\delta_0 + (1-\alpha)\delta_1 \mid \alpha \in [0,1]\}$. Let $\lambda \in \mathbb{R}$ and $\alpha_H : D - D \to \mathbb{R}$ be an even error function and assume and $f : D \to \mathbb{R}$ is a lower hemicontinuous and, for all $x, y \in D$, satisfies the following Hermite–Hadamard type inequality:*

$$\int\limits_{[0,1]} f(tx + (1-t)y) d\mu(t) \leq \lambda f(x) + (1-\lambda) f(y) + \alpha_H(x - y). \quad (14)$$

Then f is approximate Jensen-convex in the following sense:

$$f\Big(\frac{x+y}{2}\Big) \leq \frac{f(x) + f(y)}{2} + \alpha_J(x - y) \qquad (x, y \in D), \quad (15)$$

where $\alpha_J : D^* \to \mathbb{R}$ is a radially μ-integrable solution the following functional inequality:

$$\alpha_H(u) + \int\limits_{[0,1]} \alpha_J(|1 - 2t|u) d\mu(t) \leq \alpha_J(u) \qquad (u \in D^*), \quad (16)$$

providing that $\alpha_J(0) \geq \alpha_H(0)$.

The proof of this theorem is based on a sequence of lemmata.

Lemma 1. *Let $\alpha_H : D^* \to \mathbb{R}$ be even, μ is a Borel probability measure on $[0,1]$, such that $\mu \notin \{\alpha\delta_0 + (1-\alpha)\delta_1 \mid \alpha \in [0,1]\}$ and $\lambda \in \mathbb{R}$. Then every $f : D \to \mathbb{R}$ lower hemicontinuous function satisfying the approximate Hermite–Hadamard inequality (14), fulfills*

$$\frac{1}{2} \int\limits_{[0,1]} \Big(f\big(tx + (1-t)y\big) + f\big((1-t)x + ty\big)\Big) d\mu(t) \leq \frac{f(x) + f(y)}{2} + \alpha_H(x - y) \qquad (x, y \in D). \quad (17)$$

In what follows, we examine the Hermite–Hadamard inequality (17). For a sequense (t_n), $n \in \mathbb{N}$ define the following sequense by induction,

$$T_1 := t_1 \quad \text{and} \quad T_{n+1} := (1 - t_{n+1})T_n + t_{n+1}(1 - T_n) \tag{18}$$

Lemma 2. *Let T_n be definied by (18), then*

$$T_n = \tfrac{1}{2} - \tfrac{1}{2}(2t_1 - 1) * \cdots * (2t_n - 1) \tag{19}$$

Lemma 3. *Let $\alpha_H : D^* \to \mathbb{R}$ be a radially upper semicontinuous function. If $f : D \to \mathbb{R}$ is lower hemicontinuous and fulfills the approximate Hermite–Hadamard inequality (17) then, for all $n \in \mathbb{N}$, the function f also satisfies the Hermite–Hadamard inequality*

$$\frac{1}{2}\int\limits_{[0,1]} \cdots \int\limits_{[0,1]} \Big(f\big(T_n x + (1 - T_n)y\big) + f\big((1 - T_n)x + T_n y\big)\Big) d\mu(t_1) \dots d\mu(t_n)$$
$$\leq \frac{f(x) + f(y)}{2} + \alpha_n(x - y) \tag{20}$$

for all $x, y \in D$, whenever $n \in \mathbb{N}$, where the sequences T_n and $\alpha_n : D^* \to \mathbb{R}$ are defined by (18) and

$$\alpha_1 = \alpha_H, \qquad \alpha_{n+1}(u) = \int\limits_{[0,1]} \alpha_n(|1 - 2t|u) d\mu(t) + \alpha_H(u) \qquad (u \in D^*), \tag{21}$$

respectively.

Lemma 4. *Let $\alpha_H : D^* \to \mathbb{R}$ be even, μ is a Borel probability measure on $[0, 1]$, such that $\mu \notin \{\alpha\delta_0 + (1 - \alpha)\delta_1 \mid \alpha \in [0, 1]\}$. If $f : D \to \mathbb{R}$ is a lower hemicontinuous function, then*

$$\liminf_{n\to\infty} \frac{1}{2}\int\limits_{[0,1]} \cdots \int\limits_{[0,1]} \Big(f\big(T_n x + (1 - T_n)y\big) + f\big((1 - T_n)x + T_n y\big)\Big) d\mu(t_1) \dots d\mu(t_n)$$
$$\geq f\Big(\frac{x + y}{2}\Big). \tag{22}$$

Lemma 5. *Let $\alpha_H : D^* \to \mathbb{R}$ be a radially upper semicontinuous function. Then, for all $n \in \mathbb{N}$, the function $\alpha_n : D^* \to \mathbb{R}$ defined by (21) is nondecreasing [nonincreasing], whenever α_H is nonnegative [nonpositive]. Furthermore, if $\alpha_J : D^* \to \mathbb{R}$ is a radially lower semicontinuous solution of the functional inequality (16) then*

$$\limsup_{n\to\infty} \alpha_n(u) \leq \alpha_J(u) - \alpha_J(0) + \alpha_H(0) \qquad (u \in D^*). \tag{23}$$

A simple consequence of Theorem 5 is the following corollary which is a generalized form of Theorem 1 ([17]).

Corollary 1. *Let $\alpha_H : D^* \to \mathbb{R}$ be even and radially upper semicontinuous, $\rho : [0, 1] \to \mathbb{R}_+$ be integrable with $\int_0^1 \rho = 1$ and $\lambda \in \mathbb{R}$. Then every $f : D \to \mathbb{R}$ lower hemicontinuous function satisfying the approximate upper Hermite–Hadamard inequality*

$$\int_0^1 f\big(tx + (1 - t)y\big)\rho(t)dt \leq \lambda f(x) + (1 - \lambda)f(y) + \alpha_H(x - y) \qquad (x, y \in D),$$

fulfills the approximate Jensen inequality (15). Provided that $\alpha_J : D^* \to \mathbb{R}$ is a radially lower semicontinuous solution of the functional inequality

$$\alpha_J(u) \geq \int_0^1 \alpha_J(|1 - 2t|u)\rho(t)dt + \alpha_H(u) \qquad (u \in (D - D))$$

and $\alpha_J(0) \ge \alpha_H(0)$.

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