ON APPROXIMATELY (k,h)-CONVEX FUNCTIONS

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Abstract. A real valued function $f : D \to \mathbb{R}$ defined on an open convex subset D of a normed space X is called *rationally* (k,h,d)*-convex* if it satisfies

$$f\left(k(t)x + k(1-t)y\right) \leq h(t)f(x) + h(1-t)f(y) + d(x,y)$$

for all $x, y \in D$ and $t \in \mathbb{Q} \cap [0,1]$, where $d : X \times X \to \mathbb{R}$ and $k, h : [0,1] \to \mathbb{R}$ are given functions.

Our main result is of a Bernstein-Doetsch type. Namely, we prove that if f is locally bounded from above at a point of D and rationally (k,h,d)-convex then it is continuous and (k,h,d)-convex.

1. Introduction

Convexity and its generalizations are very important both in pure mathematics and in applications. It is wildly known that in applications we can not state convexity (generalized convexity) on the examined function, but we know some convexity like (generalized convexity like) behavior on the function. This means, we do not know weather the function in question is convex (generalized convex) or not, but we know that it is close to a convex (generalized convex) function. So, examining approximate convexity and approximate generalized convexity is an important task mainly in terms of optimization theory.

On the other hand we can not know regularity on the unknown function in general. Improving regularity of an unknown function is also an important and useful topic not only in applications but in pure mathematics too. Improving regularity means getting a stronger regularity property from a weaker one.

Several authors have dealt with the above mentioned questions. The first approximate convexity result is due to Hyers and Ulam [HU52], where the

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authors used a constant error term. An another possibility to make a mathematical model of ”being close to a convex function” is to use a function error term, which depends on the distance of the variables (see e.g. [LNT00] or [Pal03]).

Probably the most significant result in the early history of regularity theory of convex functions was developed by Bernstein and Doetsch [BD15]. They proved that the local boundedness from above at a point of a Jensen convex function implies its continuity and convexity as well on the whole domain.

The starting point of this work is a very recent generalization of convexity, namely (k, h)-convexity. This notion of k-convex set and (k, h)-convex functions were introduced in [MR12] by B. Micherda and T. Rajba and was investigated in [Haz12].

Let $k : (0, 1) \to \mathbb{R}$ be a given function. Then a subset D of a real linear space X will be called *k-convex* if $k(t)x + k(1-t)y \in D$ for all $x, y \in D$ and $t \in (0, 1)$.

Let $k, h : (0, 1) \to \mathbb{R}$ be two given functions and suppose that $D \subset X$ is a k-convex set. Then a function $f : D \to \mathbb{R}$ is *(k, h)-convex* if, for all $x, y \in D$ and $t \in (0, 1)$,

$$f(k(t)x + k(1-t)y) \leq h(t)f(x) + h(1-t)f(y).$$

They prove some properties of these sets and functions and gave some example too. Now we introduce notion of *(k, h)-convexity with respect to T*.

Let X be a real or complex topological vector space, T be a nonempty set, such that the following property fulfill:

$$t \in T \text{ if and only if } 1 - t \in T.$$

For example $T = \{1/2\}$, $T = \{t_0, 1 - t_0\}$, or T is an interval (or an arbitrary subset of $\mathbb{R}$) which symmetric with respect to $1/2$. Furthermore, T may be a subset of $\mathbb{C}$, but it can be a subset of $\mathbb{R}^n$ or $\mathbb{C}^n$ too.

Furthermore let $D \subset X$ be a nonempty open, k-convex set (that is, $k(t)x + k(1-t)y \in D$ whenever $x; y \in D$ and $t \in T$), and $k, h : T \to \mathbb{R}$ be given functions. We say the function $f : D \to \mathbb{R}$ is $T - (k, h)$-convex function, if

$$f(k(t)x + k(1-t)y) \leq h(t)f(x) + h(1-t)f(y), \tag{1.1}$$

for all $x; y \in D$ and $t \in T$. To avoid the trivialities and the unimportant cases, we suppose that there exists an element $t_0 \in T$ such that

$$k(t_0)k(1-t_0)h(t_0)h(1-t_0) \neq 0.$$

The concept of h convexity appeared first in [Var07] defined by Varošanec. This is a far generalization not only of convexity ($h(t) = t$) but e.g. of s-convexity ($h(t) = t^s$, $0 < s \leq 1$) due to Breckner [Bre78], and other classes of functions (see [Var07] again) too.

We examine such functions which are "close to" a (k,h)-convex function in some sense. A function, on which some natural requirements are made, measures the error. More precisely: let $d : X \times X \to \mathbb{R}$ be given. A function $f : D \to \mathbb{R}$ is said to be (k,h,d)*-convex* if

$$f\left(k(t)x + k(1-t)y\right) \le h(t)f(x) + h(1-t)f(y) + d(x,y) \tag{1.2}$$

for all $x, y \in D$ and $t \in [0,1]$. We are using two other concepts, *rational* (k,h,d)*-convexity* and t-(k,h,d)*-convexity*, which mean f fulfills (1.2) only for all rational t or a fixed t $(0 < t < 1)$ respectively.

In the earlier investigations, $\|x-y\|^p$ $(p > 0)$ or a linear combination with positive coefficients of such expressions were applied as an error term (see e.g. [LNT00], [Pal03], [HP05]). Other type of approximate convexity were investigated in [BHJ09, BHJ11, BH11, MP10, MP11a, MP12a, MP12b]. These define metrics on X, where the distance of two points does not change if we translate them, and they possess some kind of homogeneity property. Thus, generalizing and summing of the previous attributes simultaneously, we assume that d is a ψ*-subhomogeneous, translation invariant semimetric*, namely

$$\begin{array}{lrcl}
(i) & d(x,y) & \ge & 0, \\
(ii) & d(x,y) & = & d(y,x), \\
(iii) & d(x,y) & \le & d(x,z) + d(z,y), \\
(iv) & d(x+z, y+z) & = & d(x,y), \\
(v) & d(ux, uy) & \le & \psi(u)d(x,y)
\end{array}$$

for all $x, y, z \in X$ and $u > 0$, where the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is bounded. The first three property declare that d is a semimetric on X, (iv) states the translation invariance of d, and (v) is the subhomogeneity of d with respect to ψ.

In the sequel d always denotes a continuous, ψ-subhomogeneous, translation invariant semimetric.

2. Main results

We have two main results. The first one is a regularity theorem on rationally (k,h,d)-convex, and (k,h,d)-convex functions. The second one is on (k,h,d)-convexity of rationally (k,h,d)-convex functions.

It is very important to note, that we suppose some technical conditions on h, namely we assume

$$\lim_{t\to 0} h(t) = 0 \text{ and } \lim_{t\to 1} h(t) = 1 \tag{2.1}$$

henceforth. We draw the reader's attention to the fact, that the special functions mentioned in the introduction fulfill these conditions.

Theorem 1. *Assume that $d(x,x)=0$, and $f : D \to \mathbb{R}$ is a rationally (k,h,d)-convex or (k,h,d)-convex function. If f is locally bounded from above at a point of D, then it is continuous on D.*

Theorem 2. *Assume that $d(x,x)=0$, and $f : D \to \mathbb{R}$ is a rationally (k,h,d)-convex function. If f is locally bounded from above at a point of D, then it is (k,h,d)-convex.*

It is clear that we can not expect the continuity of f without continuity assumption on d. Keeping in mind the nonnegativity of d, it seems natural to assume $d(x,x)=0$.

It is worthy to mention the fact, that these theorems do not remain true with t-(k,h,d)-convexity assumption on f. On the other hand, it is an open problem, to find the smallest (in some sense, e.g. measure, category etc.) subset of the unit interval, such that the theorems remain true if f fulfills (1.2) only for all t from this subset. We know that any dense subset of $[0,1]$ is enough.

3. Proofs of the main theorems

We begin with two lemmas. In the first one we deal with boundedness of t-(k,h,d)-convex functions. We recall that a function $f : D \to \mathbb{R}$ is called locally bounded from above at a point of D, if there exists a neighborhood U of this point such that f is bounded from above on U, and f is bounded from above on D, if it is bounded from above at every point of D. One can define local lower boundedness, and boundedness at a point or on the whole domain in a similar way.

Lemma 1. *Let $k,h : T \to \mathbb{R}$ be given nonnegative functions and let $t_0 \in T$ be fixed such that $k(t_0)k(1-t_0) \neq 0$ and $k(t_0)+k(1-t_0)=1$. We assume that $h(t_0)$ and $h(1-t_0)$ are not zero simultaneously. Furthermore let $D \subset X$ be an open, nonempty, k-convex set, and let $f : D \to \mathbb{R}$ be a t_0-(k,h)-convex function. Then the t-(k,h,d)-convexity of a function and the locally boundedness from above at a point of its domain implies the locally boundedness of the function on the whole domain.*

The next result states that the local upper boundedness of a rationally (k,h,d)-convex function at a point of D yields its continuity at this point as well.

Lemma 2. *Let $T = \left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$. Let $k,h : T \to]0,\infty[$ be given continuous functions such that $k\left(\frac{1}{n}\right)+k\left(1-\frac{1}{n}\right)=1$ for every $n \in \mathbb{N}$. Let $d(x,x)=0$ and $f : D \to \mathbb{R}$ be a (k,h,d)-convex function with respect to T (or rational (k,h,d)-convex). If f is locally bounded from above at a point of its domain, then it is continuous at the same point.*

Proof of Theorem 1. According to Lemma 1, f is locally bounded at every point of D. So, we can use Lemma 2, which implies the continuity of f at every point of D. □

Proof of Theorem 2. We prove that the function f is t-(k, h, d)-convex for all $t \in [0, 1]$. Let $t \in [0, 1]$ arbitrary. Then there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $t_n \in \mathbb{Q}$ and $t_n \to t$ (when n tends to ∞). Applying rational (k, h, d)-convexity of f, we get

$$f\left(k(t_n)x + k(1 - t_n)y\right) \leq h(t_n)f(x) + h(1 - t_n)f(y) + d(x, y). \tag{3.1}$$

The local upper boundedness of f implies the continuity of f (according to Lemma 1). Therefore, taking the limit $n \to \infty$ in (3.1), we get

$$f\left(k(t)x + k(1 - t)y\right) \leq h(t)f(x) + h(1 - t)f(y) + d(x, y),$$

which proves the (k, h, d)-convexity of f. □

4. Corollaries and applications

In this section we show some applications of the previous results. We begin with a corollary, which is an immediate consequence of Lemma 1. We call f (k, h, d)-convex with respect to $S \subset]0, 1[$, if f fulfills (1.2) only for all $t \in S$.

Corollary 1. *Let $f : D \to \mathbb{R}$ be a (k, h, d)-convex function with respect to $S \subset]0, 1[$. If f is locally bounded from above at a point of D, then it is locally bounded on D.*

If the underlying space is of finite dimension, the local boundedness from above assumption can be weakened. Using Theorem 1, Theorem 2 we get the following two corollaries.

Corollary 2. *Let D be an open, convex subset of $\mathbb{R}^n$, $f : D \to \mathbb{R}$ be a rationally (k, h, d)-convex function, and $d(x, x) = 0$. If there exists a set $S \subset D$ of positive Lebesgue measure, such that f is bounded from above on S, then f is continuous, and (k, h, d)-convex on D.*

Corollary 3. *Let D be an open, convex subset of $\mathbb{R}^n$, $f : D \to \mathbb{R}$ be a rationally (k, h, d)-convex function, and $d(x, x) = 0$. If there exists a Baire-measurable set $S \subset D$ of second category, such that f is bounded from above on S, then f is continuous, and (k, h, d)-convex on D.*

Theorem 1, Theorem 2, Lemma 1, and Lemma 2 are far generalizations of earlier theorems.

We get [BHJ09, Theorem 2.] from Lemma 1 with $d \equiv 0$, and $h(t) = \lambda^s$ ($\lambda \in]0, 1[$ is fixed here).

With $d \equiv 0$, and $h(t) = t^s$, we get [Bre78, Satz 2.1] from Lemma 2. With the same casting one can deduce [Bre78, Satz 2.2 and Korollar 2.3] from Lemma 1, and Theorem 1 with Theorem 2 respectively.

We have [Kuc85, Theorem 6.2.1., Theorem 6.2.2., Theorem 6.2.3. 148 p.] from Lemma 1 with $h(t) = \frac{1}{2}$ and $d \equiv 0$.

References

[BD15] F. Bernstein and G. Doetsch, *Zur Theorie der konvexen Funktionen*, Math. Annalen **76** (1915), 514–526.

[BHJ09] P. Burai, A. Házy and T. Juhász, *Bernstein-Doetsch type results for s-convex functions* Publ. Math. Debrecen **75** (2009), vol 1-2., 23–31

[BHJ11] P. Burai, A. Házy and T. Juhász, *On approximately Breckner s-convex functions* Control and Cybernetics **40** (2011), no. 1. 91–99.

[BH11] P. Burai, A. Házy, *On approximately h-convex functions.* J Convex Anal. **18** no. 2 (2011) 447 – 454.

[Bre78] W. W. Breckner, *Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen*, Publ. Inst. Math. (Beograd) **23** (1978), 13–20.

[Haz12] A. Hzy, *Bernstein-Doetsch type results for (k,h)-convex functions.* Miskolc Math. Notes **13** no. 2, (2012) 325–336.

[HU52] D. H. Hyers and S.M. Ulam, *Approximately convex functions*, Proc. Amer. Math. Soc. **3** (1952) 821–828.

[HP05] A. Házy and Zs. Páles, *On approximately t-convex functions*, Publ. Math. Debrecen **66** (2005), no. 3-4, 489–501. Dedicated to the 75th birthday of Professor Heinz König.

[Kuc85] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities*, Państwowe Wydawnictwo Naukowe — Uniwersytet Śląski, Warszawa–Kraków–Katowice, 1985.

[LNT00] D. T. Luc H. V. Ngai and M. Théra, *Approximate convex functions*, J. Nonlinear Convex Anal. **1** (2000), no. 2, 155–176.

[MP10] J. Makó and Zs. Páles, *Approximate convexity of Takagi type functions* J. Math. Anal Appl. **369** (2010) no. 2, 545–554.

[MP11a] J. Makó and Zs. Páles *Strengthening of strong and approximate convexity*, Acta Math. Hung **132** (2011) vol. 1-2, 73–91.

[MP12a] J. Makó and Zs. Páles, *Implications between approximate convexity properties and approximate Hermite-Hadamard inequalities*, Central European Journal of Math. **10** (2012) no. 3, 1017–1041

[MP12b] J. Makó and Zs. Páles, *On φ-convexity*, Publ. Math. Debr. **80** (2012) no. 1-2, 107–126.

[MP13] J. Makó and Zs. Páles, *On Approximately convex Takagi type functions* Proc. Amer. Math. Soc. (2013)

[MR12] B. Micherda and T. Rajba: *On some Hermite-Hadamard-Fejér inequalities for (k,h)-convex functions*, Math. Ineq. Appl. **12** (2012), no 4, 931-940.

[Pal03] Zs. Páles, *On approximately convex functions*, Proc. Amer. Math. Soc. **131** (2003), no. 1, 243–252.

[Var07] S. Varošanec, *On h-convexity*, J. Math. Anal. Appl. **326** (2007), 303-311.

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