Four classes of interactions for evolutionary games

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The symmetric four-strategy games are decomposed into a linear combination of 16 basis games represented by orthogonal matrices. Among these basis games four classes can be distinguished as it is already found for the three-strategy games. The games with self-dependent (cross-dependent) payoffs are characterized by matrices consisting of uniform rows (columns). Six of 16 basis games describe coordination-type interactions among the strategy pairs and three basis games span the parameter space of the cyclic components that are analogous to the rock-paper-scissors games. In the absence of cyclic components the game is a potential game and the potential matrix is evaluated. The main features of the four classes of games are discussed separately and we illustrate some characteristic strategy distributions on a square lattice in the low noise limit if logit rule controls the strategy evolution. Analysis of the general properties indicates similar types of interactions at larger number of strategies for the symmetric matrix games.

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I. INTRODUCTION

In evolutionary games $n \times n$ payoff matrices are used to define the interactions among (equivalent) players following one of their n strategies against their coplayers defined by a connectivity network [1–3]. The systematic analysis of the games [4] and the classification of the resultant behavior are prevented by the large number of parameters (n^2) characterizing the interaction itself, particularly if $n > 3$. The first classification of the 2×2 games was suggested by Rapoport and Guyer [5] who considered the cases where the payoffs are characterized by their rank (1, 2, 3, and 4). Using this notation Liebrand [6] discussed the social dilemmas. The introduction of replicator dynamics [7] initiated a different classification based on the evolutionarily stable strategies [8–11] and phase portrait [12,13]. Very recently the games have been analyzed by distinguishing intragroup and intergroup interactions within the framework of population dynamics [14].

In recent literature of evolutionary game theory the two-strategy games are frequently characterized by four payoffs (P, R, S, and T) referring to punishment, rewards for mutual cooperation, sucker's payoff, and temptation to choose defection when the two strategies are named as defection and cooperation in the terminology of social dilemma [15,16]. In that case, however, the interaction can be well described by only two parameters (T and S) without loss of generality when the dynamics is controlled by payoff differences and if we use a suitable payoff unit by choosing $P = 0$ and $R = 1$ [17]. In the two-dimensional $S - T$ parameter space four quadrants are distinguished characterizing the harmony ($T < 1$ and $S > 0$), hawk-dove ($T > 1$ and $S > 0$), prisoner's dilemma ($T > 1$ and $S < 0$), and stag hunt ($T < 1$ and $S < 0$) games. These four types of games can be identified by the corresponding Nash equilibria. For example, in the region of prisoner's dilemma the game has only one Nash equilibrium dictating the choice of defection for both selfish players.

The existence of potential games [18,19] has raised the demand of finding another classification that allows us to distinguish clearly the potential games within the set of matrix (or normal) games. In a previous paper [20] we have shown that all the symmetric matrix games for $n = 2$ and 3 can be decomposed into the linear combinations of elementary games represented by orthogonal basis matrices. More precisely, the payoff matrices are built up from their two-dimensional Fourier components for both $n = 2$ and 3. Due to the general features of the Fourier series expansion the strength of each component can be evaluated straightforwardly. For the symmetric two-strategy games ($n = 2$) the first Fourier component can be interpreted as an irrelevant game where the equivalent payoffs eliminate the essence of games. The linear combinations of the first and second components represent games with self-dependent payoffs where the player's income is independent of the opponent's strategy. Similarly, the linear combinations of the first and third components describe games with cross-dependent payoffs when the player's income depends only on the coplayer's strategy. The direct interactions between the players are quantified by the fourth term resembling the coordination-type (or anticoordination-type) interactions on the analogy of the ferromagnetic (or antiferromagnetic) Ising model [21] with spins oriented upward or downward. In fact, these are the reasons why the multiagent games can be mapped onto an Ising-type model if the interactions among the players are described by symmetric two-strategy games. All these games are potential games that evolve into the Boltzmann distribution [19] if the strategy reversals are controlled by the so-called logit rule resembling the Glauber dynamics for the kinetic Ising model [22].

Similar concepts of decomposition were suggested previously in Refs. [23,24] without the introduction of a concrete set of basis games. The introduction of a suitable set of the orthogonal basis games, however, gives us a more sophisticated

knowledge on the anatomy of matrix games. For example, the games with self- and cross-dependent payoffs are represented by the linear combinations of three orthogonal basis games characterized by matrices with uniform elements in columns and rows for $n = 3$. Evidently, the basis game with uniform matrix elements belongs to both types of the latter classes. For $n = 3$, the subset of the self- and cross-dependent games can be defined as the linear combination of $(2n - 1) = 5$ basis matrices. Additionally, three of the nine components describe games with symmetric payoff matrices that are the linear combination of coordination-type (or anticoordination-type) interactions for the three possible strategy pairs. The ninth orthogonal component corresponds to the traditional rock-paper-scissors game, which is a zero-sum game with an antisymmetric payoff matrix, and the presence of this component prevents the existence of potential.

Now we show that the above-mentioned general features are inherited for the symmetric n-strategy games and the "dimension" of the four classes of elementary games increases with the number of strategies. Because many relevant questions are related to the existence of potential, the next section is addressed to quantify the necessary conditions. In Sec. III we show how the payoff matrix can be built up as the linear combination of orthogonal basis matrices representing elementary games for $n = 4$. The application of the Walsh-Hadamard matrices [25] simplifies the calculations and invokes the theory of directed graphs for the graphical illustration of the inherent structure of symmetric matrix games. The general properties of the four classes of elementary games are discussed separately for the level of pair interactions and also for the spatial version of multiagent evolutionary games in the consecutive sections. The summary of this analysis implies discussions of those features that may remain valid for larger number of strategies for the symmetric games.

II. EXISTENCE OF POTENTIAL IN SYMMETRIC FOUR-STRATEGY GAMES

The symmetric matrix games are used to describe quantitatively the pair interactions between two equivalent players (x and y) who have four options to choose independently of each other. In the mathematical notation of game theory these (pure) strategies are denoted by the traditional unit vectors of a four-dimensional vector space as

$$\mathbf{s}_x = \mathbf{s}_y = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \tag{1}$$

The payoffs for both players depend on the strategies they choose and are expressed by products as

$$u_x = \mathbf{s}_x \cdot \mathbf{A}\mathbf{s}_y \quad \text{and} \quad u_y = \mathbf{s}_y \cdot \mathbf{A}\mathbf{s}_x, \tag{2}$$

where the element A_{ij} of the payoff matrix $\mathbf{A}$ defines the payoff for the first player if she chooses her ith strategy, whereas the coplayer selected the jth strategy ($i, j = 1,2,3,4$).

The present symmetric two-person game is a potential game [18,19] if we can introduce a symmetric 4×4 potential matrix

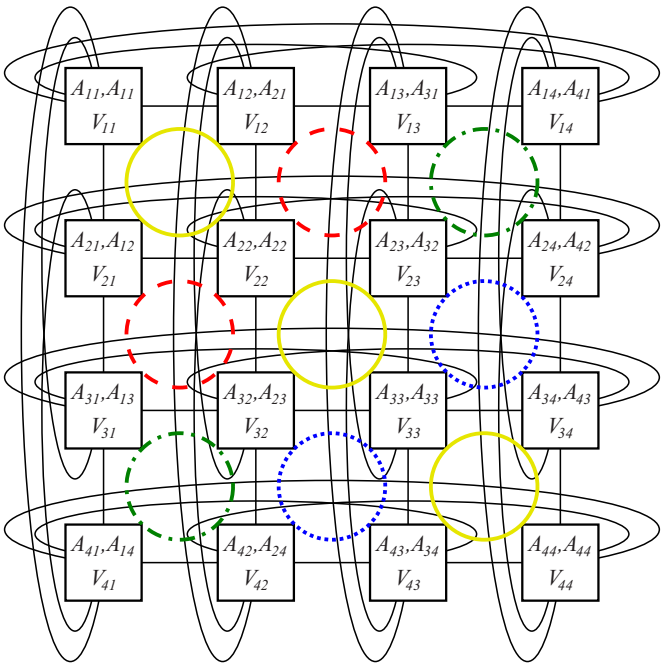

FIG. 1. (Color online) Dynamical graph with the independent and relevant loops indicated by dashed (red), dotted (blue), and dashed-dotted (green) circles. Along the main diagonal the solid (yellow) circles refer to symmetric 2×2 games.

$\mathbf{V}$ that satisfies the following conditions:

$$V_{kj} - V_{ij} = A_{kj} - A_{ij}, \tag{3}$$

that is, for unilateral strategy modification the potential variation is equivalent to the payoff variation of the active player. The above equation expresses the case when the first player modifies her strategy from the ith to the kth while the second player uses her jth strategy. Similar requirements should be satisfied when only the second player changes her strategy. However, for the symmetric two-player games the latter condition is satisfied if the potential matrix $\mathbf{V}$ is symmetric ($V_{ij} = V_{ji}$).

The potential $\mathbf{V}$ exists if the sum of the mentioned payoff variations of the active player is zero along all the closed trajectories in the space of strategy profiles where only unilateral changes are allowed. The large number of possible loops is illustrated in Fig. 1, which shows the dynamical graph for the present four-strategy game [26]. In this dynamical graph the nodes represent strategy profiles (microscopic states) and the edges connect those strategy profiles that can be transformed into each other if only one of the players modifies her strategy.

In Fig. 1 the nodes are arranged in the same order as they appear within the payoff and potential matrices. Here the nodes are denoted by large boxes allowing as to give the payoffs (upper row) for both players and also the value of potential (lower row) for all the strategy profiles. This arrangement of nodes reflects the relevant symmetries. Notice that each subgraph, consisting of nodes within a single row or column, is complete and within these subgraphs the condition of the existence of potential is satisfied because here only one player changes her strategy.

According to the Kirchhoff laws [27] we can distinguish nine independent loops (see Fig. 1) if we apply the methods used in the analysis of electric circuits [20,28]. Three of the nine loops are located along the main diagonal and each one represents a symmetric 2×2 subgame where the potential always exists. In the present system the dashed-dotted (green) circles represent an antisymmetric pair of loops that give the same condition for the existence of the potential, namely,

$$A_{41} - A_{31} + A_{24} - A_{14} + A_{32} - A_{42} + A_{13} - A_{23} = 0. \tag{4}$$

Two additional conditions can be derived from the other two pairs of antisymmetric loops. Namely, the conditions along the (blue) dotted circles in Fig. 1 correspond to

$$A_{42} - A_{32} + A_{34} - A_{24} + A_{33} - A_{43} + A_{23} - A_{33} = 0, \tag{5}$$

which can be simplified as

$$A_{23} - A_{32} + A_{34} - A_{43} + A_{42} - A_{24} = 0. \tag{6}$$

Similarly, the third condition is related to the four-edge loops indicated by dashed (red) circles in Fig. 1:

$$A_{31} - A_{21} + A_{23} - A_{13} + A_{22} - A_{32} + A_{12} - A_{22} = 0, \tag{7}$$

which obeys the following form after some algebraic manipulation:

$$A_{12} - A_{21} + A_{23} - A_{32} + A_{31} - A_{13} = 0. \tag{8}$$

In sum, the potential exists if the matrix components A_{ij} satisfy the above three conditions defined by Eqs. (4)–(7). It is emphasized that in the deduction of these three criteria we have exploited the symmetries and the interdependence of four-edge loops within the dynamical graphs.

This method predicts $(n-1)(n-2)/2$ independent and relevant pairs of four-edge loops that should be taken into consideration when deriving similar conditions for the existence of potential when $n > 3$. Monderer and Shapley [18] and Hofbauer and Sigmund [12] have proved the existence of potential if similar conditions are satisfied for all the four-edge loops or the equivalent 2×2 subgames.

The potential matrix $\mathbf{V}$ can be evaluated as detailed below and the actual value of the potential matrix for a given strategy pair $(\mathbf{s}_x, \mathbf{s}_y)$ can be expressed as $\mathbf{s}_x \cdot \mathbf{V}\mathbf{s}_y$.

Up to now we have studied symmetric two-player games. Due to the linear relationship between the payoff and potential matrices we can introduce multiagent potential games with N players if the interactions between the players are composed of equivalent two-player potential games. In these systems the microscopic state (strategy profile) is defined by the set of individual strategies, $\mathbf{S} = (\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_N)$ and the corresponding potential value is obtained as

$$U(\mathbf{S}) = \sum_{\langle x,y \rangle} \mathbf{s}_x \cdot \mathbf{V}\mathbf{s}_y, \tag{9}$$

where the summation runs over the interacting players x and y.

For the illustration of the effect of some types of interactions on the macroscopic behavior in multiagent systems, we will consider models with nearest-neighbor interactions between the players distributed on a square lattice. If the evolution of the strategy profile $\mathbf{S}$ is defined by random sequential application of the logit rule, then these systems will evolve into the Boltzmann distribution [19]. For an elementary step of this evolutionary process we choose a player (e.g., x) at random and this player is allowed to select another strategy $\mathbf{s}'_x$ favoring exponentially her higher individual payoff. For the potential games this preference can be quantified by the potential variation for unilateral strategy changes. More precisely, the probability of choosing strategy $\mathbf{s}'_x$ is expressed as

$$w(\mathbf{s}'_x) = \frac{e^{U(\mathbf{S}')/K}}{\sum_{\mathbf{S}''} e^{U(\mathbf{S}'')/K}}, \tag{10}$$

where in the microscopic states $\mathbf{S}'$ and $\mathbf{S}''$ the player at site x chooses $\mathbf{s}_x = \mathbf{s}'_x$ and $\mathbf{s}_x = \mathbf{s}''_x$, respectively, and $\mathbf{s}''_x$ runs over all the possible strategies while the strategies are fixed for all the other players. For the logit rule, K quantifies the strength of the stochastic noises. In the limit $K \to 0$, the players choose their best strategy and the system develops into one of the pure Nash equilibria characterized by the maximum value of $U(\mathbf{S})$.

In order to demonstrate the richness in the stationary states and also the close analogy to physical systems when the interaction belongs to the coordination-type games, the mentioned system will be investigated by Monte Carlo (MC) simulations on a square lattice with $L \times L$ sites under periodic boundary conditions. During these simulations we have determined the average values of strategy frequencies (ϱ_k, $k = 1, \ldots, 4$) in the stationary states. The linear system size is varied from $L = 400$ to 1400, the relaxation and sampling times are chosen between $t_r = t_s = 10^4$ and 10^6 MCS (during the time unit 1 MCS each player has a chance to modify her own strategy once on average). The larger system sizes and longer run times are selected when approaching the critical transition point in order to increase the statistical accuracy.

III. DECOMPOSITION OF THE SYMMETRIC FOUR-STRATEGY MATRIX GAMES

The idea of the matrix decomposition is based on the fact that a matrix $\mathbf{A}$ of rank n can be considered as a traditional vector of dimension n^2 if the components A_{ij} are arranged into a column. The two-dimensional Fourier decomposition worked efficiently for three strategies ($n = 3$). Here we suggest a different approach. On the analogy to the traditional vector calculus now the payoff matrices are built up as a linear combination of basis matrices that are created from a set of four-dimensional orthogonal basis vectors. For later convenience we choose the following four orthogonal vectors composed of $+1$ and -1 as:

$$\mathbf{e}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{e}^{(2)} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad \mathbf{e}^{(3)} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{e}^{(4)} = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \tag{11}$$

and the dyadic (or tensor) products of these vectors,

$$\mathbf{g}(m) = \mathbf{e}^{(k)} \otimes \mathbf{e}^{(l)}, \tag{12}$$

with elements $g_{ij}(m) = e_i^{(k)} e_j^{(l)}$ for $k,l = 1,\ldots,4$, serve as basis matrices (or elementary games) with labels ($m = 1,2,\ldots,16$) specified below. These basis matrices satisfy the conditions of orthogonality,

$$\sum_{i,j} g_{ij}(m) g_{ij}(m') = \begin{cases} 0, & \text{if } m \neq m', \\ C(m), & \text{if } m = m', \end{cases} \tag{13}$$

where $C(m) = 16$ as $|g_{ij}| = 1$ in the present case. In general, the payoff matrix can be expressed [25] as

$$\mathbf{A} = \sum_{m=1}^{16} \frac{\alpha(m)}{C(m)} \mathbf{g}(m), \tag{14}$$

where the coefficients $\alpha(m)$ are given by the scalar product of the matrices $\mathbf{A}$ and $\mathbf{g}(m)$,

$$\alpha(m) = \mathbf{A} \cdot \mathbf{g}(m) = \sum_{i,j} A_{ij} g_{ij}(m), \tag{15}$$

which is defined on the analogy of the scalar product of two linear vectors.

IV. GAMES WITH SELF- AND CROSS-DEPENDENT PAYOFFS

Following our previous notation [20] the first basis matrix (labeled with $m = 1$) is defined as

$$\mathbf{g}(1) = \mathbf{e}^{(1)} \otimes \mathbf{e}^{(1)}, \tag{16}$$

with elements $g_{ij}(1) = 1$. This all-ones matrix represents the irrelevant component of payoffs for all cases when the decision or dynamics are controlled by payoff differences. There are three additional basis matrices, namely,

$$\mathbf{g}(2) = \mathbf{e}^{(1)} \otimes \mathbf{e}^{(2)}, \tag{17}$$

$$\mathbf{g}(3) = \mathbf{e}^{(1)} \otimes \mathbf{e}^{(3)}, \tag{18}$$

$$\mathbf{g}(4) = \mathbf{e}^{(1)} \otimes \mathbf{e}^{(4)}, \tag{19}$$

which consist of columns with uniform values. The latter property is conserved for the following linear combinations:

$$\mathbf{A}^{(\text{cross})} = \sum_{m=1}^{4} \alpha'(m) \mathbf{g}(m), \tag{20}$$

representing the subset of games with cross-dependent payoffs. For this set of games the players cannot modify their own payoffs by choosing another strategy. Consequently, the games with cross-dependent payoffs do not give contributions to the potential matrix. If this type of games defines the pair interactions in a multiagent model for a logit rule then the players choose their strategy at random.

In opposition to the cross-dependent payoffs we can distinguish games with self-dependent payoffs when the payoff matrices are composed of uniform rows. The corresponding elementary games are defined as

$$\mathbf{g}(5) = \mathbf{e}^{(2)} \otimes \mathbf{e}^{(1)}, \tag{21}$$

$$\mathbf{g}(6) = \mathbf{e}^{(3)} \otimes \mathbf{e}^{(1)}, \tag{22}$$

$$\mathbf{g}(7) = \mathbf{e}^{(4)} \otimes \mathbf{e}^{(1)}, \tag{23}$$

and the subset of games with self-dependent payoff can be given as

$$\mathbf{A}^{(\text{self})} = \alpha'(1)\mathbf{g}(1) + \sum_{m=5}^{7} \alpha'(m)\mathbf{g}(m). \tag{24}$$

The reader can easily check that the following potential matrix,

$$\mathbf{V}^{(\text{self})} = \sum_{m=5}^{7} \alpha'(m)[\mathbf{g}(m) + \mathbf{g}^T(m)], \tag{25}$$

satisfies the condition Eq. (3) for the games with self-dependent payoffs.

Notice that $\mathbf{g}(m) = \mathbf{g}^T(m+3)$ (if $m = 2,3,4$) and this feature can be exploited by introducing another set of basis matrices ($\mathbf{g}'(m)$, with $m = 2,3,\ldots,6$) when we distinguish symmetric and antisymmetric basis matrices as

$$\mathbf{g}'(m = r) = \tfrac{1}{2}[\mathbf{e}^{(r)} \otimes \mathbf{e}^{(1)} + \mathbf{e}^{(1)} \otimes \mathbf{e}^{(r)}], \tag{26}$$

$$\mathbf{g}'(m = 3 + r) = \tfrac{1}{2}[\mathbf{e}^{(r)} \otimes \mathbf{e}^{(1)} - \mathbf{e}^{(1)} \otimes \mathbf{e}^{(r)}], \tag{27}$$

where $r = 2,3$, and 4. Evidently, the above basis matrices preserve the conditions of orthogonality. Accordingly, the games with self- and cross-dependent payoffs are spanned by the linear combinations of four symmetric and three antisymmetric basis matrices.

The relevant properties of the above antisymmetric basis matrices can be illustrated by

$$\mathbf{g}'(5) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix}, \tag{28}$$

which possesses two pairs of identical columns and rows. The mentioned symmetries characterize $\mathbf{g}'(6)$ and $\mathbf{g}'(7)$, as well.

V. COORDINATION GAMES

Among the 16 dyadic products of the vector Eqs. (11) there are three symmetric matrices that we use to define the following three basis matrices as

$$\mathbf{g}(8) = \mathbf{e}^{(2)} \otimes \mathbf{e}^{(2)}, \tag{29}$$

$$\mathbf{g}(9) = \mathbf{e}^{(3)} \otimes \mathbf{e}^{(3)}, \tag{30}$$

$$\mathbf{g}(10) = \mathbf{e}^{(4)} \otimes \mathbf{e}^{(4)}. \tag{31}$$

Notice that these three basis games are composed of only $+1$s and -1s in a way ensuring their orthogonality to $\mathbf{g}(m)$ for $m = 1,\ldots,7$ as the sum of payoffs is zero within each row

and column. For example,

$$\mathbf{g}(8) = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix}. \tag{32}$$

From the rest of nonsymmetric dyadic products we can derive three additional symmetric basis vectors as

$$\mathbf{g}(11) = \tfrac{1}{2}[\mathbf{e}^{(2)} \otimes \mathbf{e}^{(3)} + \mathbf{e}^{(3)} \otimes \mathbf{e}^{(2)}], \tag{33}$$

$$\mathbf{g}(12) = \tfrac{1}{2}[\mathbf{e}^{(3)} \otimes \mathbf{e}^{(4)} + \mathbf{e}^{(4)} \otimes \mathbf{e}^{(3)}], \tag{34}$$

$$\mathbf{g}(13) = \tfrac{1}{2}[\mathbf{e}^{(4)} \otimes \mathbf{e}^{(2)} + \mathbf{e}^{(2)} \otimes \mathbf{e}^{(4)}], \tag{35}$$

because $(\mathbf{e}^{(k)} \otimes \mathbf{e}^{(l)})^T = \mathbf{e}^{(l)} \otimes \mathbf{e}^{(k)}$. As all these basis matrices are symmetric, the contribution of their arbitrary linear combinations,

$$\mathbf{A}^{(\mathrm{coord})} = \sum_{m=8}^{13} \alpha(m)\mathbf{g}(m), \tag{36}$$

to the potential matrix is $\mathbf{V}^{(\mathrm{coord})} = \mathbf{A}^{(\mathrm{coord})}$. This six-dimensional subspace of matrices is closed under transformations when exchanging the same two rows and columns subsequently. The latter transformations realize the exchange of labels simultaneously without introducing fundamentally new behaviors.

The knowledge of the potential matrix $\mathbf{V}$ can be utilized to determine the preferred Nash equilibrium for both the two-player games and the spatial multiagent evolutionary games. For example, if $\max(V_{ij}) = V_{kk}$ then the strategy pair (k,k) is a Nash equilibrium for the two-player game and the homogeneous distribution of the kth strategy is a stable state of the evolutionary games (introduced in Sec. II) in the limit $K \to 0$. In such systems the K-dependence of the strategy frequencies have a typical behavior plotted in Fig. 2. The plotted MC data are obtained for a system where $\beta(m) = 1/m$ if $8 \leqslant m \leqslant 12$ and $\beta(m) = 0$ otherwise. In this model the homogeneous distribution of strategy 1 dominates the system behavior in the low noise limit.

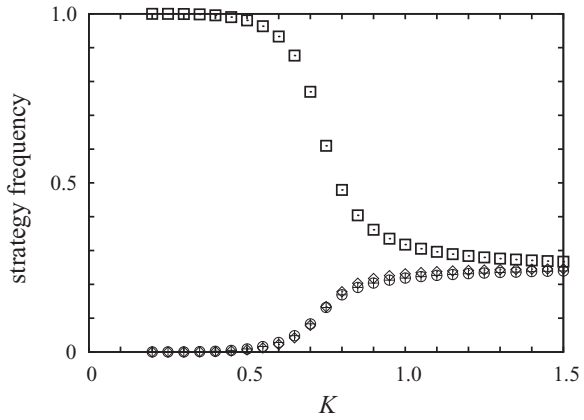

FIG. 2. Typical K-dependence of strategy frequencies for evolutionary games on the square lattice where the pair interactions are composed of $\mathbf{f}^{(pq)}$ with strength chosen at random. The MC data are denoted by open squares, diamonds, circles, and pluses for the strategies 1, 2, 3, and 4, respectively.

As $\mathbf{V}$ is a symmetric matrix, its maximum values can occur in pair, e.g., $\max(V_{ij}) = V_{kl} = V_{lk}$. In the latter cases the two-player game has two equivalent pure Nash equilibria, namely the strategy pairs (k,l) and (l,k). For the multiagent evolutionary games the system has two equivalent ordered strategy arrangements in the low noise limit on a square lattice that can be divided into two sublattices (denoted as X and Y) on the analogy of the white and black boxes of the checkerboard. For the zero noise limit, the players choose strategy k in one of the sublattices while they follow strategy l within the opposite sublattice. This situation resembles the antiferromagnetic spin arrangements for the Ising-type models [21].

It is noteworthy that if the players in one of the two sublattices exchange their strategy labels l and k (this transformation is realized by exchanging the corresponding two columns or rows in the payoff matrix), then the resultant system has two equivalent homogeneous ordered states as it occurs in the ferromagnetic Ising model in the absence of external magnetic field. The mentioned transformations can be used to justify similar K-dependence (including phase transition(s), thermodynamic derivatives, responses to perturbations, etc.) in a group of systems satisfying some symmetries. Similar phenomena are illustrated in the three-strategy potential games [20] and are expected to be present in systems of larger number of strategies.

Within this six-dimensional subspace of matrix games one can find directions realizing clearly the coordination type 2×2 subgames with payoff matrices $\mathbf{f}^{(pq)}$ ($p < q = 2,3$, and 4) if both players are constrained to choose either their pth or qth strategies. The components of matrices $\mathbf{f}^{(pq)}$ with attractive Ising type (or coordination type) interactions between the strategy pair (p,q) can be defined as

$$f_{ij}^{(pq)} = \begin{cases} -1, & \text{if } i = p \text{ and } j = q, \text{ or } i = q \text{ and } j = p, \\ 0, & \text{if } i,j \neq p,q, \\ 1, & \text{if } i = j = p \text{ or } i = j = q. \end{cases} \tag{37}$$

These matrices contain two rows and columns composed of 0s and each one can be obtained from

$$\mathbf{f}^{(12)} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{38}$$

by exchanging the same two rows and columns subsequently. The above six matrices $\mathbf{f}^{(pq)}$ are not orthogonal to each other in the sense defined by Eq. (13). At the same time these components span the whole subspace of coordination-type games. In fact, each $\mathbf{f}^{(pq)}$ can be expressed as a linear combination of three of six $\mathbf{g}(m)$ ($m = 8, \ldots, 13$) basis matrices. For example,

$$\mathbf{f}^{(12)} = \tfrac{1}{4}[\mathbf{g}(9) + \mathbf{g}(10) + \mathbf{g}(13)]. \tag{39}$$

In light of the above feature the coordination type interactions can be considered as the linear combinations of symmetric two-strategy subgames where the strength of coordination is defined for each symmetric strategy pair. This set of games includes cases when some of the $\mathbf{f}^{(pq)}$ basis

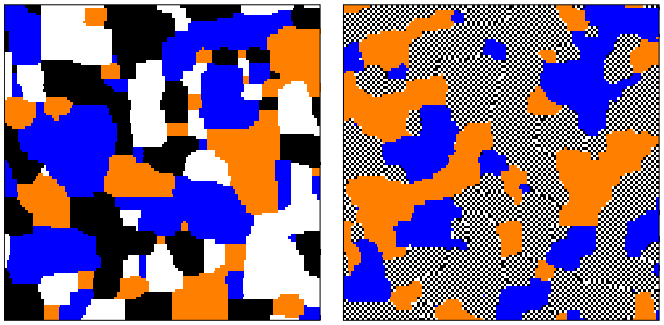

FIG. 3. (Color online) Two snapshots illustrate strategy distributions during the domain growing process on a square lattice at $K \simeq K_c/2$ if the interaction is described by the four-state Potts model (left) and by one of its relatives (right) obtained by exchanging the third and fourth rows of the payoff matrix.

games can be present with negative weight factors that refer to anticoordination-type interactions. The anticoordination interactions enforce sublattice ordering when the players on sublattice X follow the first strategy and they choose the opposite one within the sublattice Y. There exists an other equivalent sublattice ordered state where the strategies are exchanged.

In the literature of physics the most frequently investigated system within this subset of games is the four-state Potts model (for a survey see [29]) that represents a universality class of critical phase transitions [30]. The Potts models [31] were introduced to study systems with n equivalent (homogeneous) ordered states that are transformed to a disordered strategy distribution above a critical noise level ($K > K_c$). The resultant order-disorder transition is continuous and the frequency ϱ_i of states i converges algebraically to $1/4$ when approaching the critical point K_c. More precisely, $|\varrho_i - 1/4| \propto (K_c - K)^\beta$ where $\beta = 1/12$ and $(K_c - K) \to +0$. In the present notations the four-state Potts model is composed of all the $\mathbf{f}^{(pq)}$ matrices with equal weight factors.

For the two-state magnetic Ising model the equivalence between the ferromagnetic and antiferromagnetic ordering phenomena on the square lattice is related to the fact that the spin reversal on one of the sublattices transforms the attractive interactions into repulsive ones. Here $\mathbf{f}^{12}$ is transformed into $-\mathbf{f}^{12}$ if strategy labels 1 and 2 are exchanged on one of the sublattices. Evidently, a similar exchange of a strategy pair on one of the sublattices for the four-state Potts model creates an additional set of models exhibiting equivalent order-disorder phase transitions. The analogous behavior of these models is illustrated in Fig. 3 that draws a parallel between the snapshots obtained during the domain growing for the four-state Potts model and for one of its relatives. Similar equivalence between the three-state Potts model and its relatives was reported and discussed in detail for the evolutionary three-strategy spatial games [20].

The above-mentioned family members of the four-state Potts model are located along disjunct directions (half-lines) within the six-dimensional subspace of the coordination-type games. Within this subset of games, however, there exist many other combinations of elementary coordination games that exhibit different symmetries and order-disorder phase

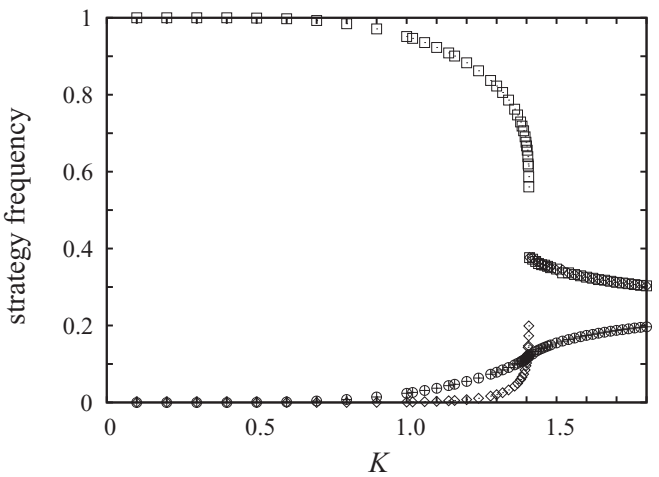

FIG. 4. Strategy frequencies as a function of noise K for evolutionary games with pair interactions $\mathbf{f}^{(12)}$ on the square lattice. The MC data are denoted by the same symbols used in Fig. 2.

transitions. For example, the game represented by the matrices $\mathbf{f}^{(12)}$ perform an Ising-type order-disorder phase transition as it is shown in Fig. 4. The numerical results show an Ising-type order-disorder critical transition for the strategy frequencies ϱ_1 and ϱ_2, whereas $\varrho_3 = \varrho_4$ vary smoothly from 0 to $1/4$ when K is increased. Our numerical data are consistent with the theoretical expectation predicting $(\varrho_1 - \varrho_2) \simeq (K_c - K)^\beta$ with $\beta = 1/8$ if $(K_c - K) \to +0$ for $K_c = 1.4077(1)$. Evidently, similar K-dependence of the strategy frequencies occur within the sublattices X and Y when the pair interactions are given by $-\mathbf{f}^{(12)}$.

On the analogy of the above model, Fig. 5 shows another universal critical transition occurring when the pair interaction is defined by the matrix $\mathbf{A} = \mathbf{f}^{(12)} + \mathbf{f}^{(23)} + \mathbf{f}^{(13)}$. In that case the corresponding potential matrix has three equivalent maximal values ($V_{11} = V_{22} = V_{33}$) that prescribes the existence of three (equivalent) homogeneous ordered states in the limit

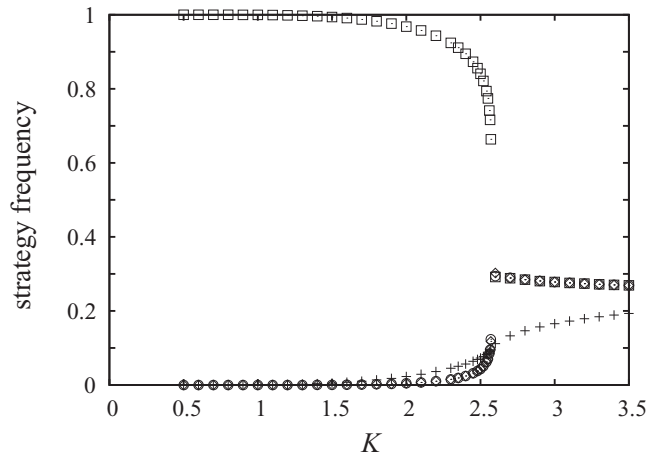

FIG. 5. Monte Carlo data for the strategy frequencies versus noise on the square lattice when the pair interaction is defined as $\mathbf{A} = \mathbf{f}^{(12)} + \mathbf{f}^{(23)} + \mathbf{f}^{(13)}$. Symbols agree with those used in Fig. 2 (here the circles and diamonds coincide).

$K \to 0$ on the square lattice. The MC results show that ϱ_4 increases smoothly with K from 0 to 1/4 and $\varrho_2 = \varrho_3$ if $\varrho_1 \to 1$ in the limit $K \to 0$.

The preliminary MC simulations indicate more complex behavior for a payoff matrix $\mathbf{A} = \mathbf{f}^{(12)} + \mathbf{f}^{(34)}$ that ensures four equivalent ordered strategy arrangements in the zero-noise limit. Evidently, the latter system is equivalent to those defined, for example, by $\mathbf{A} = \mathbf{f}^{(13)} + \mathbf{f}^{(24)}$. Similar richness in the behavior and phase diagrams is reported for the Ashkin-Teller model [32] and for other systems exhibiting fourfold degenerated ground states [33].

VI. CYCLIC GAMES

In this section we discuss the last three basis games characterized by the following antisymmetric matrices:

$$\mathbf{g}(14) = \tfrac{1}{2}[\mathbf{e}^{(2)} \otimes \mathbf{e}^{(3)} - \mathbf{e}^{(3)} \otimes \mathbf{e}^{(2)}], \tag{40}$$

$$\mathbf{g}(15) = \tfrac{1}{2}[\mathbf{e}^{(3)} \otimes \mathbf{e}^{(4)} - \mathbf{e}^{(4)} \otimes \mathbf{e}^{(3)}], \tag{41}$$

$$\mathbf{g}(16) = \tfrac{1}{2}[\mathbf{e}^{(4)} \otimes \mathbf{e}^{(2)} - \mathbf{e}^{(2)} \otimes \mathbf{e}^{(4)}], \tag{42}$$

where, for example,

$$\mathbf{g}(16) = \begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \tag{43}$$

illustrates their general properties. First we emphasize that $\mathbf{g}(16)$ can be interpreted as a straightforward extension of the rock-paper-scissors game. Here the four strategies cyclically dominate each other, which can be illustrated by the directed graph c in Fig. 6 because its adjacency matrix is identical to $\mathbf{g}(16)$. In graph theory [34], the simple directed graphs with n nodes are characterized by an $n \times n$ adjacency matrix $\mathbf{C}$ where $C_{ij} = 0$ if the nodes i and j are not connected and $C_{ij} = -C_{ji} = 1$ if there exists a directed edge from node i to j. Figure 6 shows three directed graphs representing Hamilton cycles (directed loops including all edges). In fact, there exist three additional Hamilton cycles that can be obtained by reversing all edge directions simultaneously and are described by the matrices $-\mathbf{g}(m)$ for $m = 14, 15$, and 16. Notice that the eight linear combinations of the cyclic basis games, namely,

$$\mathbf{e} = \pm\mathbf{g}(14) \pm \mathbf{g}(15) \pm \mathbf{g}(16), \tag{44}$$

define rock-paper-scissors-type three-strategy subgames when the use of one of the four strategies is prohibited. The corresponding directed graphs are given by the eight (possible) directed three-edge loops with one isolated nodes.

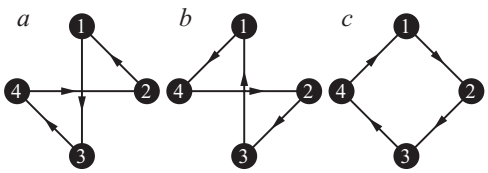

FIG. 6. The three cyclic basis games, Eqs. (40)–(42), are defined by adjacency matrices of the directed graphs a, b, and c, respectively.

Notice that all antisymmetric $n \times n$ matrices can be described as combinations of the adjacency matrices of the possible directed graphs of n nodes with only a single directed edge. Within this subset of matrices the cyclic games are orthogonal to both the self- and cross-dependent matrices that restrict the analysis to those directed graphs where the numbers of outgoing and ingoing edges are equivalent for each node. The latter requirements are satisfied for graphs with a directed loop and also for those that are composed of directed loops without common edges. The above mentioned three- and four-edge directed loops represent graphically some inherent properties of the cyclic basis games for $n = 4$.

The presence of cyclic basis games $[\beta(14), \beta(15), \beta(16) \neq 0]$ in a payoff matrix $\mathbf{A}$ prevents the existence of potential because for each cyclic component there exist closed trajectories in the space of strategy profiles where the preferred directions form directed loops, therefore the Kirchhoff laws cannot be satisfied. Conversely, potential exists if $\beta(14) = \beta(15) = \beta(16) = 0$. The latter condition coincides with the criteria given by Eqs. (4), (5), and (7). More precisely, the criterium Eq. (4) is equivalent to the orthogonality condition $\beta(16) = \mathbf{A} \cdot \mathbf{g}(16) = 0$ as defined by Eq. (15). Additionally, the condition of $\beta(16) = 0$ can be interpreted as the vanishing of the sum of payoff variations of the active player along the four-state loops $(1,3) \to (1,4) \to (2,4) \to (2,3) \to (1,3)$ or $(3,1) \to (3,2) \to (4,2) \to (4,1) \to (2,1)$ within the dynamical graph. Similar relationships can be deduced for the other two terms dictating $(\beta(14) = \beta(15) = 0)$.

In fact, for the symmetric four-strategy games we can distinguish three pairs of four-state loops where the first player uses either strategy i or j, whereas the second player can select one of the other two strategies, namely, either strategy k or l (for the above discussed situation $i = 1$, $j = 2$, $k = 3$, and $l = 4$). In these cases the two players use two different pairs of strategies.

Among the four-state loops of the dynamical graph (see Fig. 1) we can distinguish two additional classes. The first class includes those loops where both players are constrained to use the same strategy pair. Three of the possible six symmetric 2×2 subgames are indicated by solid (yellow) circles in Fig. 1. Along these loops the symmetry of the game ensures that the sum of payoff variations vanishes as it happens to the symmetric two-strategy games [20].

Within the second class of loops the players have one common strategy and the second ones are distinct. The investigation of these cases can be mapped onto the analysis of three-strategy games where potential can exist in the absence of the corresponding rock-paper-scissors component that can be built up as a suitable linear combination of the three four-state cyclic basis games mentioned above. More precisely, Eq. (6) is equivalent to the orthogonality condition $\mathbf{A} \cdot [\mathbf{g}(14) + \mathbf{g}(15) + \mathbf{g}(16)] = 0$ where the second term describes rock-paper-scissors-type cyclic dominance between the strategies 2, 3, and 4.

The three cyclic basis games [$\mathbf{g}(14)$, $\mathbf{g}(15)$, and $\mathbf{g}(16)$] can be mapped onto each other by relabeling the strategies. This is the reason why the corresponding games exhibit similar behaviors resembling those observed for the rock-paper-scissors games on square lattice at different evolutionary rules. Due to the "cyclic" symmetries for all the three basis

games the four strategies are present with the same frequency [35–39]. The strategy frequencies can be tuned by varying the strength of these components [40–42]. In the spatial systems the small domains invades each other cyclically along irregular interfaces. The size of domains can be enhanced by introducing additional coordination-type interactions [43].

VII. GENERALIZATION FOR SYMMETRIC n-STRATEGY GAMES

Most of the above features remain valid for all symmetric matrix games with $n > 4$ strategies. Namely, the all-ones matrix as well as the basis games with self- and cross-dependent payoffs exhibit similar properties to those described in Sec. IV. Within this [$(2n-1)$-dimensional] parameter space of games there is no direct interactions between the players. The potential is determined by the self-dependent components on the analogy of Eq. (25) and we can distinguish $(n-1)$ antisymmetric basis matrices.

The parameter space of the coordination-type interactions is spanned by $n(n-1)/2$ symmetric $\mathbf{f}^{(pq)}$ matrices defined on the analogy of Eq. (37) for $p < q = 2,\ldots,n$. The space of the whole symmetric $n \times n$ matrices is spanned by the linear combination of the coordination-type games and by the n symmetric matrices derived from the self- and cross-dependent components as it happened for $n = 4$. The rest of the $(n-1)(n-2)/2$-dimensional subspace of the antisymmetric matrices involves the linear combination of the four-strategy cyclic subgames. The latter cyclic components can be derived from $\mathbf{g}(16)$ by adding all-zeros column(s) and row(s), as represented by the matrix

$$\mathbf{e}^{(c-1234)} = \begin{pmatrix} 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{45}$$

for $n = 5$, which is the adjacency matrix of a directed graph with one directed four-edge loop (through the strategies 1, 2, 3, and 4) and one isolated node (representing strategy 5). On the analogy to $\mathbf{e}^{(c-1234)}$ we can introduce many other cyclic games (e.g., $\mathbf{e}^{(c-1234)}$) for four of n strategies that are orthogonal to all $\mathbf{f}^{(pq)}$ as well as to $\mathbf{A}^{(\mathrm{self})}$ and $\mathbf{A}^{(\mathrm{cross})}$. The relevance of the basis matrices $\mathbf{e}^{(c-ijkl)}$ is justified by the fact that the scalar products $\mathbf{A} \cdot \mathbf{e}^{(c-1234)} = 0$ is equivalent to the general conditions required for the existence of potential (see Refs. [12,18,20,24]). More precisely, the condition $\mathbf{A} \cdot \mathbf{e}^{(c-ijkl)} = 0$ ensures that the Kirchhoff law is satisfied along the four-edge loop $(i,j) \to (k,j) \to (k,l) \to (i,l) \to (i,j)$ in the space of pure strategy profiles.

Among the general properties of the $n \times n$ symmetric matrix games we have to underline the importance of the symmetric components of the payoff matrix $\mathbf{A}$. Collaborating players can agree to share the accumulated payoff equally as it happens between fraternal players, friends, or family members [1,17,44–46]. In that case the effective payoff matrix $\mathbf{A}^{(\mathrm{eff})} = (\mathbf{A} + \mathbf{A}^T)/2$ does not contain the contribution of the antisymmetric components. In other words, the decision of the collaborating players is affected by neither the cyclic nor antisymmetric portion of the self- and cross-dependent components. Evidently, $\mathbf{A}^{(\mathrm{eff})}$ is a potential game that always has at least one pure Nash equilibrium and the preferred one provides the maximal value of the potential when $\mathbf{V}^{(\mathrm{eff})} = \mathbf{A}^{(\mathrm{eff})}$. The resultant payoff can serve as a reference when considering the additional effect of social dilemmas or cyclic dominance.

VIII. SUMMARY

We have studied the decomposition of the symmetric 4×4 matrix games into the linear combination of elementary games defined by diadic products of the column vectors of the Walsh-Hadamard matrices. For $n = 2^k$ (k is an integer) these matrices are composed of elements of $+1$ or -1 and have indicated clearly the inherent properties of matrices representing possible pair interactions in evolutionary games. Using this formalism we could distinguish four classes of elementary interactions from which any symmetric matrix games can be built up. The first class of interactions involves games with self-dependent payoffs that are defined by matrices with identical elements within each row. The direct interaction between the players is also missing for the second class of interactions called games with cross-dependent payoffs as here the player's income depends only on the decision of her coplayer and the payoff matrix contains columns of identical values. The third class of interactions defines the strength of coordination for each the possible strategy pair that may be either positive (attractive) or negative (repulsive). The fourth class of games summarize the effects of cyclic dominance and it can be considered as the extension of the traditional rock-paper-scissors game.

The research of the decomposition was originally motivated by the identification of potential games and by developing a method for the evaluation of potential if it exists. It is found that the potential exists in the absence of the cyclic components and the potential matrix itself can be expressed by a simple formula in the knowledge of the first three components.

One of the main advantages of the potential games is the fact that in multiagent evolutionary games the maximal value of the potential is achieved by a strategy profile that resembles the ground state in physical systems. In several cases, the four-strategy games are constructed from a two-strategy game by adding a new option (punishment, reward, reputation, etc.) and payoffs to each strategy via the introduction of a few parameters [47–49]. These latter models become potential games if the set of payoff parameters satisfy only one equation.

For $n = 4$ the application of the present diadic products has highlighted some hidden feature of interactions described by payoff matrices. It turned out that the most relevant cyclic basis games can be illustrated graphically by directed graphs with a single directed loop. This picture supports the extension of the above-described properties for symmetric games with $n > 4$ strategies.

The systematic analysis of spatial evolutionary games for logit rules goes beyond the scope of the present work. Now our investigations are restricted to some particular combinations of a few basis games within the subset of coordination-type interactions. The Monte Carlo simulations have indicated a

richness in the stationary behaviors. More curious behaviors are expected when considering more complex evolutionary games, including the other three classes of interactions and modifying the dynamical rule and/or the connectivity structure.

ACKNOWLEDGMENTS

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