A least-square spectral element method for stability analysis of time delay systems

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Abstract: This paper presents a numerical method for the stability analysis of retarded functional differential equations with time-periodic coefficients. The method approximates the solution segments, corresponding to the endpoints of the principal period, by their piecewise Lagrange interpolants. Then a mapping between these solution segments is obtained by the minimization of the least-square integral of the residual error. Finally, stability properties are determined using the matrix approximation of the monodromy operator, obtained by this mapping, according to the Floquet theorem. The formulation of the method is presented for an equation of general type while results are shown for the delayed oscillator.

Keywords: time delay, time-varying system, stability analysis, stability domains, least-squares method, numerical method, spectral element

1. INTRODUCTION

In the recent decades the significance of time-delay has been discovered in an increasing number of applications in engineering and biosciences. Machine tool vibrations (??), wheel dynamics of vehicles (?), traffic dynamics (?), control (?), human balancing (?) and epidemiology (?) can be mentioned as examples. In the majority of the above applications it is desired to keep the system in the proximity of a particular state or motion by a proper tuning of system parameters. Therefore, stability properties of time-delay systems has been given an increasing attention and, as a result, many analytical and numerical methods can be found on their linear stability analysis. A commonly used analytical method is the D-subdivision method (?), while for numerical methods are the semi-discretization method (?), spectral element method (?), pseudospectral collocation method (?) or methods based on the truncation of Hill’s infinite matrix (e.g. ??).

Although time dependency of the system parameters are often neglected, therefore the particular phenomenon is described by time-invariant (autonomous) differential equations, this simplification cannot be justified in all cases. For instance, milling operations, turning operations with spindle speed variation or digitally controlled machines are inevitably time-varying systems. The governing equations for these systems are usually modelled as differential equations with time-periodic parameters, and their stability can be determined using the Floquet theory.

This paper deals with the formulation of a method for the numerical stability analysis of linear retarded functional differential equations (RFDE) with time-periodic coefficients. First, the main steps of the method are presented for a RFDE of general type. Then it is applied to a particular example: the delayed oscillator. For this example stability charts are constructed and the convergence of stability boundaries are demonstrated. Finally, based on these results, some conclusions are drawn.

2. METHOD

In this section the formulation of the method is presented for a RFDE involving point delays and distributed delay with periodic coefficients of principal period T, having the form

\[ \dot{y}(t) = A(t)y(t) + \sum_{l=1}^{\sigma} B_l(t)y(t-\tau_l) + \int_{a}^{b} \gamma(t, \theta)y(t+\theta)d\theta, \] (1)

where \( y : \mathbb{R} \to \mathbb{R}^n; A, B_l : \mathbb{R} \to \mathbb{R}^{n \times n}; 0 < \tau_l, l = 1, 2, \ldots; \gamma : \mathbb{R}^2 \to \mathbb{R}^{n \times n}; \) and \( A(t) = A(t + T); B_l(t) = B_l(t + T), l = 1, 2, \ldots; \gamma(t, \theta) = \gamma(t + T, \theta), \forall \theta \) for all \( t. \) The mapping between a solution segment \( y^{-\tau+t_1}_{t_1} \) and an initial function \( y^{a,b} \) is given by the solution operator as \( y^{-\tau+t_1}_{t_1} = U(t_1)y^{a,b}, \) where \( \tau > \max\{\tau_1, \tau_2, \ldots, \tau_\sigma\}. \) Here and in the following, notation \( y^{a,b} \) refers to \( \{y(t) : [a, b]\}. \) According to the Floquet theory the stability of (1) can be determined from the eigenvalues of the monodromy operator \( U(T). \) Namely, (1) is stable if and only if all the eigenvalues of \( U(T) \) have modulus less than one (see Chapter 8 in ? for details). Although there exist some counterexamples (see e.g. (?)), in general, the eigenvalues of the monodromy operator cannot be determined in closed form, therefore an approximation of \( U(T) \) is necessary. In this section a method is presented for the calculation of the matrix approximation \( \overline{U} \) of \( U(T), \) for which the above stability condition can easily be checked.

After performing numerical integration using Gaussian quadrature, (1) can be approximated by the RFDE

This work was supported by the Hungarian National Science Foundation under grant OTKA-K105433.
\[ \dot{y}(t) = A(t)y(t) + \sum_{p=1}^{m} B_p(t)y(t - \tau_p) \] 

(2)

which contains point delays only. Here

\[ B_p(t) = \frac{a-b}{2} \gamma(t, \theta_{p-\rho}) w_{p-\rho} \ldots, \] 

(23)

\[ k = -EK + 1, -EK + 2, \ldots, E, \] 

where \( y_{k,j} = y_k(\zeta_j) \) and \( \phi_j(\zeta) \) are the Lagrange base polynomials. The node

\[ \dot{y}(t) = A(t)y(t) + \sum_{p=1}^{m} B_p(t)y(t - \tau_p) \] 

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\[ k = -EK + 1, -EK + 2, \ldots, E, \] 

where \( y_{k,j} = y_k(\zeta_j) \) and \( \phi_j(\zeta) \) are the Lagrange base polynomials. The node
set \( \{ \zeta_j \}_{j=1}^N \) of interpolation is chosen to be of Lobatto-type, therefore there are nodes on the endpoints of interval \( \zeta \in [-1, 1] \), which gives

\[
y_{k,N} = y_{k+1,1},
\]

\[k = -EK + 1, -EK + 2, \ldots, E - 1 \] for (17). The approximation (23) of the elements discretizes the variational problem (22) as

\[
\sum_{k=1}^E \left( \Gamma^k(y_{e,j}), \frac{\partial \Gamma^k(y_{e,j})}{\partial y_{s,i}} \right) = 0
\]

where \( s = 1, 2, \ldots, E; \ i = 2, 3, \ldots, N; \ e = 1, 2, \ldots, E; \ j = 2, 3, \ldots, N \). Here the residual error functions of the elements can be calculated as

\[
\Gamma^k(y_{e,j}) = \sum_{j=1}^N S_j^k y_{k,j} - \sum_{j=1}^N \sum_{j=1}^N \sum_{m=m+1}^{M_{s}} \left( \tilde{Q}^k_{j} \phi_{y_{k-d_{j-1},j}} + \tilde{R}^k_{j} \phi_{y_{k-d_{j},j}} \right),
\]

(26)

while

\[
\tilde{S}^k_j = S^k \phi_j, \quad \tilde{Q}^k_{j} = Q^k \phi_j, \quad \tilde{R}^k_{j} = R^k \phi_j.
\]

(29)

Note that formula (25) takes into consideration (24) for \( k = 0, 1, \ldots, E - 1 \), but the summation with respect to \( j \) does not. After carrying out the differentiation in (25) one obtains

\[
\Pi_{s,i} = 0; \ s = 1, 2, \ldots, E; \ i = 2, 3, \ldots, N - 1
\]

\[
\Pi_{s,N} + \Pi_{s+1,1} = 0; \ s = 1, 2, \ldots, E - 1; \ i = N
\]

where

\[
\Pi_{s,i} = \left( \Gamma^s, \tilde{S}^s_i \right) + \sum_{i=1}^{M^R} \left( \Gamma^{s+d_{i}}, \sum_{m=m_{i-1}+1}^{M_s} \tilde{Q}^{s+d_{i},p} \right)
\]

\[
+ \sum_{i=1}^{M^Q} \left( \Gamma^{s+d_{i}+1,p}, \sum_{m=m_{i-1}+1}^{M_s} \tilde{Q}^{s+d_{i}+1,p} \right).
\]

(31)

where \( M^R_s \) and \( M^Q_s \) are defined according to

\[
\begin{cases}
    s + d_u \leq E & \text{if } \nu \leq M^R_s + s + d_{i+1} \\
    s + d_u > E & \text{if } \nu > M^R_s
\end{cases}
\]

\[
\begin{cases}
    E & \text{if } \nu \leq M^Q_s \\
    > E & \text{if } \nu > M^Q_s
\end{cases}
\]

(32)

Note that (30) and (24) together define the mapping

\[
H = G Y^{-\tau,0}
\]

(33)

between point sets \( \{ y_{k,j} \}_{k=1,j=1}^{E,N} \) and \( \{ y_{k,j} \}_{k=-KE+1,j=1}^{0,N} \).

Here \( H \) is a square matrix, while \( Y^{-\tau,0} \in \mathbb{R}^{n(E(N-1)+1) \times 1} \), and their elements are given by

\[
Y^{Ah,Bh} = \{ Y_u^{Ah,Bh} \}_{u=1}^{B(A-N)(N-1)+1}
\]

(34)

and

\[
Y_u^{Ah,Bh} = y_{k,j}
\]

(35)

with

\[
\tilde{U} = \begin{bmatrix}
\Theta \quad \begin{bmatrix} I \\ \vdots \\ 0 \end{bmatrix} \\
\begin{bmatrix} n(E-N-1) \\ \vdots \\ n(E-N-n) \end{bmatrix} \\
\begin{bmatrix} n(E-N-1) \\ \vdots \\ n(E-N-n) \end{bmatrix}
\end{bmatrix}
\]

\[
H^{-1} G
\]

(36)

(37)

Since the elements of \( Y^{-\tau+T,T} \) and \( Y^{-\tau,0} \) precisely define the piecewise Lagrange interpolant of \( y^{-\tau+T,T} \) and \( y^{-\tau,0} \), respectively, therefore the approximation of the monodromy operator is given by the mapping

\[
Y^{-\tau+T,T} = \tilde{U} Y^{-\tau,0}.
\]

(38)

For the case \( T < \tau_m \), the structure of \( \tilde{U} \) is shown in Figure 1, where \( I \) and \( \Theta \) are identity and null matrices, respectively, depicted with their corresponding dimensions. For the case \( T \geq \tau_m \), the matrix approximation of the monodromy operator is simply

\[
\tilde{U} = H^{-1} G.
\]

(39)

For the computation of \( \tilde{U} \) it is useful to apply Lobatto-type Chebyshev or Lobatto-type Legendre points as the node set of interpolation, since they provide fast convergence of the interpolant. It is also beneficial to carry out integration numerically, by using Lobatto-type Gaussian quadrature. This increases the speed of the method while keeps the error of numerical integration low. Note also that the Lobatto-type Gaussian quadrature integrates on the Lobatto-type Legendre nodes therefore, by choosing this point set as nodes of interpolation the computational effort can further be decreased.

### 3. CASE STUDY

In this section some results are shown for the stability of the delayed oscillator, which is governed by

\[
y(t) = A_0 y(t) + B_0 y(t - \tau),
\]

(40)

where

\[
y(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 & 0 \\ b_0 & 0 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 0 & 1 \\ -a_0 & 0 \end{bmatrix}.
\]

(41)

Since this equation is autonomous, the principal period can be defined as an arbitrary positive number. Choosing the principal period as \( T = \tau \) and utilizing the discretization detailed in the previous section, one can obtain

\[
\tilde{S}_j(\zeta) = \frac{1}{\zeta} I \phi^j(\zeta) - A_0 \phi^j(\zeta),
\]

(42)

\[
\tilde{R}_j(\zeta) = B_0 \phi^j(\zeta).
\]

(43)
Figures 2 and 3 show the convergence of stability boundaries under the increase of interpolation degree \( N \) and element number \( E \), respectively. Stability charts are determined using \( 500 \times 500 \) gridpoints.

4. CONCLUSIONS

In this paper a new method was presented for the stability analysis of time-periodic RFDEs. After the derivation of the method for RFDEs of general type it was applied to a particular system. The method shows convergence for the stability boundaries of the delayed oscillator both under the increase of element number and interpolation order.

REFERENCES


