ON APPROXIMATE HERMITE–HADAMARD TYPE INEQUALITIES

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Abstract. The main results of this paper offer sufficient conditions in order that an approximate lower Hermite–Hadamard type inequality implies an approximate Jensen convexity property. The key for the proof of the main result is a Korovkin type theorem.

1. Introduction

Throughout this paper $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{N}$ and $\mathbb{Z}$ denote the sets of real, nonnegative real, natural and integer numbers respectively. Let X be a real linear space and $D \subset X$ be a convex set. Denote by D^* the set $(D - D)$.

One can easily see that, for any constant $\varepsilon \geq 0$, the ε-convexity of f (cf. [7]), i.e., the validity of

$$f\big(tx + (1-t)y\big) \leq tf(x) + (1-t)f(y) + \varepsilon \qquad (x, y \in D,\, t \in [0,1]),$$

implies the following lower and upper ε-Hermite–Hadamard inequalities

$$f\Big(\frac{x+y}{2}\Big) \leq \int_0^1 f\big(tx + (1-t)y\big)dt + \varepsilon \qquad (x, y \in D), \tag{1}$$

and

$$\int_0^1 f\big(tx + (1-t)y\big)dt \leq \frac{f(x)+f(y)}{2} + \varepsilon \qquad (x, y \in D). \tag{2}$$

The above implication was discovered if $\varepsilon = 0$ by Hadamard [5] in 1893. (See also [16], [9] and [19] for a historical account). For $\varepsilon = 0$, the converse is also known to be true (cf. [18], [19]), i.e., if a function $f : D \to \mathbb{R}$ which is continuous over the segments of D satisfies (1) or (2) with $\varepsilon = 0$, then it is also convex. Concerning the reversed implication for the case $\varepsilon > 0$, Nikodem, Riedel, and Sahoo in [20] have recently shown that the ε-Hermite–Hadamard inequalities (1) and (2) do not imply the $c\varepsilon$-convexity of f (with any $c > 0$). Thus, in order to obtain results that establish implications between the approximate Hermite–Hadamard inequalities and the approximate Jensen inequality, one has to consider these inequalities with nonconstant error terms.

Let $\alpha_J : D^* \to \mathbb{R}$ be an even error function. We say that a function $f : D \to \mathbb{R}$ is α_J-Jensen convex, if for all $x, y \in D$,

$$f\Big(\frac{x+y}{2}\Big) \leq \frac{f(x)+f(y)}{2} + \alpha_J(x-y) \qquad (x, y \in D). \tag{3}$$

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In order to describe the old and new results about the connection of an approximate Jensen convexity inequality and the approximate Hermite–Hadamard inequality with variable error terms, we need to introduce the following terminology.

For a function $f : D \to \mathbb{R}$, we say that f is *hemi-P*, if, for all $x, y \in D$, the mapping

$$t \mapsto f((1-t)x + ty) \qquad (t \in [0,1]) \tag{4}$$

has property P. For example f is hemiintegrable, if for all $x, y \in D$ the mapping defined by (4) is integrable. Analogously, we say that a function $h : D^* \to \mathbb{R}$ is *radially-P*, if for all $u \in D^*$, the mapping

$$t \mapsto h(tu) \qquad (t \in [0,1])$$

has property P on $[0,1]$.

In [12] the authors established the connections between an upper Hermite–Hadamard type inequality and a Jensen type inequality, which were stated in the following theorem.

Theorem A. *Let $\alpha_H : D^* \to \mathbb{R}$ be even and radially upper semicontinuous, $\rho : [0,1] \to \mathbb{R}_+$ be integrable with $\int_0^1 \rho = 1$ and there exist $c \geq 0$ and $p > 0$ such that*

$$\rho(t) \leq c(-\ln|1-2t|)^{p-1} \qquad (t \in]0, \tfrac{1}{2}[\cup]\tfrac{1}{2}, 1[),$$

and $\lambda \in [0,1]$. Then every $f : D \to \mathbb{R}$ lower hemicontinuous function satisfying the approximate upper Hermite–Hadamard inequality

$$\int_0^1 f\big(tx + (1-t)y\big)\rho(t)dt \leq \lambda f(x) + (1-\lambda)f(y) + \alpha_H(x-y) \qquad (x, y \in D),$$

fulfills the approximate Jensen inequality (3), provided that $\alpha_J : D^* \to \mathbb{R}$ is a radially lower semicontinuous solution of the functional inequality

$$\alpha_J(u) \geq \int_0^1 \alpha_J(|1-2t|u)\rho(t)dt + \alpha_H(u) \qquad (u \in D^*)$$

and $\alpha_J(0) \geq \alpha_H(0)$.

In [6], A. Házy and Zs. Páles established a connection between a lower Hermite–Hadamard type inequality and a Jensen type inequality by proving the following result.

Theorem B. *Let $\alpha_H : D^* \to \mathbb{R}_+$ be a nonnegative even function. Assume that $f : D \to \mathbb{R}$ is an upper hemicontinuous function satisfying the approximate lower Hermite–Hadamard inequality*

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f\big(tx + (1-t)y\big)dt + \alpha_H(x-y) \qquad (x, y \in D), \tag{5}$$

Then f is α_J-Jensen convex, where $\alpha_J : 2D^* \to \mathbb{R}_+$ is a nonnegative radially lower semicontinuous, radially increasing solution of the functional inequality

$$\alpha_J(u) \geq \int_0^1 \alpha_J(2tu)dt + \alpha_H(u) \qquad (u \in D^*). \tag{6}$$

In [13] using a Korokvkin type theorem the authors prove the following theorem.

Theorem C. *Let μ be a Borel probability measure on $[0,1]$ with a non-singleton support. Let $\varepsilon : D^2 \to \mathbb{R}$ such that $\varepsilon(x,x) = 0$ for all $x \in D$ and $\varepsilon^* : D^2 \times [0,1] \to \mathbb{R}$ be a function such that, for all $x, y \in D$, $\varepsilon^*(x,y,0) = \varepsilon^*(x,y,1) = 0$ and*

$$\varepsilon^*(x,y,s) \geq \begin{cases} \int\limits_{[0,1]} \varepsilon^*\big(x,y,\frac{st}{\mu_1}\big)d\mu(t) + \varepsilon\big(x, \frac{\mu_1 - s}{\mu_1}x + \frac{s}{\mu_1}y\big) & s \in [0,\mu_1], \\ \int\limits_{[0,1]} \varepsilon^*\big(x,y,\frac{t+s-st-\mu_1}{1-\mu_1}\big)d\mu(t) + \varepsilon\big(\frac{1-s}{1-\mu_1}x + \frac{s-\mu_1}{1-\mu_1}y, y\big) & s \in [\mu_1, 1]. \end{cases}$$

Then every $f : D \to \mathbb{R}$ upper hemi-continuous solution of the following lower Hermite–Hadamard type functional inequality

$$f(\mu_1 x + (1-\mu_1)y) \leq \int\limits_{[0,1]} f(tx + (1-t)y)d\mu(t) + \varepsilon(x,y) \qquad (x, y \in D)$$

also fulfills

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon^*(x,y,t) \qquad (x, y \in D,\ t \in [0,1]).$$

In this paper we examine the implication from an upper Hermite–Hadamard type inequality to a Jensen type inequality. Thus in this paper, we are searching connections between the approximate upper Hermite–Hadamard inequality

$$\int\limits_{[0,1]} f(tx + (1-t)y)d\mu(t) \leq \lambda f(x) + (1-\lambda)f(y) + \alpha_H(x-y) \tag{7}$$

and the approximate Jensen inequality (3), where $f : D \to \mathbb{R}$, $\alpha_H, \alpha_J : D^* \to \mathbb{R}$ are given even functions, $\lambda \in \mathbb{R}$ and μ is a Borel probability measure on $[0,1]$. First we prove a Korovkin-type theorem (Theorem 1, Proposition 4), then in Theorem 5 below, we generalize Theorem A replacing the Lebesgue–Stieltjes integral by an integral with respect to an arbitrary Borel probability measure.

Throughout this paper, the notation δ_t stands for the Dirac measure concentrated at the point $t \in [0,1]$.

2. Korovkin type theorems

In the sequel, denote by $C([0,1])$ and $B([0,1])$ the space of continuous and bounded Borel measurable real valued functions defined on the interval $[0,1]$ equipped with the usual supremum norm. Denote by $p_i : [0,1] \to \mathbb{R}$ the following polynomials:

$$p_i(u) := u^i, \qquad (i = 0, 1, 2)$$

Let μ be a Borel probability measure on $[0,1]$ and denote by μ_1 the first momentum of μ, namely $\int_{[0,1]} td\mu(t)$. In this section, we will prove a Korovkin type theorem, which will play an important role in the proof of the main result Theorem 5. To see the historical background of these theorems, we recall the classical Korovkin theorem ([8], [1]), which play an important role in the functional analysis.

Theorem D. *Let $\mathcal{T}_n : B([0,1]) \to B([0,1])$ be a sequence of positive operators such that for all $u \in [a,b]$ and $i \in \{0,1,2\}$*

$$\lim_{n\to\infty} (\mathcal{T}_n p_i)(u) = p_i(u).$$

Then, for all bounded upper semicontinuous function $h : [0,1] \to \mathbb{R}$

$$\limsup_{n\to\infty} \mathcal{T}_n h(u) \le h(u) \qquad (u \in [a,b]). \tag{8}$$

A simple example of the sequence of this operator is the classical Bernstein-operators, namely

$$(B_n h)(u) := \sum_{k=0}^{n} \binom{n}{k} h\Big(\frac{k}{n}\Big) u^k (1-u)^{n-k} \qquad (u \in [0,1]). \tag{9}$$

It is easy to see that

$$B_n p_0 = p_0, \quad B_n p_1 = p_1 \quad \text{and} \quad B_n p_2 = \tfrac{1}{n} p_1 + \tfrac{n-1}{n} p_2.$$

The following theorem is the main application of Theorem D, this is Weierstrass's first approximation theorem, which says that for all continuous function on a compact interval can be approximated by polynomials:

Theorem E. *Using the above notations, for all* $h \in C([0,1])$, $\lim_{n\to\infty} B_n h = h$.

In [13] Zs. Páles and the first author got the following Korovkin type theorem.

Theorem F. *Let* $\mathcal{T}_n : B([0,1]) \to B([0,1])$ *be a sequence of positive operators such that for all* $u \in [a,b]$ *and* $i \in \{0,1\}$

$$\lim_{n\to\infty} (\mathcal{T}_n p_i)(u) = p_i(u).$$

Suppose that there exists a strictly convex $g \in C([0,1])$*, such that*

$$\mathcal{T}_n g \to g(0)p_0 + (g(1) - g(0))p_1$$

Then, for all bounded upper semicontinuous function $h : [0,1] \to \mathbb{R}$

$$\limsup_{n\to\infty} (\mathcal{T}_n h) \le h(0)p_0 + (h(1) - h(0))p_1. \tag{10}$$

In [13], an important example has also been given, namely let

$$(\mathcal{T}_\mu h)(u) := \begin{cases} \int\limits_{[0,1]} h\big(\frac{tu}{\mu_1}\big) d\mu(t) & \text{if} \quad 0 \le u \le \mu_1, \\ \int\limits_{[0,1]} h\big(1 - \frac{(1-t)(1-u)}{1-\mu_1}\big) d\mu(t) & \text{if} \quad \mu_1 \le u \le 1. \end{cases} \quad \text{and} \quad \mathcal{T}_n := \mathcal{T}_\mu^n. \tag{11}$$

It can be proved that the sequence of these operators has the property (10). Using these facts Theorem C can be also proved. (See [13] for more details.)

The first main result of this paper the following Korovkin type result.

Theorem 1. *Let* $\mathcal{T}_n : B([0,1]) \to B([0,1])$ $(n \in \mathbb{N})$ *be a sequence of positive linear operators such that*

$$\lim_{n\to\infty} (\mathcal{T}_n p_0) = p_0. \tag{12}$$

Suppose that there exists a function $g \in C([0,1])$ *with* $g(\frac{1}{2}) = 0$ *and* $g > 0$ *on* $[0,1] \setminus \{\frac{1}{2}\}$ *such that* $\lim_{n\to\infty} (\mathcal{T}_n g) = 0p_0$. *Then, for all bounded lower semicontinuous function* $h : [0,1] \to \mathbb{R}$,

$$\liminf_{n\to\infty} \mathcal{T}_n h \ge h\big(\tfrac{1}{2}\big) p_0. \tag{13}$$

Remark 2. It can be easy to see (under the same conditions of the previous one), if $h:[0,1]\to\mathbb{R}$ is upper semicontinuous, then we get:

$$\limsup_{n\to\infty} \mathcal{T}_n h \le h\big(\tfrac12\big)p_0. \tag{14}$$

It easily follows from the above theorem that, if f is continuous, then (13) holds with equality and the "liminf" can be replaced by "lim".

Proof. Let $h\in B([0,1])$ be a lower semicontinuous function and $\varepsilon>0$ be arbitrary.

$$\phi(u) := \frac{-\varepsilon + (h(u)-h(\frac12))}{g(u)}, \qquad (u\in[0,1]\setminus\{\tfrac12\}).$$

Since h is lower semicontinous continuous in $\frac12$, there exists $0<\delta<\frac12$ such that:

$$-\varepsilon < h(u)-h(\tfrac12) \quad \text{if} \quad |u-\tfrac12|<\delta.$$

Thus

$$\phi>0 \quad \text{on} \quad]\delta-\tfrac12,\tfrac12[\cup]\tfrac12,\delta+\tfrac12[.$$

The function ϕ is lower semicontinuous and the set $[0,\delta-\frac12]\cup[\delta+\frac12,1]$ is compact, therefore there exists a $K\in\mathbb{R}$, such that:

$$\phi>K \quad \text{on} \quad [0,\delta-\tfrac12]\cup[\delta+\tfrac12,1].$$

Thus, there exists L such that $\phi>L$ on $[0,1]\setminus\{\frac12\}$. Therefore,

$$-\varepsilon - Lg(u) < h(u)-h(\tfrac12) \qquad (u\in[0,1]).$$

Using that $\mathcal{T}_n$ is a positive linear functional on $B([0,1])$ we can get that

$$-\varepsilon\mathcal{T}_n p_0 - L\mathcal{T}_n g < \mathcal{T}_n h - h(\tfrac12)\mathcal{T}_n p_0.$$

Taking the limit $n\to\infty$, we get

$$-\varepsilon \le \liminf_{n\to\infty}(\mathcal{T}_n h - h(\tfrac12)\mathcal{T}_n p_0).$$

Since $\varepsilon>0$ is arbitrary, this implies that

$$0 \le \liminf_{n\to\infty}(\mathcal{T}_n h - h(\tfrac12)\mathcal{T}_n p_0).$$

which means that (13) holds. $\square$

In what follows, we construct a large family of positive linear operators on $B([a,b])$ which satisfies the assumptions of the previous results and will be instrumental in the investigation of approximate convexity. Let μ be a Borel probability measure on $[0,1]$ and define a sequence of linear operators $\mathcal{T}_n^\mu: B([a,b])\to B([a,b])$ by the following formula:

$$\mathcal{T}_n^\mu h := \int\limits_{[0,1]}\dots\int\limits_{[0,1]} h\big(\tfrac12+\tfrac12(2t_1-1)\cdot\ldots\cdot(2t_n-1)\big)d\mu(t_1)\dots d\mu(t_n)p_0. \tag{15}$$

Proposition 3. *Assume that μ is a Borel probability measure on $[0,1]$ and define $\mathcal{T}_n^\mu$ by (15). Then, for all $n\in\mathbb{N}$, $\mathcal{T}_n^\mu: B([a,b])\to B([a,b])$ is a bounded positive linear operator with*

$$\|\mathcal{T}_n^\mu\| \le 1. \tag{16}$$

In addition, for all $h\in B([0,1])$

$$\mathcal{T}_n^\mu p_0 = p_0. \tag{17}$$

Proof. If $h \in B([a,b])$ then, for all fixed $u \in [0,1]$ and $n \in \mathbb{N}$,

$$\begin{aligned}|(\mathcal{T}_n^\mu h)(u)| &\le \int_{[0,1]} \dots \int_{[0,1]} \left|h\left(\tfrac{1}{2} + \tfrac{1}{2}(2t_1-1)\cdot \dots \cdot (2t_n-1)\right)\right| d\mu(t_1)\dots d\mu(t_n) p_0(u) \\ &\le \|h\| \int_{[0,1]} \dots \int_{[0,1]} p_0(u) d\mu(t_1)\dots d\mu(t_n) p_0(u) = \|h\| p_0,\end{aligned}$$

which proves the boundedness of $\mathcal{T}_n^\mu$ and (16). The linearity and positivity of $\mathcal{T}_n^\mu$ is obvious. □

Proposition 4. *Assume that μ is a Borel probability measure on $[0,1]$, such that $\mu \notin \{\alpha\delta_0 + (1-\alpha)\delta_1 \mid \alpha \in [0,1]\}$ and for all $n \in \mathbb{N}$ define $\mathcal{T}_n^\mu$ by (15). Then, for all lower semicontinuous $h \in B([0,1])$,*

$$h\left(\tfrac{1}{2}\right) \le \liminf_{n\to\infty} (\mathcal{T}_n^\mu h)(u) \qquad (u \in [0,1]). \tag{18}$$

Proof. By Proposition 3 we can get that $\mathcal{T}_n^\mu : B([0,1]) \to B([0,1])$ is a positive linear operator which satisfies (12). Let $g : [0,1] \to \mathbb{R}$ be defined by

$$g(t) := \left(t - \tfrac{1}{2}\right)^2 \qquad t \in [0,1].$$

Then, $g(\frac{1}{2}) = 0$ and g is positive on $[0,1] \setminus \{\frac{1}{2}\}$. On the other hand,

$$\mathcal{T}_n^\mu h = \frac{1}{4} \int_{[0,1]} \dots \int_{[0,1]} \left((2t_1-1)\cdot \dots \cdot (2t_n-1)\right)^2 d\mu(t_1)\dots d\mu(t_n) p_0 = \frac{1}{4}\left(\int_{[0,1]} (2t-1)^2 d\mu(t)\right)^n p_0.$$

Since $h \le 1$, it is easy to see that

$$\int_{[0,1]} (2t-1)^2 d\mu(t) \le 1.$$

To prove that this inequality is strict, assume that $\int_{[0,1]} (2t-1)^2 d\mu(t) = 1$. Then $(2t-1)^2 = 1$ μ-almost every where $t \in [0,1]$, which means that $t = 0$ or $t = 1$ μ-almost every where $t \in [0,1]$. This implies that $\mu(\{0,1\}) = 1$, which is a contradiction by assumptions. Thus, $\int_{[0,1]} (2t-1)^2 d\mu(t) < 1$ and we can get that $\lim_{n\to\infty} \mathcal{T}_n^\mu h = 0$. Therefore, by Theorem 1 we get (18). □

3. Hermite–Hadamard type inequalities

The next theorem gives a connection between an approximate upper Hermite–Hadamard type inequality and a Jensen type inequality.

Theorem 5. *Assume that μ is a Borel probability measure on $[0,1]$, such that $\mu \notin \{\alpha\delta_0 + (1-\alpha)\delta_1 \mid \alpha \in [0,1]\}$. Let $\lambda \in \mathbb{R}$ and $\alpha_H : D^* \to \mathbb{R}$ be an even error function and assume and $f : D \to \mathbb{R}$ is a hemi-bounded, lower hemicontinuous and, for all $x, y \in D$, satisfies the following Hermite–Hadamard type inequality (7). Then f is approximate Jensen-convex in the sense of (3), where $\alpha_J : D^* \to \mathbb{R}$ is a radially-bounded and radially upper semicontinuous solution the following functional inequality:*

$$\alpha_H(u) + \int_{[0,1]} \alpha_J(|1-2t|u) d\mu(t) \le \alpha_J(u) \qquad (u \in D^*), \tag{19}$$

providing that $\alpha_J(0) \ge \alpha_H(0)$.

The proof of Theorem 5 is similar as the proof of the main theorem of [12] and it is based on a sequence of the following lemmata.

Lemma 6. *Let $\alpha_H : D^* \to \mathbb{R}$ be even, μ is a Borel probability measure on $[0,1]$, such that $\mu \notin \{\alpha\delta_0 + (1-\alpha)\delta_1 \mid \alpha \in [0,1]\}$ and $\lambda \in \mathbb{R}$. Then every $f : D \to \mathbb{R}$ lower hemicontinuous function satisfying the approximate Hermite–Hadamard inequality* (32)*, fulfills*

$$\int\limits_{[0,1]} \frac{f\big(tx+(1-t)y\big)+f\big((1-t)x+ty\big)}{2}d\mu(t) \le \frac{f(x)+f(y)}{2} + \alpha_H(x-y) \qquad (x,y\in D). \tag{20}$$

Proof. Changing the role of x and y in (32), then adding the inequality so obtained to the original inequality (32), by the evenness of α_H, we get that (20) □

In what follows, we examine the Hermite–Hadamard inequality (20). For a sequense (t_n), $n\in\mathbb{N}$ define the following sequense by induction,

$$T_1 := t_1 \quad \text{and} \quad T_{n+1} := (1-t_{n+1})T_n + t_{n+1}(1-T_n). \tag{21}$$

Lemma 7. *Let T_n be definied by* (21)*, then*

$$T_n = \tfrac{1}{2} - \tfrac{1}{2}(2t_1-1)\cdot\dots\cdot(2t_n-1). \tag{22}$$

Proof. We prove by induction on n. If $n=1$, $T_1=t_1$ and $\frac12+\frac12(2t_1-1)=t_1$, so the statement is obvious. Assume that (22) is true for $n=k$ and prove for $n=k+1$. By (21) and the inductive assumption we can get that

$$\begin{aligned}
T_{k+1} &= (1-t_{k+1})T_k + t_{k+1}(1-T_k)\\
&= (1-t_{k+1})\Big(\tfrac12+\tfrac12(2t_1-1)\cdot\ldots\cdot(2t_k-1)\Big) + t_{k+1}\Big(\tfrac12+\tfrac12(1-2t_1)\cdot\ldots\cdot(1-2t_k)\Big)\\
&= \tfrac12+\tfrac12(1-2t_1)\cdot\ldots\cdot(1-2t_k)\cdot(1-2t_{k+1}),
\end{aligned}$$

which proves this lemma. □

Lemma 8. *Let $\alpha_H : D^* \to \mathbb{R}$ be a radially upper semicontinuous function. If $f : D \to \mathbb{R}$ fulfills the approximate Hermite–Hadamard inequality* (20) *then, for all $n\in\mathbb{N}$, the function f also satisfies the Hermite–Hadamard inequality*

$$\begin{aligned}
&\frac12 \int\limits_{[0,1]}\dots\int\limits_{[0,1]} \Big(f\big(T_nx+(1-T_n)y\big)+f\big((1-T_n)x+T_ny\big)\Big)d\mu(t_1)\dots d\mu(t_n)\\
&\qquad\qquad \le \frac{f(x)+f(y)}{2}+\alpha_n(x-y)
\end{aligned} \tag{23}$$

for all $x,y\in D$, whenever $n\in\mathbb{N}$, where the sequences T_n and $\alpha_n : D^*\to\mathbb{R}$ are defined by (21) *and*

$$\alpha_1 = \alpha_H, \qquad \alpha_{n+1}(u) = \int\limits_{[0,1]} \alpha_n(|1-2t|u)d\mu(t) + \alpha_H(u) \qquad (u\in D^*), \tag{24}$$

respectively.

Proof. We prove by induction on n. If $n=1$, by the definition of α_1, (23) holds. Let $x,y\in D$ and assume that (23) holds for some $n\in\mathbb{N}$. Write x by $(1-t_{n+1})x+t_{n+1}y$ and y by $t_{n+1}x+(1-t_{n+1})y$

in (23). Using the definition of T_{n+1} and α_{n+1} ((21),(24)) and using also that α_H is even, then we obtain:

$$\frac{1}{2}\int\limits_{[0,1]}\dots\int\limits_{[0,1]}\Big(f\big(T_{n+1}x+(1-T_{n+1})y\big)+f\big((1-T_{n+1})x+T_{n+1}y\big)\Big)d\mu(t_1)\dots d\mu(t_n)$$
$$\leq \frac{f((1-t_{n+1})x+t_{n+1}y)+f(t_{n+1}x+(1-t_{n+1})y)}{2}+\alpha_n(|1-2t_{n+1}|(x-y)).$$

Integrating with respect to t_{n+1}, and applying the inductive assumption, (20) and finally (24), we obtain that

$$\begin{aligned}
&\frac{1}{2}\int\limits_{[0,1]}\dots\int\limits_{[0,1]}\int\limits_{[0,1]}\Big(f\big(T_{n+1}x+(1-T_{n+1})y\big)+f\big((1-T_{n+1})x+T_{n+1}y\big)d\mu(t_1)\dots d\mu(t_n)d\mu(t_{n+1})\\
&\leq \frac{1}{2}\int\limits_{[0,1]}\Big(f((1-t_{n+1})x+t_{n+1}y)+f(t_{n+1}x+(1-t_{n+1})y)\Big)d\mu(t_{n+1})\\
&\qquad\qquad+\int\limits_{[0,1]}\alpha_n(|1-2t_{n+1}|(x-y))d\mu(t_{n+1})\\
&\leq \frac{f(x)+f(y)}{2}+\alpha_H(x-y)+\int\limits_{[0,1]}\alpha_n(|1-2t_{n+1}|(x-y))d\mu(t_{n+1})\\
&\leq \frac{f(x)+f(y)}{2}+\alpha_{n+1}(x-y),
\end{aligned}$$

which proves the statement. □

Lemma 9. *Let $\alpha_H : D^* \to \mathbb{R}$ be even, μ is a Borel probability measure on $[0,1]$, such that $\mu \notin \{\alpha\delta_0 + (1-\alpha)\delta_1 \mid \alpha \in [0,1]\}$. If $f : D \to \mathbb{R}$ is hemibounded and lower hemicontinuous function, then*

$$\liminf_{n\to\infty}\frac{1}{2}\int\limits_{[0,1]}\dots\int\limits_{[0,1]}\Big(f\big(T_nx+(1-T_n)y\big)+f\big((1-T_n)x+T_ny\big)\Big)d\mu(t_1)\dots d\mu(t_n)$$
$$\geq f\Big(\frac{x+y}{2}\Big). \tag{25}$$

Proof. Let $x, y \in D$ be fixed and define $h : [0,1] \to \mathbb{R}$ by

$$h(t) := \frac{1}{2}\big(f\big((1-t)x+ty\big)+f\big(tx+(1-t)y\big)\big).$$

Since f is hemibounded and lower hemicontinuous, h is lower semicontinuous and $h \in B([0,1])$. Using also Lemma 7 we have that the operator $\mathcal{T}_n^\mu$ defined by (15) can be expressed as,

$$\mathcal{T}_n^\mu h = \int\limits_{[0,1]}\dots\int\limits_{[0,1]} h(T_n)d\mu(t_1)\dots d\mu(t_n).$$

By Proposition 4, (18) holds, which means that (25) that also holds. □

Lemma 10. *Let $\alpha_H : D^* \to \mathbb{R}$ be a radially upper semicontinuous function. Then, for all $n \in \mathbb{N}$, the function $\alpha_n : D^* \to \mathbb{R}$ defined by (24) is nondecreasing [nonincreasing], whenever α_H*

is nonnegative [nonpositive]. Furthermore, if $\alpha_J : D^* \to \mathbb{R}$ *is a radially bounded and radially upper semicontinuous solution of the functional inequality* (19) *then*

$$\limsup_{n\to\infty} \alpha_n(u) \le \alpha_J(u) - \alpha_J(0) + \alpha_H(0) \qquad (u \in D^*). \tag{26}$$

Proof. Assume first that α_H is nonnegative. We will prove by induction on $n \in \mathbb{N}$, that the sequence (α_n) is nondecreasing, i.e.,

$$\alpha_{n+1} \ge \alpha_n \qquad (n \in \mathbb{N}). \tag{27}$$

For $n = 1$, by the nonnegativity of $\alpha_1 = \alpha_H$, we have that, for all $u \in D^*$,

$$\alpha_2(u) = \int\limits_{[0,1]} \alpha_1(|1-2t|u)d\mu(t) + \alpha_H(u) \ge \alpha_1(u).$$

Assume that (27) holds for some $n \in \mathbb{N}$ and consider the case $n+1$. Using the definition of α_{n+1}, the inductive assumption and the nonnegativity of α_n, we get, for all $u \in D^*$, that

$$\alpha_{n+2}(u) = \int\limits_{[0,1]} \alpha_{n+1}(|1-2t|u)d\mu(t) + \alpha_H(u) \ge \int\limits_{[0,1]} \alpha_n(|1-2t|u)d\mu(t) + \alpha_H(u) = \alpha_{n+1}(u).$$

Analogously, if α_H is nonpositive, we can obtain that the sequence (α_n) is nonincreasing.

To prove (26), let $\alpha_J : D^* \to \mathbb{R}$ be a radially upper semicontinuous solution of (19). Then, for the sequence of functions $g_n := \alpha_n - \alpha_J$, we obtain

$$g_{n+1}(u) \le \int\limits_{[0,1]} g_n(|1-2t|u)d\mu(t) \qquad (u \in D^*,\ n \in \mathbb{N}),$$

Iterating this inequality, similarly as in Lemma 9 and Lemma 8, it can be proved that

$$g_{n+1}(u) \le \int\limits_{[0,1]} \dots \int\limits_{[0,1]} g_1(|1-2T_n|u)d\mu(t_1)\cdots d\mu(t_n) \qquad (u \in D^*,\ n \in \mathbb{N}), \tag{28}$$

where T_n is defined by (21). Taking the limsup as $n \to \infty$ in (28), by Proposition 4, we get that, for all $u \in D^*$,

$$\limsup_{n\to\infty} g_{n+1}(u) \le \limsup_{n\to\infty} \int\limits_{[0,1]} \dots \int\limits_{[0,1]} g_1(|1-2T_n|u)d\mu(t_1)\cdots d\mu(t_n) \le g_1(0) = \alpha_H(0) - \alpha_J(0),$$

which immediately yields (26). □

Proof of Theorem 5. Assume that the conditions of Theorem 5 hold. Using Lemma 8, we obtain (23). Then taking the $\liminf$ in (23) and using the fact that $\liminf \alpha_n \le \limsup \alpha_n$, then applying Lemma 9 and Lemma 10 we obtain that the function f is α_J-Jensen convex, i.e. (3) holds. □

A simple consequence of Theorem 5 is the following corollary which is a generalized form of Theorem A ([12]).

Corollary 11. *Let* $\alpha_H : D^* \to \mathbb{R}$ *be even and radially bounded and radially upper semicontinuous,* $\rho : [0,1] \to \mathbb{R}_+$ *be integrable with* $\int_0^1 \rho = 1$ *and* $\lambda \in \mathbb{R}$. *Then every* $f : D \to \mathbb{R}$ *hemi-bounded* *and lower hemicontinuous function satisfying the approximate upper Hermite–Hadamard inequality*

$$\int_0^1 f\big(tx+(1-t)y\big)\rho(t)dt \le \lambda f(x)+(1-\lambda)f(y)+\alpha_H(x-y) \qquad (x,y\in D),$$

fulfills the approximate Jensen inequality (3) *provided that* $\alpha_J : D^* \to \mathbb{R}$ *is a radially lower semicontinuous solution of the functional inequality*

$$\alpha_J(u) \ge \int_0^1 \alpha_J(|1-2t|u)\rho(t)dt + \alpha_H(u) \qquad (u\in D^*)$$

and $\alpha_J(0)\ge\alpha_H(0)$.

In what follows let X be a normed space, ν be a signed Borel measure on $]0,\infty[$ and define the error function by the following way:

$$\alpha_H(u) := \int\limits_{]0,\infty[} \|u\|^q d\nu(q) \tag{29}$$

These error functions determine a large class of the error functions.

Corollary 12. *Assume that* ν *is a signed Borel measure on* $]0,\infty[$, *such that*

$$\int\limits_{]0,\infty[} \|u\|^q d\nu(q) < \infty \qquad (u\in D^*) \tag{30}$$

and

$$\int\limits_{]0,\infty[} \Bigg(1-\int\limits_{[0,1]} |1-2t|^q d\mu(t)\Bigg)^{-1} d|\nu|(q) < \infty. \tag{31}$$

Let μ *be a Borel probability measure on* $[0,1]$, *such that* $\mu \notin \{\alpha\delta_0+(1-\alpha)\delta_1 \mid \alpha\in[0,1]\}$. *Let* $\lambda\in\mathbb{R}$ *and assume and* $f : D\to\mathbb{R}$ *is hemi-bounded and lower hemicontinuous and, for all* $x,y\in D$, *satisfies the following Hermite–Hadamard type inequality:*

$$\int\limits_{[0,1]} f(tx+(1-t)y)d\mu(t) \le \lambda f(x)+(1-\lambda)f(y) + \int\limits_{]0,\infty[} \|x-y\|^q d\nu(q). \tag{32}$$

Then f *is approximate Jensen-convex in the following sense:*

$$f\Big(\frac{x+y}{2}\Big) \le \frac{f(x)+f(y)}{2} + \int\limits_{]0,\infty[} \Bigg(1-\int\limits_{[0,1]} |1-2t|^q d\mu(t)\Bigg)^{-1} \|x-y\|^q d\nu(q) \qquad (x,y\in D).$$

Proof. By Theorem 5, it suffices to show that the function

$$\alpha_J(u) := \int\limits_{]0,\infty[} \Bigg(\int_0^1 \big(1-|1-2t|^q\big)d\mu(t)\Bigg)^{-1} \|u\|^q d\nu(q) \quad (u\in D^*)$$

is well-defined and satisfies (19) with equality where $\alpha_H : D^*\to\mathbb{R}$ is defined by

$$\alpha_H(u) := \int\limits_{]0,\infty[} \|u\|^q d\nu(q) \qquad (u\in D^*).$$

To see that, for all $u \in D^*$, $\alpha(u)$ is finite, we distinguish two cases. If $\|u\| \le 1$, then $\|u\|^q \le 1$ for all $q > 0$, and hence, by assumption (31),

$$|\alpha_J(u)| \le \int\limits_{]0,\infty[} \Big(1 - \int\limits_{[0,1]} |1-2t|^q d\mu(t)\Big)^{-1} d|\nu|(q) < \infty.$$

Now let $\|u\| > 1$. Then, the functions $q \mapsto \|u\|^q$ and $q \mapsto 1 - \int\limits_{[0,1]} |1-2t|^q d\mu(t)$ are increasing functions, hence

$$|\alpha_J(u)| \le \|u\| \int\limits_{]0,\infty[} \Big(1 - \int\limits_{[0,1]} |1-2t|^q d\mu(t)\Big)^{-1} d|\nu|(q) + \Big(\int\limits_{[0,1]} \big(1 - |1-2t|\big) d\mu(t)\Big)^{-1} \int\limits_{]1,\infty[} \|u\|^q d|\nu|(q),$$

which is again finite by conditions (30) and (31). To prove that α_J satisfies (19), using that μ is a probability measure, we compute

$$\begin{aligned}
&\int\limits_{[0,1]} \alpha_H(|1-2s|u) d\mu(s) + \alpha_J(u) \\
&= \int\limits_{[0,1]} \int\limits_{]0,\infty[} \Big(\int\limits_{[0,1]} \big(1 - |1-2t|^q\big) d\mu(t)\Big)^{-1} \||1-2s|u\|^q d\nu(q) d\mu(s) + \int\limits_{]0,\infty[} \|u\|^q d\nu(q) \\
&= \int\limits_{]0,\infty[} \left(\frac{\int\limits_{[0,1]} |1-2s|^q d\mu(s)}{\int\limits_{[0,1]} \big(1 - |1-2t|^q\big) d\mu(t)} + 1\right) \|u\|^q d\nu(q) = \alpha_J(u),
\end{aligned}$$

which proves that (19) holds with equality. □

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