ON \((\alpha, \beta, a, b)\)-CONVEX FUNCTIONS

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Abstract. In this paper we investigate the \((\alpha, \beta, a, b)\)-convex functions which is a common generalization of the usual convexity, the \(s\)-convexity in first and second sense, the \(h\)-convexity, the Godunova-Levin functions and the \(P\)-functions. This notion of convexity was introduced by Maksa and Páles in [16] in the following way: an \((\alpha, \beta, a, b)\)-convex function is defined as a function \(f : D \to \mathbb{R}\) (where \(D\) is an open, \((\alpha, \beta)\)-convex, nonempty subset of a real or complex topological vector space) which satisfies the inequality

\[
f(\alpha(t) x + \beta(t) y) \leq a(t) f(x) + b(t) f(y) \quad (x, y \in D; t \in [0, 1]).
\]

The main goal of the paper is to prove some regularity and Bernstein-Doetsch type results for \((\alpha, \beta, a, b)\)-convex functions.

1. Introduction

Maksa and Páles in [16] dealt with the following problem:

Let \(X\) be a real or complex topological vector space, \(D \subset X\) be a nonempty open set, \(T\) be a nonempty set, and \(\alpha, \beta, a, b : T \to \mathbb{R}\) be given functions. The problem is to find all the solutions \(f : D \to \mathbb{R}\) of the functional equation

\[
f(\alpha(t) x + \beta(t) y) = a(t) f(x) + b(t) f(y) \quad (x, y \in D; t \in T) \tag{1}
\]

provided that \(D\) is \((\alpha; \beta)\)-convex, that is, \(\alpha(t) x + \beta(t) y \in D\) whenever \(x, y \in D\) and \(t \in T\). To avoid the trivialities and the unimportant cases, we suppose that there exists an element \(t_0 \in T\) such that

\[
a(t_0) \beta(t_0) a(t_0) b(t_0) \neq 0.
\]

The solutions of (1) as \((\alpha; \beta; a; b)\)-affine functions and the solutions \(f\) of the corresponding inequality

\[
f(\alpha(t) x + \beta(t) y) \leq a(t) f(x) + b(t) f(y) \quad (x, y \in D; t \in T) \tag{2}
\]

will be called \((\alpha; \beta; a; b)\)-convex functions.

In our paper we investigate the \((\alpha; \beta; a; b)\)-convex functions. This notion of convexity is a common generalization of the usual convexity, the \(s\)-convexity in first and second sense, the \(h\)-convexity, the Godunova-Levin functions and the \(P\)-functions.

In the special cases when \(T = \{1/2\}\), \(T = \{t_0\}\) or \(T = \mathbb{Q} \cap [0, 1]\), the corresponding convex functions are said to be \(J\)ensen-(\(\alpha; \beta; a; b)\)-convex, \(t_0 - (\alpha; \beta; a; b)\)-convex and \(rationally-(\alpha; \beta; a; b)\)-convex.

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Let $h : [0, 1] \rightarrow \mathbb{R}$ be a given function. In the case, when $\alpha(t) = t, \beta(t) = 1 - t, a(t) = h(t), b(t) = h(1 - t)$ we get the so called $h$-convex functions, which was introduced by Varošanec [29] and was generalized by Házy [11]. We say that $f : D \rightarrow \mathbb{R}$ is an $h$-convex function if, for all $x, y \in D$ and $t \in [0, 1]$, we have
\[
 f(tx + (1 - t)y) \leq h(t)f(x) + h(1 - t)f(y). \tag{3}
\]

The Godunova-Levin functions was investigated by Godunova-Levin [7]. We say that $f : I \rightarrow \mathbb{R}$ (where $I$ is a real interval) is a Godunova-Levin function, if $f$ is nonnegative and for all $x, y \in I$ and $t \in (0, 1)$ we have
\[
 f(tx + (1 - t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1 - t}.
\]

Some properties of this type of functions are given in Dragomir, Pečarić and Persson [6] Mitro- nović and Pečarić [27], Mitrošnović, Pečarić and Fink [28]. Among others, it is proved that nonnegative monotone and nonnegative convex functions belong to this class of functions. The Godunova-Levin functions are $(\alpha; \beta; a; b)$-convex functions, with $\alpha(t) = t, \beta(t) = 1 - t, a(t) = 1/t, b(t) = 1/(1 - t)$.

The concept of $s$-convexity in the first sense was introduced by Orlicz [21]. A real valued function $f : D \rightarrow \mathbb{R}$ is called Orlicz $s$-convex or $s$-convex in the first sense, if
\[
 f(t^s x + (1 - t)^s y) \leq tf(x) + (1 - t)f(y)
\]
for every $x, y \in D, t \in [0, 1]$, where $s \in [1, \infty]$ is fixed number. The Orlicz $s$-convex functions are $(\alpha; \beta; a; b)$-convex functions, with $\alpha(t) = t^s, \beta(t) = (1 - t)^s, a(t) = t, b(t) = 1 - t$.

The concept of $s$-convexity in the second sense was introduced by Breckner [4]. A real valued function $f : D \rightarrow \mathbb{R}$ is called Breckner $s$-convex or $s$-convex in the second sense, if
\[
 f(tx + (1 - t)y) \leq t^sf(x) + (1 - t)^sf(y)
\]
for every $x, y \in D$ and $t \in [0, 1]$, where $s \in [0, 1]$ is a fixed number. The Breckner $s$-convex functions are $(\alpha; \beta; a; b)$-convex functions, with $\alpha(t) = t, \beta(t) = 1 - t, a(t) = t^s, b(t) = (1 - t)^s$.

The case $s = 1$ means the usual convexity of $f$.

In Breckner [4] and Breckner and Orban [5] Berstein-Doetsch type results were proved on rationally $s$-convex functions, moreover, for the $s$-Hölder property of $s$-convex functions. Pycia [26] gives a new proof of the latter statement, when $f$ is defined on a nonempty, convex subset of a finite dimensional vector space. In the paper Hudzik and Maligranda [14] the authors collect some properties of $s$-convex functions defined on the nonnegative reals. In the paper Burai, Házy and Juhász [2] there are some Berstein-Doetsch type result on $(H, s)$-convex functions.

The $P$-functions was investigated in Dragomir, Pečarić and Persson [6]. A real valued function $f : D \rightarrow \mathbb{R}$ (where $D$ is a convex, open, nonempty subset of a real (complex) linear space $X$) is called $P$-function, if for every $x, y \in D$ and $t \in [0, 1]$ we have
\[
 f(tx + (1 - t)y) \leq f(x) + f(y).
\]

Some results about the $P$-functions there are in Pearce and Rubinov [25], Tseng, Yang and Dragomir [28]. The $P$-functions are $(\alpha; \beta; a; b)$-convex functions, with $\alpha(t) = t, \beta(t) = 1 - t, a(t) = 1, b(t) = 1$. 
In Bernstein and Doetsch [1] proved that if a function \( f : D \to \mathbb{R} \) (where \( D \) is a convex, open, nonempty subset of a real (complex) linear space \( X \)) is locally bounded from above at a point of \( D \), then the Jensen-convexity of the function yields its local boundedness and continuity as well, which implies the convexity of the function \( f \) (see Kuczma [15] for further references). This result has been generalized by several authors. The first such type results are due to Nikodem and Ng [20] for the approximately Jensen-convex functions (the so-called \( \epsilon \)-Jensen-convexity), which was extended by Páles (Páles [22] and [23]) to approximately \( t \)-convex functions. Further generalizations can be found in papers Mrowiec [19], Házy [9] and [10], Házy and Páles [12] and [13]. In the paper Gilányi, Nikodem and Páles [8] there are some Bernstein-Doetsch type results for quasiconvex functions.

2. Main results

In this section we assume that \( (X, \| \cdot \|) \) is a real (complex) normed space. We recall that a function \( f : D \to \mathbb{R} \) is called locally bounded from above on \( D \) if, for each point of \( p \in D \), there exist \( g > 0 \) and a neighborhood \( U(p, g) := \{ x \in X : \|x - p\| < g \} \) such that \( f \) is bounded from above on \( U(p, g) \). We assume that \( a, b : [0, 1] \to \mathbb{R} \) are nonnegative.

**Proposition 1.** Let \( t_0 \in [0, 1] \) be fixed such that \( \alpha(t_0) + \beta(t_0) = 1 \) and \( f : D \to \mathbb{R} \) be an \((\alpha; \beta; a; b)\)-convex function. Then

(i) if \( a(t_0) + b(t_0) > 1 \) then \( f \) is nonnegative.

(ii) if \( a(t_0) + b(t_0) < 1 \) then \( f \) is nonpositive.

**Proof.** Let \( x \) be an arbitrary element of \( D \). Using \((\alpha; \beta; a; b)\)-convexity of \( f \)

\[
 f(x) = f(\alpha(t_0)x + \beta(t_0)x) \leq a(t_0)f(x) + b(t_0)f(x) = (a(t_0) + b(t_0))f(x),
\]

which implies

\[
 0 \leq (a(t_0) + b(t_0)) - 1)f(x).
\]

If \( a(t_0) + b(t_0) - 1 > 0 \), then we have \( f(x) \geq 0 \) and if \( a(t_0) + b(t_0) - 1 < 0 \), then we have \( f(x) \leq 0 \).

\[\square\]

**Theorem 1.** Let \( [0, 1] \subset T, \alpha, \beta, a, b : T \to \mathbb{R} \) be given nonnegative functions and let \( t_0 \in [0, 1] \) be fixed such that \( \alpha(t_0)\beta(t_0)a(t_0)b(t_0) \neq 0 \) and \( \alpha(t_0) + \beta(t_0) = 1 \). Furthermore let \( D \subset X \) be open, nonempty, \((\alpha; \beta)\)-convex set, let \( f : D \to \mathbb{R} \) be a \( \alpha - (\alpha; \beta; a; b) \)-convex function, which is locally bounded from above at a point of \( D \). Then \( f \) is locally bounded at every point of \( D \).

**Proof.** Since \( \alpha(t_0)\beta(t_0) \neq 0 \) therefore we get \( \alpha(t_0), \beta(t_0) > 0 \). We prove that \( f \) is locally bounded from above on \( D \).

First we prove that \( f \) is locally bounded from above on \( D \). Define the sequence of sets \( D_n \) by

\[
 D_0 := \{ p \}, \quad D_{n+1} := \alpha(t_0)D_n + \beta(t_0)D.
\]

Using induction on \( n \), we prove that \( f \) is locally upper bounded at each point of \( D_n \). By assumption, \( f \) is locally bounded from above at \( p \in D_0 \). Assume that \( f \) is locally upper bounded at each point of \( D_n \). For \( x \in D_{n+1} \), there exist \( x_0 \in D_n \) and \( y_0 \in D \) such that \( x = \alpha(t_0)x_0 + b(t_0)y_0 \). By the inductive assumption, there exist \( r > 0 \) and a constant \( M_0 \geq 0 \) such that \( f(x') \leq M_0 \) for \( \|x_0 - x'\| < r \). Then, by the \( t_0 - (\alpha; \beta; a; b) \)-convexity of \( f \), for \( x' \in U_0 := U(x_0, r) \) we have

\[
 f(\alpha(t_0)x' + \beta(t_0)y_0) \leq a(t_0)f(x') + b(t_0)f(y_0) \leq a(t_0)M_0 + b(t_0)f(y_0) =: M.
\]
Therefore, for 
\[ y \in U := \alpha(t_0)U_0 + \beta(t_0)y_0 = U(\alpha(t_0)x_0 + \beta(t_0)y_0, t_0r) = U(x, t_0r), \]
we get that \( f(y) \leq M \). Thus \( f \) is locally bounded from above on \( D_{n+1} \).

On the other hand, we show that 
\[ D = \bigcup_{n=1}^{\infty} D_n. \]

From the definition of \( D_n \), it follows by induction that \( D_n = (\alpha(t_0))^n p + (1 - (\alpha(t_0))^n)D \). For fixed \( x \in D \), define the sequence \( x_n \) by 
\[ x_n := \frac{x - (\alpha(t_0))^n p}{1 - (\alpha(t_0))^n}. \]

Then \( x_n \to x \) if \( n \to \infty \). As \( D \) is open, \( x_n \in D \) for some \( n \). Therefore 
\[ x = \alpha(t_0)^n p + (1 - (\alpha(t_0))^n) x_n \in (\alpha(t_0))^n p + (1 - (\alpha(t_0))^n) D = D_n. \]

Thus \( f \) is locally bounded from above on \( D \).

Now, we prove that \( f \) is locally bounded from below. Let \( q \in D \) be arbitrary. Since \( f \) is locally bounded from above at the point \( q \), there exist \( \rho > 0 \) and \( M > 0 \) such that 
\[ \sup_{U(q, \rho)} f \leq M. \]

Let \( x \in U(q, \beta(t_0)q) \) and \( y := \frac{q - \alpha(t_0)x}{\beta(t_0)} \). Then \( y \) is in \( U(q, \rho) \). By \( t_0 - (\alpha; \beta; a; b) \)-convexity, 
\[ f(q) \leq a(t_0)f(x) + b(t_0)f(y), \]

which implies 
\[ f(x) \geq \frac{f(q) - b(t_0)f(y)}{a(t_0)} \geq \frac{f(q) - b(t_0)M}{a(t_0)} =: M'. \]

Therefore \( f \) is locally bounded from below at any point of \( D \). \( \square \)

**Corollary 1.** Let \( f : D \to \mathbb{R} \) be a Jensen-convex or \( t_0 \)-convex function. If \( f \) is locally bounded from above at a point of \( D \), then \( f \) is locally bounded at every point of \( D \).

**Corollary 2.** Let \( f : D \to \mathbb{R} \) be a Breckner \( (t_0, s) \)-convex function. If \( f \) is locally bounded from above at a point of \( D \), then \( f \) is locally bounded at every point of \( D \).

**Corollary 3.** Let \( f : D \to \mathbb{R} \) be a \( (t_0, h) \)-convex function such that \( h(t_0) \) and \( h(1 - t_0) \) are not zero simultaneously. If \( f \) is locally bounded from above at a point of \( D \), then \( f \) is locally bounded at every point of \( D \).

**Remark 1.** It is a well-known fact that if a Jensen-convex function \( f \) is locally bounded above at a point of its domain (see [1], [15]), then it is continuous on its domain. This is not true for \( (Jensen, h) \)-convex functions, which implies is not true for \( Jensen - (\alpha; \beta; a; b) \). Indeed, in the case \( h(\lambda) = \lambda^s \) (where \( 0 < s < 1 \) is a fixed number), in [2] we give an example which is \( (Jensen, h) \)-convex, bounded and nowhere continuous.

Next theorem gives a sufficient condition when local boundedness implies continuity.
**Theorem 2.** Let $\alpha, \beta, a, b$ be given nonnegative, continuous functions satisfying the limit conditions
\[
\lim_{t \to 0} a(t) = 0 \quad \text{and} \quad \lim_{t \to 0} b(t) = 1.
\]
and $\alpha(t) + \beta(t) = 1$.

Let the sequence $\{t_n\}_{n \in \mathbb{N}}$ be such that $t_n \in [0, 1]$ and $t_n$ tends to 0 (when $n \to \infty$) and assume that $a(t_n)$ and $b(t_n)$ are not simultaneously zero. Let $T = \{t_n\}_{n \in \mathbb{N}}$.

If $f : D \to \mathbb{R}$ is $T - (\alpha; \beta; a; b)$-convex function and $f$ is locally bounded from above at a point of $D$. Then $f$ is continuous on $D$.

**Proof.** Since $a(t_0)$ and $b(t_0)$ are not zero simultaneously, therefore, without loss generality, we may assume that $b(t_0) > 0$.

Since $f$ is locally bounded from above at a point $x_0 \in D$, there exists a neighborhood $U$ at $x_0$ and a constant $K \geq 0$ such that $f(x) \leq K$ for every $x \in U$. Let $\varepsilon$ be an arbitrary nonnegative constant. Then there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then
\[
a(t_n)K + [b(t_n) - 1] f(x_0) < \varepsilon,
\]
whence
\[
\frac{a(t_n)}{b(t_n)} K + \left[ 1 - \frac{1}{b(t_n)} \right] f(x_0) < \varepsilon.
\]

Let $V$ be a neighborhood of 0 such that $x_0 + V \subseteq U$, and let $U' = x_0 + \alpha(t_n)V$. We prove that
\[
|f(x) - f(x_0)| < \varepsilon \quad (x \in U').
\]

For $x \in U'$ there exist $y, z \in x_0 + V$ such that
\[
x = \alpha(t_n)y + \beta(t_n)x_0,
\]
\[
x_0 = \alpha(t_n)z + \beta(t_n)x.
\]

Indeed,
\[
y - x_0 = \frac{1}{\alpha(t_n)}(x - x_0) \in \frac{1}{\beta(t_n)}\alpha(t_n)V = V,
\]
and
\[
z - x_0 = \frac{1 - \alpha(t_n)}{\alpha(t_n)}(x_0 - x) \in \frac{1 - \alpha(t_n)}{\alpha(t_n)}\alpha(t_n)V = (1 - \alpha(t_n))V \subseteq V.
\]

According to $T - (\alpha; \beta; a; b)$-convexity of $f$,
\[
f(x) \leq a(t_n)f(y) + b(t_n)f(x_0) \leq a(t_n)K + b(t_n)f(x_0),
\]
\[
f(x_0) \leq a(t_n)f(z) + b(t_n)f(x) \leq a(t_n)K + b(t_n)f(x).
\]

We get
\[
f(x) - f(x_0) \leq a(t_n)K + [b(t_n) - 1] f(x_0) < \varepsilon \quad (4)
\]
and
\[
f(x) \geq \frac{f(x_0) - a(t_n)K}{b(t_n)},
\]
which implies
\[
f(x) - f(x_0) \geq \left[ \frac{1}{b(t_n)} - 1 \right] f(x_0) - \frac{a(t_n)}{b(t_n)} K > -\varepsilon. \quad (5)
\]

The inequalities (4) and (5) show that $|f(x) - f(x_0)| < \varepsilon$, that is $f$ is continuous at $x_0$, which was to be proved.  \(\square\)
Remark 2. The previous limit conditions are not necessary, since in the case of Jensen-convexity are not fulfilled. However, the result of Bernstein and Doetsch is valid for Jensen-convex functions. In contrary, the nonnegative monotone functions - which are not necessary continuous - belongs to a special class of the \((\alpha; \beta; a; b)\)-convex functions, to the class of Godunova-Levin functions. Therefore, in this setting, the limit conditions in question cannot be ignored.

3. Convexity property of rationally-\((\alpha; \beta; a; b)\)-convex

The following result offers a generalization of the theorem of Bernstein-Doetsch [1], Breckner [4], Burai-Házy-Juhász [2] and Házy [11] for rationally-\((\alpha; \beta; a; b)\)-convex functions.

Theorem 3. Let \(\alpha, \beta, a, b\) be given nonnegative, continuous functions satisfying the limit conditions

\[\lim_{t \to 0} a(t) = 0 \quad \text{and} \quad \lim_{t \to 0} b(t) = 1.\]

and \(\alpha(t) + \beta(t) = 1\).

Assume that \(a(t_0)\) and \(b(t_0)\) are not zero simultaneously for all \(t_0 \in \mathbb{Q} \cap [0, 1]\). If \(f : D \to \mathbb{R}\) is rationally-\((\alpha, \beta, a, b)\)-convex and locally bounded from above at a point of \(D\), then \(f\) is continuous and \((\alpha, \beta, a, b)\)-convex.

Proof. We prove that the function \(f\) is \(t_0 - (\alpha; \beta; a; b)\)-convex for all \(t_0 \in [0, 1]\). Let \(t_0 \in [0, 1]\) arbitrary. Then there exists a sequence \(\{t_n\}_{n \in \mathbb{N}}\) such that \(t_n \in \mathbb{Q}\) and \(t_n \to t_0\) (when \(n\) tends to \(\infty\)). Applying rationally-\((\alpha, \beta, a, b)\)-convexity of \(f\), we get

\[f(\alpha(t_n)x + \beta(t_n)y) \leq a(t_n)f(x) + b(t_n)f(y).\]

The local upper boundedness of \(f\) implies the continuity of \(f\) (according to Theorem 2). Therefore, taking the limit \(n \to \infty\) in (6), we get

\[f(\alpha(t_0)x + \beta(t_0)y) \leq a(t_0)f(x) + b(t_0)f(y),\]

which proves the \((\alpha, \beta, a, b)\)-convexity of \(f\). \(\square\)

Corollary 4. Let \(D \subset X\) be a nonempty, convex, open set and let \(h : [0, 1] \to \mathbb{R}\) be a given nonnegative, continuous function satisfying the limit conditions

\[\lim_{t \to 0} h(t) = 0 \quad \text{and} \quad \lim_{t \to 1} h(t) = 1.\]

and assume that \(h(t_0)\) and \(h(1 - t_0)\) are not simultaneously zero for all \(t_0 \in \mathbb{Q} \cap [0, 1]\).

If \(f : D \to \mathbb{R}\) is rationally-\(h\)-convex and \(f\) is locally bounded from above at a point \(D\), then \(f\) is continuous on \(D\) and \(h\)-convex.

Corollary 5. Let \(D \subset X\) be a nonempty, convex, open set. If \(f : D \to \mathbb{R}\) is rationally-Breckner \(s\)-convex and locally bounded from above at a point \(D\), then \(f\) is continuous on \(D\) and Breckner \(s\)-convex.

Theorem 4. Let \(T = [0, 1]\), \(\alpha, \beta, a, b : T \to \mathbb{R}\) be given nonnegative functions such that \(\alpha, \beta\) continuous on \(T\) and \(a(t) + b(t) = 1\). Let \(f : \mathbb{R}_+ \to \mathbb{R}\) an \((\alpha, \beta, a, b)\)-convex function. Then

(i) if \((\alpha + \beta)(T) = [r, 1]\) (where \(r < 1\)), then \(f\) is nondecreasing.

(ii) if \((\alpha + \beta)(T) = [1, r]\) (where \(r > 1\)), then \(f\) is nonincreasing.

(iii) if \((\alpha + \beta)(T) = [r_1, r_2]\) (where \(r_1 < 1 < r_2\)), then \(f\) is constant.
Proof. We have, for \( x > 0 \) and \( t \in [0, 1] \)
\[
f(\alpha(t)x + \beta(t)x) \leq a(t)f(x) + b(t)f(x) = f(x).
\]
Let \( \gamma = \alpha + \beta \). Then \( \gamma \) is continuous on \([0, 1]\).

In the case (i) we get \( \gamma(T) = [r, 1] \), where \( r > 1 \). Let \( u \in [r, 1] \) be arbitrary. Then there exists a \( t \in [0, 1] \) such that \( \gamma(t) = u \). This yields that
\[
f(ux) \leq f(x) \quad (x \in \mathbb{R}_+, u \in [r, 1]).
\]

If now \( u \in [r^2, 1] \) then \( u^{1/2} \in [r, 1] \). Therefore, by the fact that (7) holds for all \( x \in \mathbb{R}_+ \), we get
\[
f(ux) = f(u^{1/2}(u^{1/2}x)) \leq f(u^{1/2}x) \leq f(x)
\]
for all \( x \in \mathbb{R}_+ \). By induction we then obtain that
\[
f(ux) \leq f(x) \quad (x \in \mathbb{R}_+, u \in [0, 1]).
\]
Therefore, taking \( 0 < u < v \) and applying (8), we get
\[
f(u) = f((u/v)v) \leq f(v),
\]
which means that \( f \) is nondecreasing on \( \mathbb{R}_+ \).

The proof of the cases (ii) and (iii) are similar. \( \square \)

The above results do not hold, in general, in the case of convex functions, because a convex function \( f : \mathbb{R}_+ \to \mathbb{R} \), need not be non-decreasing. But in the case of Orlicz s-convex function this is true.

**Corollary 6.** Let \( 0 < s < 1 \). Let \( f : \mathbb{R}_+ \to \mathbb{R} \) an Orlicz s-convex function. Then \( f \) is nondecreasing.

**Remark 3.** In the paper of Hudzik and Maligranda [14] is gave an example which shows that the Orlicz s-convex function is nondecreasing on \( \mathbb{R}_+ \), but not necessarily on \([0, \infty)\). For the readers convenience we recall the example: let \( a, b, c \in \mathbb{R} \) and let
\[
f(x) = \begin{cases} 
a & \text{if } x = 0 \\
bx^s + c & \text{if } x \neq 0.
\end{cases}
\]
Then if \( b > 0 \) and \( c < a \) then \( f \) is non-decreasing on \((0, \infty)\) but not on \([0, \infty)\).

### 4. Optimization

It is a very well known fact that every local minimizer of a convex function is a global one. The same is true for \((\alpha, \beta, a, b)\)-convex functions under some assumptions.

**Theorem 5.** Let \( X \) be a real or complex topological vector space, \( D \subset X \) be a nonempty open \((\alpha; \beta)\)-convex set, where \( \alpha, \beta, a, b : [0, 1] \to \mathbb{R} \) be given nonnegative, continuous functions satisfying the limit conditions
\[
\lim_{t \to 0} \alpha(t) = 0 \quad \text{and} \quad \lim_{t \to 0} \beta(t) = 1.
\]
and assume that \( a(t) + b(t) = 1 \).

Then every local minimizer \( x_0 \in D \) of an \((\alpha, \beta, a, b)\)-convex function \( f : D \to \mathbb{R} \) is a global one.
Proof. Let \( x_0 \in D \) be a local minimizer of \( f \). Then there exists a positive real number \( r \), such that
\[
  f(x_0) \leq f(y), \quad y \in U(x_0, r).
\]
Assume that \( x_0 \) is not a global minimizer. Then there exists \( z \in D \), such that \( f(x_0) > f(z) \). Using this and the \((\alpha, \beta, a, b)\)-convexity of \( f \), we have
\[
  f(\alpha(t)x + \beta(t)y) < a(t)f(x) + b(t)f(y) = f(x),
\]
which is a contradiction.

\[\square\]

Corollary 7. Every local minimizer of an Orlicz-convex function \( f : D \to \mathbb{R} \) is a global one. If the function \( f \) is strictly Orlicz-convex, then there is at most one global minimum.

Corollary 8. Every local minimizer of a convex function \( f : D \to \mathbb{R} \) is a global one. If the function \( f \) is strictly convex, then there is at most one global minimum.

References


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