Intersection cuts – standard versus restricted

Egon Balas*[1] and Tamás Kis[2]

[1]Tepper School of Business, Carnegie Mellon University, Pittsburgh, Pennsylvania 15213, eb17@andrew.cmu.edu

[2]Institute for Computer Science and Control, H-1518 Budapest, Hungary, kis.tamas@sztaki.mta.hu

Abstract

This note is meant to elucidate the difference between intersection cuts as originally defined, and intersection cuts as defined in the more recent literature. It also states a basic property of intersection cuts under their original definition.

Intersection cuts for mixed integer programs were introduced in the early 1970's [1, 2] as inequalities obtained by intersecting the extreme rays of the polyhedral cone $C(B)$, where B is a basis of the linear programming relaxation P, with the boundary of some convex set T whose interior contains the vertex $v(B)$ of P but no *feasible integer point*. Such a set T will be called P_I*-free*, where P_I is the set of feasible integer points.

In particular, if the simplex tableau associated with the basis B is

$$x_B = \bar{x}_B - \sum_{j \in J} \bar{a}_j x_j,$$

where J indexes the co-basis of B (i.e. the set of nonbasic variables) and if the extreme rays

$$\begin{pmatrix} \bar{x}_B \\ 0 \end{pmatrix} + \begin{pmatrix} \bar{a}_j \\ e_j \end{pmatrix} \lambda_j, \quad j \in J$$

of the cone $C(B)$ (where e_j is the j-th unit vector) intersect the boundary of T at the points defined by $\lambda_j = \lambda_j^*$, $j \in J$, then the hyperplane through these n points defines the intersection cut

$$\sum_{j \in J} \frac{1}{\lambda_j^*} x_j \geq 1. \tag{1}$$

More recently, intersection cuts became the focus of renewed interest as a result of the seminal paper by Andersen, Louveaux, Weismantel and Wolsey [3], which highlights their significance in the context of cut generation from multiple rows of the simplex tableau.

*The research of Egon Balas was supported by NSF grant CMMI-1263239 and ONR contract N000141210032.

However, this paper and the ensuing voluminous literature used a narrower definition of intersection cuts, namely as inequalities obtained by intersecting the extreme rays of $C(B)$ with the boundary of some convex set T' whose interior contains $v(B)$ but *no integer point*. Such a set T' is called *lattice-free*. This definition is more restrictive than the original one, since it excludes intersection cuts obtained from P_I-free sets that are not lattice-free, whereas the original definition includes all intersection cuts from convex lattice-free sets, as these are all P_I-free. In the sequel we will refer to intersection cuts obtained from P_I-free convex sets as standard (SIC), and to those obtained from lattice-free convex sets as restricted (RIC).

In most of the specific cases considered so far in the literature this difference does not matter, since the lattice-free sets used to generate cuts are P_I-free. This is the case with split cuts and cuts obtained by combining splits, like cuts from triangles or qualdrilaterals. But if the lattice-free set T' has a facet whose relative interior contains only infeasible integer points, then switching to a P_I-free set T larger than T' may yield a stronger cut. Furthermore, intersection cuts from a lattice-free set T', when expressed in terms of the nonbasic variables, have all their coefficients nonnegative, as is easily seen from the definition (1) of the cut. On the other hand, intersection cuts from a P_I-free set may have negative coefficients in terms of the nonbasic variables. This is easiest to see if we express the intersection cut from the P_I-free polyhedron T with facets defined by $\sum_{j \in J} d_{ij} x_j \leq d_{i0}$, $i \in Q$, as disjunctive cuts, $\delta x \geq 1$ from $\vee_{i \in Q}(\sum_{j \in J} d_{ij} x_j \geq d_{i0})$, having coefficients

$$\delta_j = \max_{i \in Q} \frac{d_{ij}}{d_{i0}}, \quad j \in J.$$

Clearly, if $d_{ij} < 0$ for all $i \in Q$, then $\delta_j < 0$. This cannot occur for a lattice-free convex set T', since in the case of the latter, the only rays that do no intersect the boundary of T' are those parallel to some facet of T', in which case they have $d_{ij} = 0$ in the inequality defining that facet.

Example. Consider the instance

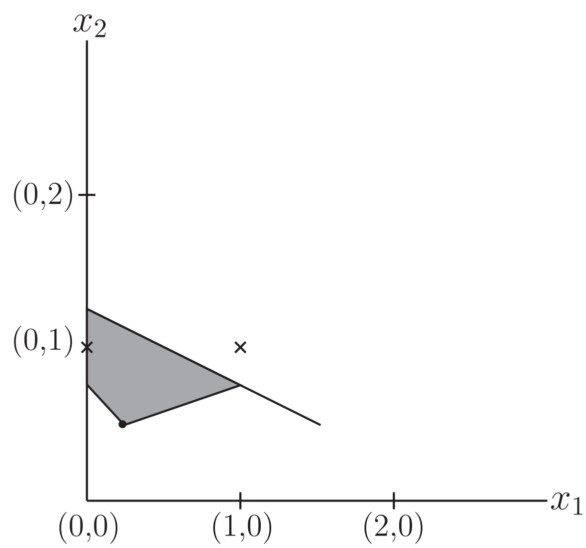

Figure 1:

$$\begin{array}{rrrrrl}
\min & x_1 & + & 2x_2 & & \\
& 4x_1 & + & 4x_2 & \geq & 3 \\
& -x_1 & + & 3x_2 & \geq & \frac{5}{4} \\
& 2x_1 & + & 4x_2 & \leq & 5 \\
& x_1, & & x_2 & \geq & 0 \quad \text{integer}
\end{array}$$

whose linear programming relaxation is the shaded area in Figure 1. The optimal LP solution is $\bar{x} = (\frac{1}{4}, \frac{2}{4})$, and the associated simplex tableau is

		x_1	x_2	s_1	s_2	s_3
x_1	$\frac{1}{4}$	1		$\frac{3}{16}$	$-\frac{1}{4}$	
x_2	$\frac{1}{2}$		1	$\frac{1}{16}$	$\frac{1}{4}$	
s_3	$\frac{5}{2}$			$-\frac{5}{8}$	$-\frac{1}{2}$	1

The intersection cut from the lattice-free triangle $T' := \{x \in \mathbb{R}_2 : x_1 \geq 0, x_2 \geq 0, x_1+x_2 \leq 2\}$, shown in Figure 2, is $(-\frac{1}{19})x_1 + x_2 \geq \frac{3}{4}$, defined by the two intersection points $\left(0, \frac{3}{4}\right)$ and $\left(\frac{19}{16}, \frac{13}{16}\right)$. But since the integer point $(1, 1)$ is infeasible, the lattice-free set T' can be replaced

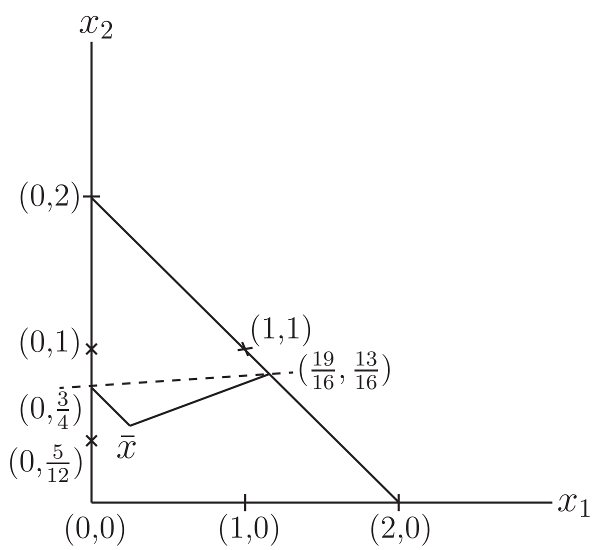

Figure 2:

with the P_I-free set $T := \{x \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}$ which yields the cut $x_1 \leq 0$, defined by the two intersection points $\left(0, \frac{3}{4}\right)$ and $\left(0, \frac{5}{12}\right)$. Note that the latter point is obtained by intersecting $\operatorname{bd} S$ with the *negative* extension of the edge $\binom{\bar{x}_1}{\bar{x}_2} + \binom{\frac{3}{16}}{\frac{1}{16}}\lambda$.

More generally, when considering the *class* of cuts from lattice-free convex sets versus the *class* of cuts from P_I-free convex sets, the two are significantly different and the latter, as it is to be expected, is considerably larger than the former and its strongest members dominate those of the former. In the context of cut-generating functions and group relaxations of mixed integer programs, several authors have considered cuts from "S-free" convex sets, where S

is some arbitrary set [6, 4]. Of course, this category includes as a special case intersection cuts from P_I-free convex sets. Nevertheless, these papers do not make the connection with the original definition of intersection cuts, which is replaced with the narrower definition of intersection cuts from lattice-free convex sets. This has led to the discovery of some basic properties of RIC's described as properties of intersection cuts, although they do not apply to intersection cuts as originally defined. In particular, an interesting feature of RIC's described in [5] is that if corner(B) denotes the corner polyhedron associated with the basis B, i.e. the convex hull of integer points in $C(B)$, then every nontrivial inequality defining a facet of corner(B) is an intersection cut. The proof uses the definition of intersection cuts from lattice-free sets, and it breaks down if we replace the lattice-free set by a P_I-free set. In other words, the result, while correct for RIC's, is not valid for SIC's. On the other hand, SIC's have a much stronger property: they define the facets of the integer hull itself:

Theorem 1. *Every facet of conv P_I that cuts off some vertex of P is defined by a standard intersection cut.*

Proof. Let F be a facet of conv P_I defined by the inequality $\varphi x \geq \varphi_0$ satisfied by all $x \in P_I$, but violated by some $x \in P$. Then F contains $\dim \operatorname{conv} P_I$ affinely independent integer points of conv P_I, and

$$\{x \in \mathbb{R}^n : \varphi x < \varphi_0\} \cap P_I = \emptyset.$$

Hence the interior of the set $T := \{x \in \mathbb{R} : \varphi x \leq \varphi_0\}$ contains no point of P_I, i.e. T is a P_I-free convex set. On the other hand, $\operatorname{int} T$ contains some vertex v of P cut off by F. Hence the standard intersection cut from $C(B(v))$, the cone associated with the basis B defining the vertex v, is precisely $\varphi x \geq \varphi_0$. □

Corollary 2. *Every vertex v of a corner polyhedron such that $v \notin$ conv P_I is cut off by some SIC.*

The authors have benefitted from helpful comments by Gérard Cornuéjols and an anonymous referee.

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