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On strongly convex functions

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ABSTRACT. The main results of this paper give a connection between strong Jensen convexity and strong convexity type inequalities. We are also looking for the optimal Takagi type function of strong convexity. Finally a connection will be proved between the Jensen error term and an useful error function.

1. INTRODUCTION

Throughout this paper $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{N}$ and $\mathbb{Z}$ denote the sets of real, nonnegative real, natural and integer numbers respectively.

Let X be a normed space and $D \subseteq X$ be a nonempty convex subset of X. Denote by D^* the set $\{\|x-y\|, x, y \in D\}$. It can be seen that D^* is an interval. Let $\alpha : D^* \to \mathbb{R}_+$ be a nonnegative error function. We say that a function $f : D \to \mathbb{R}$ is *strongly α-Jensen convex*, if, for all $x, y \in D$,

$$f\Big(\frac{x+y}{2}\Big) \leq \frac{f(x)+f(y)}{2} - \alpha(\|x-y\|). \tag{1.1}$$

Observe that if $\alpha \equiv 0$, we can get the classical definition of Jensen-convexity. When $\alpha(u) = cu^2$, we can get a kind of notion of strong convexity introduced by Polyak in [16] and examined by Azócar, Giménez, Nikodem and Sánchez (in [1]), Merentes and Nikodem, (in [12]) and Nikodem and Páles [14]. If $\alpha(u) = \varepsilon u^p$, then f is called *strongly (ε, p)-Jensen convex function*. In Section 2, we are looking connection between strong α-convexity and strong convexity type inequalities. Then, we are looking for the optimal error function. In Section 3, we will establish the connections between strong α-convexity and strong α-Jensen convexity, moreover the connections between strong convexity and Hermite–Hadamard type inequalities will be shown. These results will be the generalization of previous results of [1] and [12]. We say that $f : D \to \mathbb{R}$ is *locally upper bounded*, if for all $x, y \in D$, there exists a $K_{x,y}$ such that $f \leq K_{x,y}$ on $[x, y]$, where $[x, y] = \{tx + (1-t)y \mid t \in [0, 1]\}$.

In the sequel, we need the famous Bernstein–Doetsch theorem.

Theorem 1.1. *Let $f : D \to \mathbb{R}$ be locally upper bounded and Jensen-convex, then f is convex and continuous.*

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Recently, some results concerning approximate convexity were proved. It is a natural questions, what happens when we consider a nonpositive error function, namely a strong convexity inequality.

In what follows we recall some Bernstein-Doetsch type theorem for approximately convex functions. A function $f : D \to \mathbb{R}$ is said to be *approximately α-Jensen convex* on D, if, for all $x, y \in D$,

$$f\Big(\frac{x+y}{2}\Big) \le \frac{f(x)+f(y)}{2} + \alpha(\|x-y\|). \tag{1.2}$$

Let introduce the Takagi type functions $\mathcal{T}_\alpha : \mathbb{R} \times D^* \to \mathbb{R}_+$ and $\mathcal{S}_\alpha : \mathbb{R} \times D^* \to \mathbb{R}_+$ by

$$\mathcal{T}_\alpha(t,u) := \sum_{n=0}^{\infty} \frac{1}{2^n}\alpha\big(d_{\mathbb{Z}}(2^n t)u\big) \qquad \big((t,u) \in \mathbb{R} \times D^*\big) \tag{1.3}$$

and

$$\mathcal{S}_\alpha(t,u) := \sum_{n=0}^{\infty} \alpha\Big(\frac{u}{2^n}\Big) d_{\mathbb{Z}}(2^n t) \qquad \big((t,u) \in \mathbb{R} \times D^*\big), \tag{1.4}$$

where $d_{\mathbb{Z}}(t) := 2\,\mathrm{dist}(t, \mathbb{Z})$. Note that the first series converges uniformly if α is bounded, on the other hand, for the uniform convergence of the second series, it is sufficient if $\sum_{n=n_0}^{\infty} \alpha\big(2^{-n}\big) < \infty$ for some $n_0 \in \mathbb{N}$. The importance of the function $\mathcal{T}_\alpha$ introduced above is enlightened by the following result ([9], [18]) which can be considered as a generalization of the celebrated Bernstein-Doetsch theorem [2].

Theorem 1.2. *Let $f : D \to \mathbb{R}$ be locally upper bounded on D and let $\alpha : D^* \to \mathbb{R}_+$. Then f is α-Jensen convex on D if and only if*

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \mathcal{T}_\alpha(t, \|x-y\|) \tag{1.5}$$

for all $x, y \in D$ and $t \in [0, 1]$.

The other Takagi type function $\mathcal{S}_\alpha$ was introduced by Jacek Tabor and Józef Tabor ([18]). Its role and importance in the theory of approximate convexity is shown by the next theorem.

Theorem 1.3. *Let $f : D \to \mathbb{R}$ be upper semicontinuous on D and let $\alpha : D^* \to \mathbb{R}_+$ be nondecreasing such that $\sum_{n=n_0}^{\infty} \alpha(2^{-n}) < \infty$ for some $n_0 \in \mathbb{N}$. Then f is α-Jensen convex on D if and only if*

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \mathcal{S}_\alpha(t, \|x-y\|) \tag{1.6}$$

for all $x, y \in D$ and $t \in [0, 1]$.

Let $\varepsilon, q \ge 0$ be arbitrary constants. When $\alpha(u) := \varepsilon u^q, (u \in D^*)$, the two corollaries below (see [6] and [18]) are immediately consequences of the previous theorems. For $q \ge 0$, define the Takagi type functions S_q and T_q by

$$T_q(t) := \sum_{n=0}^{\infty} \frac{\big(d_{\mathbb{Z}}(2^n t)\big)^q}{2^n}, \qquad S_q(t) := \sum_{n=0}^{\infty} \frac{d_{\mathbb{Z}}(2^n t)}{2^{nq}} \qquad (t \in \mathbb{R}). \tag{1.7}$$

They generalize the classical Takagi function

$$T(t) := \sum_{n=0}^{\infty} \frac{\mathrm{dist}(2^n t, \mathbb{Z})}{2^n} \qquad (t \in \mathbb{R})$$

in two ways, because $T_1 = S_1 = 2T$ holds obviously. This function was introduced by Takagi in [19] and it is a well-known example of a continuous but nowhere differentiable real function. It is less trivial, but it can be proved that $T_2(t) = S_2(t) = 4t(1-t)$ for $t \in [0,1]$.

Corollary 1.1. *Let $f : D \to \mathbb{R}$ be locally upper bounded on D and $\varepsilon, q \geq 0$. Then f is (ε, q)-Jensen convex on D, if and only if*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon T_q(t)\|x-y\|^q \tag{1.8}$$

for all $x, y \in D$ and $t \in [0,1]$.

Corollary 1.2. *Let $f : D \to \mathbb{R}$ be upper semicontinuous on D and $\varepsilon, q \geq 0$. Then f is (ε, q)-Jensen convex on D if and only if*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon S_q(t)\|x-y\|^q \tag{1.9}$$

for all $x, y \in D$ and $t \in [0,1]$.

In [3], Boros proved that if $q = 1$ and $t \in [0,1]$ is fixed, then $S_1(t) = T_1(t) = 2T(t)$ is the smallest possible. In [17] Tabor and Tabor showed that if $1 \leq q \leq 2$ and $t \in [0,1]$ is fixed, then $S_q(t)$ is the smallest possible value so that (1.9) be valid for all (ε, q)-Jensen convex functions f on D. Later in [8] and [11], the authors examined whether the error terms $\mathcal{T}_\alpha(t, \|x-y\|)$, $\mathcal{S}_\alpha(t, \|x-y\|)$ in (1.5) in (1.6) and $T_q(t)$ in (1.8) are the smallest possible ones. In other words, for all fixed $x, y \in D$, the exact upper bound of the convexity-difference of α-Jensen convex functions defined by

$$C_\alpha(x,y,t) := \sup_{f \in \mathcal{JC}_\alpha(D)} \{f(tx + (1-t)y) - tf(x) - (1-t)f(y)\}, \tag{1.10}$$

where

$$\mathcal{JC}_\alpha(D) := \{f : D \to \mathbb{R} \,|\, f \text{ is } \alpha\text{-Jensen convex on } D\}$$

was examined. The statement of Theorem **1.2**, Theorem **1.3**, Corollary 1.1, and Corollary 1.2 can be stated as

$$C_\alpha(x,y,t) \leq \tau(t, \|x-y\|), \tag{1.11}$$

where $\tau : \mathbb{R} \times D^* \to \mathbb{R}_+$ is given by

$$\tau := \mathcal{T}_\alpha, \quad \tau := \mathcal{S}_\alpha, \quad \tau(t,u) := \varepsilon T_q(t) u^q, \quad \text{and} \quad \tau(t,u) := \varepsilon S_q(t) u^q,$$

respectively. To obtain also a lower bound for $C_\alpha(x,y,t)$, (and thus to prove the sharpness of the inequality (1.11)), the following important observation was done by Páles in [15].

Theorem 1.4. *Let $\alpha : D^* \to \mathbb{R}$ be continuous. Let $\tau : \mathbb{R} \times D^* \to \mathbb{R}$ be continuous in its first variable, with $\tau(0,u) = \tau(1,u) = 0$ for all $u \in D^*$, which is Jensen convex in the following sense, for all $u \in D^*$ and $s, t \in [0,1]$,*

$$\tau\Big(\frac{t+s}{2}, u\Big) \leq \frac{\tau(t,u) + \tau(s,u)}{2} + \alpha(|t-s|u).$$

Then,

$$C_\alpha(x,y,t) \geq \tau(t, \|x-y\|).$$

2. From strong α-Jensen convexity to strong convexity

With the help of the following theorem, we can "strengthen" our error function α. (See in [7].)

Theorem 2.5. *Let $f : D \to \mathbb{R}$ be a strongly α-Jensen convex function. Then f is strongly $\widetilde{\alpha}$-Jensen convex on D, where*

$$\widetilde{\alpha}(u) := \sup\left\{n^2\alpha\left(\tfrac{u}{n}\right) \mid n \in \mathbb{N}\right\} \qquad (u \in D^*). \tag{2.12}$$

This means that, we can assume that $\alpha(2u) \geq 4\alpha(u)$ for all $u \in D^*$. In the case of strong (ε, q)-convexity, this means that $q \geq 2$. Similarly as in Theorem **1.2** and Theorem **1.3**, it can be proved two Bernstein–Doetsch type results for locally upper bounded strongly Jensen convex functions. Thus, these theorems give us connections between strong α-Jensen convexity and convexity type inequalities. See also in [4].

Theorem 2.6. *Let $f : D \to \mathbb{R}$ be locally upper bounded on D and let $\alpha : D^* \to \mathbb{R}_+$. Then f is strongly α-Jensen convex on D if and only if*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - \mathcal{T}_\alpha(t, \|x-y\|) \tag{2.13}$$

for all $x, y \in D$ and $t \in [0,1]$.

Theorem 2.7. *Let $f : D \to \mathbb{R}$ be upper semicontinuous on D and let $\alpha : D^* \to \mathbb{R}_+$ be $\sum_{n=n_0}^{\infty} \alpha(2^{-n}) < \infty$ for some $n_0 \in \mathbb{N}$. Then f is α-Jensen convex on D if and only if*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - \mathcal{S}_\alpha(t, \|x-y\|) \tag{2.14}$$

for all $x, y \in D$ and $t \in [0,1]$.

We can also look for the optimal Takagi type function. In other words, for all fixed $x, y \in D$, we want to obtain the exact upper bound of the convexity-difference of strongly α-Jensen convex functions defined by

$$SC_\alpha(x,y,t) := \sup_{f \in \mathcal{SJC}_\alpha(D)} \{f(tx + (1-t)y) - tf(x) - (1-t)f(y)\}, \tag{2.15}$$

where

$\mathcal{SJC}_\alpha(D) := \{f : D \to \mathbb{R} \mid f$ is locally upper bounded and strongly α-Jensen convex on $D\}$.

By Theorem **1.4**, it is enough to prove the Jensen-convexity of $S_\alpha(\cdot, u)$ or $\mathcal{T}_\alpha(\cdot, u)$. We shall prove that the Takagi type function $S_\alpha(\cdot, u)$ will be the optimal choice. To show this suspicion let introduce the following Takagi type function $S_\varphi : [0,1] \to \mathbb{R}$ defined by

$$S_\varphi(x) = \sum_{n=0}^{\infty} \varphi\left(\tfrac{1}{2^n}\right) d_{\mathbb{Z}}(2^n x), \tag{2.16}$$

where $P := \{1, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}, \ldots,\}$ and $\varphi : P \to \mathbb{R}_+$ is a nonnegative function. In fact, the proof of these results are very similar as in [11], so we ignore it. The main results of this section state that, under certain assumptions on the function $\varphi : P \to \mathbb{R}$, $(-S_\varphi)$ is well-defined and strongly Jensen convex in the following sense: For all $x, y \in [0,1]$,

$$-S_\varphi\left(\frac{x+y}{2}\right) \leq \frac{-S_\varphi(x) - S_\varphi(y)}{2} - \varphi \circ d_{\mathbb{Z}}\left(\frac{x-y}{2}\right). \tag{2.17}$$

First we describe the situation when the definition of S_φ is correct.

Lemma 2.3. *Let $\varphi : P \to \mathbb{R}_+$ be a nonnegative function. Then S_φ is well-defined, i.e., the series on the right hand side of* (2.16) *is convergent everywhere if and only if*

$$\sum_{n=0}^{\infty} \varphi\big(\tfrac{1}{2^n}\big) < \infty. \tag{2.18}$$

In the sequel, the class of nonnegative functions $\varphi : P \to \mathbb{R}_+$ satisfying the condition (2.18) will be denoted by $\mathcal{H}$:

$$\mathcal{H} := \Big\{\varphi : P \to \mathbb{R}_+ \mid \sum_{n=0}^{\infty} \varphi\big(\tfrac{1}{2^n}\big) < \infty\Big\}.$$

The next theorem, which was discovered by Jacek Tabor and Józef Tabor, has an important role in the proof of the main theorem of this section.

Theorem 2.8. *For every $x, y \in \mathbb{R}$*

$$S_2\Big(\frac{x+y}{2}\Big) \leq \frac{S_2(x) + S_2(y)}{2} + d_{\mathbb{Z}}^2\Big(\frac{x-y}{2}\Big).$$

The following simple observation is a direct consequence of the previous theorem.

Corollary 2.4. *For every $x, y \in [0, 1]$*

$$-S_2\Big(\frac{x+y}{2}\Big) = \frac{-S_2(x) - S_2(y)}{2} - d_{\mathbb{Z}}^2\Big(\frac{x-y}{2}\Big).$$

In the next result we give a representation of $S_\varphi(x)$ as an infinite linear combination of the values $S_2(2^n x)$, $n = 1, 2, \ldots$.

Theorem 2.9. *Let $\varphi \in \mathcal{H}$. Then, for every $x \in \mathbb{R}$,*

$$S_\varphi(x) = \varphi(1) S_2(x) + \sum_{n=1}^{\infty} \Big(\varphi\big(\tfrac{1}{2^n}\big) - \tfrac{1}{4}\varphi\big(\tfrac{1}{2^{n-1}}\big)\Big) S_2(2^n x). \tag{2.19}$$

An immediate consequence of the previous two theorems is the next result which states the strong convexity of $(-S_\varphi)$.

Theorem 2.10. *Let $\varphi \in \mathcal{H}$ such that, for all $u \in \frac{1}{2}P$, $\varphi(2u) \geq 4\varphi(u)$. Then, for all $x, y \in [0, 1]$,*

$$-S_\varphi\Big(\frac{x+y}{2}\Big) \leq \frac{-S_\varphi(x) - S_\varphi(y)}{2} - \Phi_2\Big(\frac{x-y}{2}\Big), \tag{2.20}$$

where $\Phi_2 : \mathbb{R} \to \mathbb{R}$ is defined by

$$\Phi_2(u) := \sum_{n=0}^{\infty} \varphi\big(\tfrac{1}{2^n}\big)\Big(d_{\mathbb{Z}}^2(2^n u) - \frac{1}{4} d_{\mathbb{Z}}^2(2^{n+1} u)\Big). \tag{2.21}$$

In the next proposition we describe a decomposition property of the function Φ_2.

Proposition 2.5. *For $\varphi \in \mathcal{H}$, for all $u \in]0, \frac{1}{2}]$,*

$$\Phi_2(u) = \Phi_2\Big(\frac{1}{2^{[\log_2 \frac{1}{u}]}} - u\Big) + \varphi\Big(\frac{1}{2^{[\log_2 \frac{1}{u}]-1}}\Big)\Big(1 - 2 \cdot 2^{[\log_2 \frac{1}{u}]} u\Big). \tag{2.22}$$

The next theorem has an important role in the proof of our subsequent main results.

Theorem 2.11. *Let $\varphi : [0, 1] \to \mathbb{R}_+$. Assume that $\varphi(0) = 0$ and the mapping $x \mapsto \frac{\varphi(x)}{x}$ is convex on $]0, 1]$, then, for all $u \in [0, 1]$,*

$$-\Phi_2(u) \leq -\varphi \circ d_{\mathbb{Z}}(u). \tag{2.23}$$

The main result of this section is stated in the following theorem. The proof of is this theorem is based on the previous propositions and lemmas.

Theorem 2.12. *Let $\varphi : [0,1] \to \mathbb{R}_+$. Assume that $\varphi(0) = 0$ and the mapping $x \mapsto \frac{\varphi(x)}{x}$ is convex on $]0,1]$. Then $(-S_\varphi)$ is strongly Jensen convex in the sense of* (2.17).

We shall prove that the error terms $-\mathcal{S}_\alpha(t, \|x-y\|)$ in (1.6) under certain assumptions on the error function α is the smallest possible one. In other words, the next theorem will provide exact upper bound for the convexity-difference of strongly α-Jensen convex functions defined by (2.15).

Theorem 2.13. *Let $\alpha : D^* \to \mathbb{R}$ be an error function such that $\alpha(0) = 0$ and the map $u \mapsto \frac{\alpha(u)}{u}$ is convex on $D^* \setminus \{0\}$. Then, for all $x, y \in D$ and $t \in [0,1]$,*

$$SC_\alpha(x,y,t) = -\mathcal{S}_\alpha(t, \|x-y\|). \tag{2.24}$$

Taking an error function α which is a combination of power functions of exponents from $[2,\infty[$, we obtain the following result.

Theorem 2.14. *Let ν be a nonnegative bounded Borel measure on $[2,\infty[$. Define the error function $\alpha_\nu : D^* \to \mathbb{R}_+$ by*

$$\alpha_\nu(u) := \int\limits_{[2,\infty[} u^q d\nu(q) \qquad (u \in D^*).$$

Then, for all $x, y \in D$ and $t \in [0,1]$,

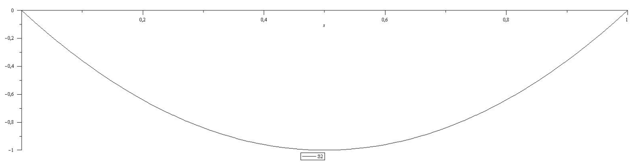
$$SC_{\alpha_\nu}(x,y,t) = -\int\limits_{[2,\infty[} S_q(t)\|x-y\|^q d\nu(q),$$

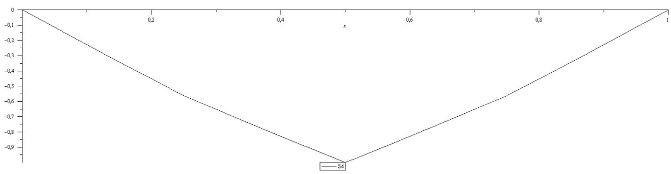
where $S_q : \mathbb{R} \to \mathbb{R}$ is given by (1.7).

Corollary 2.6. *Let $q \in [2,\infty[$ and $\varepsilon \geq 0$. Define the error function $\alpha : D^* \to \mathbb{R}_+$ by $\alpha(u) := \varepsilon u^q$. Then, for all $x, y \in D$ and $t \in [0,1]$,*

$$SC_\alpha(x,y,t) = -\varepsilon S_q(t)\|x-y\|^q.$$

The next figures demonstrate the strong convexity of $(-S_q)$, when $q = 2$ and $q = 4$.





3. On a strong convexity type inequality

Given a nonnegative function $\alpha : D^* \to \mathbb{R}_+$, we say that a map $f : D \to \mathbb{R}$ is *strongly α-convex*, if for all $x, y \in D$ and $t \in [0,1]$,

$$f(tx+(1-t)y) \leq tf(x) + (1-t)f(y) - t\alpha\big((1-t)\|x-y\|\big) - (1-t)\alpha\big(t\|y-x\|\big) \tag{3.25}$$

holds. In [9], a similar approximate convexity type inequality was examined. If (3.25) holds with a $t \in]0,1[$, we say that f is *strongly* (t,α)*-convex* on D. If (3.25) holds with $t = \frac{1}{2}$, we can get the strong $\alpha(\frac{\cdot}{2})$-Jensen convexity of the function f. By the nonnegativity of α, we have that strongly α-Jensen convex and strongly α-convex functions are always convex in the same sense, respectively.

In [7], strong α-Jensen convexity was examined and the following result was established:

Theorem 3.15. *For any function $f : D \to \mathbb{R}$, the following conditions are equivalent:*

(i) f is strongly α-convex.

(ii) f is directionally differentiable at every point of D, and for all $x_0 \in D$, the map $h \mapsto f'(x_0, h)$ is sublinear on X, furthermore for all $x_0, x \in D$,

$$f(x) \geq f(x_0) + f'(x_0, x - x_0) + \alpha(\|x - x_0\|). \tag{3.26}$$

(iii) For all $x_0 \in D$, there exits an element $A \in X'$ such that

$$f(x) \geq f(x_0) + A(x - x_0) + \alpha(\|x - x_0\|) \quad \textit{for all} \quad x \in D. \tag{3.27}$$

Thus, it can be important to look for connections between the strong (λ,α)-convexity and strong α-convexity.

Theorem 3.16. *If $f : D \to \mathbb{R}$ is locally upper bounded and strongly (λ,α)-convex, with $\lambda \in]0,1[$ then f is strongly $\frac{1}{\lambda}\alpha$-convex on D.*

Proof. Since f is strongly (λ,α)-convex and locally upper bounded, we immediately have that f is convex. Let $x, y \in D$ be arbitrary. First using that the directional derivative of $f'(y,\cdot)$ is positive homogeneuos, then appying Theorem **3.15**, with $\alpha = 0$, finally using the strong (λ,α)-convexity of f, we can get that:

$$\begin{aligned} f'(y, x-y) &= \frac{1}{\lambda} f'\big(y, \lambda(x-y)\big) \leq \frac{1}{\lambda}\big(f\big(y + \lambda(x-y)\big) - f(y)\big) \\ &\leq \frac{1}{\lambda}\big(\lambda f(x) + (1-\lambda) f(y) - f(y) - \alpha(\|x-y\|)\big) = f(x) - f(y) - \frac{1}{\lambda}\alpha(\|x-y\|), \end{aligned}$$

which is (by Theorem **3.15**) equivalent to the strong $\frac{1}{\lambda}\alpha$-convexity of f. □

It is not difficult to prove Hermite–Hadamard type inequalities for strongly (λ,α)-convex function. If $f : D \to \mathbb{R}$ is strongly (λ,α)-convex, we can get that f is strongly $\frac{1}{\lambda}\alpha$-convex. Applying Theorem **2.5** from [10], we get the following theorem.

Theorem 3.17. *Let μ be a probability Borel measure on $[0,1]$ and $\alpha : D^* \to \mathbb{R}$ be bounded and Borel measurable function. If $f : D \to \mathbb{R}$ is locally upper bounded and strongly (λ,α)-convex, then, for all $x, y \in D$, f satisfies the following lower Hermite–Hadamard type inequality*

$$\begin{aligned} f(\mu_1 x + (1-\mu_1)y) \leq &\int_{[0,1]} f(tx + (1-t)y)\,d\mu(t) \\ &- \frac{1}{\lambda}\int_{[0,1]} \Big(t\alpha((1-t)\|x-y\|) + (1-t)\alpha(t\|x-y\|)\Big)\,d\mu(t) \end{aligned} \tag{3.28}$$

with $\mu_1 = \int_{[0,1]} t\,d\mu(t)$.

Applying Theorem **3.14** from [10], we can get the following upper Hermite–Hadamard type inequality.

Theorem 3.18. *Let $\mathcal{A}$ be a sigma algebra containing the Borel subsets of $[0,1]$ and μ be a probability measure on the measure space $([0,1],\mathcal{A})$ such that the support of μ is not a singleton. Denote*

$$\mu_1 := \int\limits_{[0,1]} t d\mu(t) \qquad \text{and} \qquad S(\mu) := \mu\big([0,\mu_1]\big) \int\limits_{]\mu_1,1]} t d\mu(t) - \mu\big(]\mu_1,1]\big) \int\limits_{[0,\mu_1]} t d\mu(t).$$

Assume that $f : D \to \mathbb{R}$ is μ-integrable and strongly (λ,α)-convex. Moreover, for all $(x,y) \in D^2$,

$$I(x,y) := \int\limits_{]\mu_1,1]} \int\limits_{[0,\mu_1]} (t'' - \mu_1)\alpha\big((\mu_1 - t')\|x-y\|\big) + (\mu_1 - t')\alpha\big((t''-\mu_1)\|x-y\|\big) d\mu(t') d\mu(t'')$$

exists in $[0,\infty]$. Then, for all $(x,y) \in D^2$, the function f also satisfies the lower Hermite–Hadamard type inequality

$$f((1-\mu_1)x + \mu_1 y) \le \int\limits_{[0,1]} f\big((1-t)x + ty\big) d\mu(t) - \frac{1}{\lambda S(\mu)} I(x,y).$$

In the following theorems, we have established relations between Hermite–Hadamard type inequalities and strong (Jensen) convexity.

Theorem 3.19. *Let μ be a Borel probability measure on $[0,1]$ and assume that $\alpha : D^* \to \mathbb{R}_+$ be a given error function. Denote $\mu_1 := \int_{[0,1]} t d\mu(t)$. If $f : D \to \mathbb{R}$ is continuous and satisfies the following upper Hermite–Hadamard type inequality*

$$\int\limits_{[0,1]} f(tx + (1-t)y) d\mu(t) dt \le \mu_1 f(x) + (1-\mu_1) f(y) - \alpha(\|x-y\|), \quad (x,y \in D), \tag{3.29}$$

then f is strongly $\frac{1}{\mu_1}\alpha$-convex on D.

Proof. Let $x, y \in D$ arbitrary. By (3.29), we have that

$$\int\limits_{[0,1]} (f(y + t(x-y)) - f(y)) d\mu(t) \le \mu_1 (f(y) - f(x)) - \alpha(\|x-y\|), \qquad (x,y \in D).$$

Since α is nonnegative, f satisfies (3.29) with $\alpha = 0$ and hence f is convex, which implies

$$f'(y, t(x-y)) \le f(y + t(x-y)) - f(y), \qquad (t \in [0,1]).$$

Combining the above two inequalities, and using the positive homogeneity of the directional derivative the proof is complete. □

Theorem 3.20. *Let μ be a Borel probability measure on $[0,1]$ and assume that $\alpha : D^* \to \mathbb{R}_+$ be a given error function. Denote $\mu_1 := \int_{[0,1]} t d\mu(t)$. If $f : D \to \mathbb{R}$ is continuous and satisfies the following lower Hermite–Hadamard type inequality,*

$$f(\mu_1 x + +(1-\mu_1)y) \le \int\limits_{[0,1]} f(tx + (1-t)y) d\mu(t) - \alpha(\|x-y\|) \tag{3.30}$$

then f is $\frac{1}{\mu_1}\alpha$-convex on D.

Proof. Using again the convexity of f in (3.30), we can have that f is strongly (μ_1,α)-convex on D. Applying Theorem **3.16**, with $\lambda = \mu_1$, we have $\frac{1}{\mu_1}\alpha$-convexity of f. □

Remark 3.1. A lot of theorems and propositions are true in linear space, but we would like to work in normed space in the whole paper.

REFERENCES

[1] A. Azócar and J. Giménez and K. Nikodem and J.L. Sánchez On sztongly midconvex functions *Opuscula Mathematica* 31:15–26, 2011.

[2] F. Bernstein and G. Doetsch. Zur Theorie der konvexen Funktionen. *Math. Ann.*, 76(4):514–526, 1915.

[3] Z. Boros. An inequality for the Takagi function. *Math. Inequal. Appl.*, 11(4):757–765, 2008.

[4] C. Conzález and K. Nikodem and Zs. Páles and G. Roa Bernstein–Doetsch type theorems for set-valued maps of strongly and approximately convex and concave type *Publ. Math. Debrecen*, 84:229—252, 2014.

[5] J. Hadamard. Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann. *J. Math. Pures Appl.*, 58:171–215, 1893.

[6] A. Házy and Zs. Páles. On approximately t-convex functions. *Publ. Math. Debrecen*, 66:489–501, 2005.

[7] J. Makó and K. Nikodem and Zs. Páles, On strong $(\alpha, \mathbb{F})$-convexity. *Math. Inequal. Appl.*, 15:289–299, 2012.

[8] J. Makó and Zs. Páles. Approximate convexity of Takagi type functions. *J. Math. Anal. Appl.*, 369:545–554, 2010.

[9] J. Makó and Zs. Páles. On φ-convexity. *Publ. Math. Debrecen*, 80:107–126, 2012.

[10] J. Makó and Zs. Páles. Approximate Hermite–Hadamard type inequalities for approximately convex functions. *Math. Inequal. Appl.*, 16:507–526, 2013.

[11] J. Makó and Zs. Páles. On approximately convex Takagi type functions. *Proc. Amer. Math. Soc.*, 141: 2069-2080 , 2013.

[12] N. Merentes and K. Nikodem. Remarks on strongly convex functions. *Aequat. Math.*, 80:193–199, 2010.

[13] C. T. Ng and K. Nikodem. On approximately convex functions. *Proc. Amer. Math. Soc.*, 118(1):103–108, 1993.

[14] K. Nikodem and Zs. Páles Characterizations of inner product spaces by strongly convex functions *Banach J. Math. Anal.*, 5(1):83–87, 2011.

[15] Zs. Páles. The Forty-first International Symposium on Functional Equations, June 8–15, 2003, Noszvaj, Hungary. *Aequat. Math.*, 67:285–320, 2004.

[16] B.T. Polyak Existence theorems and convergence of minimizing sequences in extremum problems with restrictions. *Soviet Math. Dokl.*, 7:72—75, 1966.

[17] Ja. Tabor and Jó. Tabor. Generalized approximate midconvexity. *Control Cybernet.*, 38(3):655–669, 2009.

[18] Ja. Tabor and Jó. Tabor. Takagi functions and approximate midconvexity. *J. Math. Anal. Appl.*, 356(2):729–737, 2009.

[19] T. Takagi. A simple example of the continuous function without derivative. *J. Phys. Math. Soc. Japan*, 1:176–177, 1903.

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