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Seven mutually touching infinite cylinders

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Abstract

We confirm a conjecture of Littlewood: there exist seven infinite circular cylinders of unit radius which mutually touch each other. In fact, we exhibit two such sets of cylinders. Our approach is algebraic and uses symbolic and numerical computational techniques. We consider a system of polynomial equations describing the position of the axes of the cylinders in the 3 dimensional space. To have the same number of equations (namely 20) as the number of variables, the angle of the first two cylinders is fixed to 90 degrees, and a small family of direction vectors is left out of consideration. Homotopy continuation method has been applied to solve the system. The number of paths is about 121 billion, it is hopeless to follow them all. However, after checking 80 million paths, two solutions are found. Their validity, i.e., the existence of exact real solutions close to the approximate solutions at hand, was verified with the alphaCertified method as well as by the interval Krawczyk method.

Keywords: touching cylinders, line-line distance, polynomial system, homotopy method, certified solutions, alpha theory, interval methods
MSC 2010: 52C17, 52A40, 65H04, 65H20, 65G40.

1 Littlewood's conjecture on the maximal number of touching cylinders

John Edensor Littlewood ([11], Problem 7 on p. 20) proposed that

> "*Is it possible in 3-space for seven infinite circular cylinders of unit radius each to touch all the others? Seven is the number suggested by constants.*"

Two cylinders touch each other if their intersection is either a point or a line.

Finite versions of the problem are discussed as puzzles by Gardner and they are well known as 6 touching cigarettes [5, Figure 54 on page 115] and 7 touching cigarettes [5, Figure 55 on page 115]. The latter works for a ratio of length/radius greater than $7\sqrt{3}/2$. However, as it is noted by Bezdek [2] it is still open whether it is possible to find 8 or more touching finite identical cylinders. An arrangement of 4 touching coins (with a *small* ratio of length/diameter) is also known [5, Figure 49 on page 110] and this fact suggests that intermediate ratios of length/diameter could also be analyzed.

Bezdek [2] showed that 24 is an upper bound for the number of mutually touching congruent infinite cylinders. Ambrus and Bezdek [1] investigated the proposal of Kuperberg from the early 1990's that contained 8 congruent infinite cylinders. It is shown that they do not mutually touch each other, see [1, Theorem 1 and Figure 1 on page 1804] for details. Brass, Moser and Pach discuss an arrangement of 6 mutually touching infinite cylinders [3, page 98]. In the paper this lower bound is improved to 7.

Hereafter, it is assumed that cylinders are infinite and congruent, their radius is set to 1. Two cylinders of unit radius touch each other if and only if the distance of their axes is 2. Let C_i and ℓ_i denote the i-th cylinder and its axis, respectively. In the paper, $i = 1, 2, \ldots, 7$. The case of parallel cylinders (lines) is excluded from our analysis. It is left to the reader to show that if two cylinders are parallel, then the maximum number of mutually touching cylinders is four.

We intend to apply the well-known formula for the distance of two lines in $\mathbb{R}^3$. Let

$$\ell_i(s) = \mathbf{P}_i + s\,\mathbf{w}_i$$

be a parametric representation of line ℓ_i for $i = 1, \ldots, 7$. Here $\mathbf{P}_i \in \mathbb{R}^3$ is a point of ℓ_i, $\mathbf{w}_i \in \mathbb{R}^3$ is a direction vector and s is a real parameter. If lines ℓ_i and ℓ_j are skew, then their distance can be obtained as

$$d(\ell_i, \ell_j) = \frac{|(\overrightarrow{\mathbf{P}_i\mathbf{P}_j}) \cdot (\mathbf{w}_i \times \mathbf{w}_j)|}{||\mathbf{w}_i \times \mathbf{w}_j||}, \tag{1}$$

where $\cdot$ denotes dot product, $\times$ denotes cross product and $||\ ||$ denotes the Euclidean norm [6, 15]. Since the cylinders have unit radius, $d(\ell_i, \ell_j) = 2$ for all $i, j = 1, 2, \ldots, 7$, $i \neq j$, we can write equations (1) as

$$|(\overrightarrow{\mathbf{P}_i\mathbf{P}_j}) \cdot (\mathbf{w}_i \times \mathbf{w}_j)|^2 - 4||\mathbf{w}_i \times \mathbf{w}_j||^2 = 0. \tag{2}$$

In this form we avoid taking square roots. Let us introduce coordinates:

$$\mathbf{P}_i = (x_i, y_i, z_i), \qquad \mathbf{w}_i = (t_i, u_i, v_i).$$

Then we have

$$\begin{aligned} \overrightarrow{\mathbf{P}_i\mathbf{P}_j} &= (x_j - x_i, y_j - y_i, z_j - z_i), && (3) \\ \mathbf{w}_i \times \mathbf{w}_j &= (u_i v_j - v_i u_j, v_i t_j - t_i v_j, t_i u_j - u_i t_j). && (4) \end{aligned}$$

Now we substitute (3)-(4) into (2), and by using the well-known determinantal form of the triple product, we obtain the equation

$$\det \begin{bmatrix} x_j - x_i & y_j - y_i & z_j - z_i \\ t_i & u_i & v_i \\ t_j & u_j & v_j \end{bmatrix}^2 - 4\left((u_i v_j - v_i u_j)^2 + \right.$$
$$\left. + (v_i t_j - t_i v_j)^2 + (t_i u_j - u_i t_j)^2\right) = 0. \tag{5}$$

This is a polynomial equation of degree 6 in 12 variables. The polynomial on the left is a linear combination of 84 monomials.

We call a line horizontal if it is parallel to the plane $z = 0$. Any arrangement of seven lines can be translated and rotated to a position in which one of the lines (ℓ_1) is horizontal, with direction vector $\mathbf{w}_1 = (1, 0, 0)$, and it goes through the point $\mathbf{P}_1(0, 0, -1)$. It can also be assumed that the touching point of cylinders C_1 and C_2 is $(0, 0, 0)$, that is, ℓ_2 goes through the point $\mathbf{P}_2(0, 0, 1)$. The direction of (ℓ_2) is the only degree of freedom when the first two lines are considered. We shall assume, and this is explained later, that (ℓ_2) will be chosen to be orthogonal to the first line. We have so far

$$x_1 = 0,\ y_1 = 0,\ z_1 = -1, \qquad t_1 = 1,\ u_1 = 0,\ v_1 = 0; \tag{6}$$
$$x_2 = 0,\ y_2 = 0,\ z_2 = 1, \qquad t_2 = 0,\ u_2 = 1,\ v_2 = 0. \tag{7}$$

We can make some further simplifications. We may assume without loss of generality that ℓ_i ($i = 3, \ldots, 7$) is not horizontal (otherwise it would be parallel to ℓ_1 or ℓ_2), consequently, it goes through the plane $z = k$ for any $k \in \mathbb{R}$. Let us choose $k = 0$ and set

$$z_i = 0 \quad \text{for } i = 3, \ldots, 7. \tag{8}$$

Finally, the normalization of the direction vector of line ℓ_i is chosen to be $t_i + u_i + v_i = 1$ for $i = 3, \ldots, 7$. This is equivalent to

$$v_i = 1 - t_i - u_i, \qquad i = 3, \ldots, 7. \tag{9}$$

The normalization is restrictive, it may rule out some valid solutions, as it excludes all nonzero direction vectors fulfilling $t_i + u_i + v_i = 0$. However, our aim to find *one* solution rather than an analysis of all solutions. At this point we leave it open whether the excluded direction vectors may produce a valid solution.

The distance of ℓ_1 and ℓ_2 is guaranteed to be 2 by (6)-(7). Substitute (6), (8) and (9) into (5) to set the distance of ℓ_1 and ℓ_j ($3 \leq j \leq 7$):

$$\begin{aligned}
&y_j^2t_j^2 + 2y_j^2t_ju_j - 2y_j^2t_j + y_j^2u_j^2 - 2y_j^2u_j + y_j^2 + 2y_jt_ju_j + 2y_ju_j^2 - 2y_ju_j \\
&-4t_j^2 - 8t_ju_j + 8t_j - 7u_j^2 + 8u_j - 4 = 0, \qquad j = 3, \ldots, 7.
\end{aligned} \tag{10}$$

Substitute (7), (8) and (9) into (5) to set the distance of ℓ_2 and ℓ_j ($3 \leq j \leq 7$):

$$\begin{aligned}
&x_j^2t_j^2 + 2x_j^2t_ju_j - 2x_j^2t_j + x_j^2u_j^2 - 2x_j^2u_j + x_j^2 - 2x_jt_ju_j - 2x_jt_j^2 + 2x_jt_j \\
&-4u_j^2 - 8t_ju_j + 8t_j - 7t_j^2 + 8u_j - 4 = 0, \qquad j = 3, \ldots, 7.
\end{aligned} \tag{11}$$

Finally, substitute (8) and (9) into (5) to set the distance of ℓ_i and ℓ_j ($3 \leq i < j \leq 7$):

$$\begin{aligned}
&-4x_iy_it_iu_it_ju_j +4x_ix_jt_iu_it_ju_j +4x_iy_jt_iu_it_ju_j +4y_ix_jt_iu_it_ju_j +4y_iy_jt_iu_it_ju_j \\
&-4x_jy_jt_iu_it_ju_j -2x_i^2t_iu_it_ju_j -2y_i^2t_iu_it_ju_j -2x_j^2t_iu_it_ju_j -2y_j^2t_iu_it_ju_j \\
&-4x_ix_jt_iu_iu_j +4x_ix_jt_iu_j^2 +4x_ix_ju_i^2t_j -4x_ix_ju_it_ju_j +4y_iy_jt_i^2u_j \\
&-4y_iy_jt_iu_it_j -4y_iy_jt_it_ju_j +4y_iy_ju_it_j^2 +4x_ix_ju_iu_j +4y_iy_jt_it_j \\
&+x_i^2t_i^2u_j^2 +x_i^2u_i^2t_j^2 +y_i^2t_i^2u_j^2 +y_i^2u_i^2t_j^2 +x_j^2t_i^2u_j^2 +x_j^2u_i^2t_j^2 +y_j^2t_i^2u_j^2 +y_j^2u_i^2t_j^2 \\
&+2x_iy_it_i^2u_j^2 +2x_iy_iu_i^2t_j^2 -2x_ix_jt_i^2u_j^2 -2x_ix_ju_i^2t_j^2 -2x_iy_jt_i^2u_j^2 -2x_iy_ju_i^2t_j^2 \\
&-2y_ix_jt_i^2u_j^2 -2y_ix_ju_i^2t_j^2 -2y_iy_jt_i^2u_j^2 -2y_iy_ju_i^2t_j^2 +2x_jy_jt_i^2u_j^2 +2x_jy_ju_i^2t_j^2 \\
&-2x_iy_it_i^2u_j -2x_iy_it_iu_j^2 +2x_iy_jt_i^2u_j +2x_iy_jt_iu_j^2 +2x_iy_ju_i^2t_j +2x_iy_ju_it_j^2 \\
&-2x_iy_iu_i^2t_j -2x_iy_iu_it_j^2 +2y_ix_jt_i^2u_j +2y_ix_jt_iu_j^2 +2y_ix_ju_i^2t_j +2y_ix_ju_it_j^2 \\
&-2x_jy_jt_i^2u_j -2x_jy_jt_iu_j^2 -2x_jy_ju_i^2t_j -2x_jy_ju_it_j^2 -2x_i^2t_iu_j^2 -2x_i^2u_i^2t_j -2y_i^2t_i^2u_j \\
&-2y_i^2u_it_j^2 -2x_j^2t_iu_j^2 -2x_j^2u_i^2t_j -2y_j^2t_i^2u_j -2y_j^2u_it_j^2 +2x_i^2t_iu_iu_j +2x_i^2u_it_ju_j \\
&+2y_i^2t_iu_it_j +2y_i^2t_it_ju_j +2x_j^2t_iu_iu_j +2x_j^2u_it_ju_j +2y_j^2t_iu_it_j +2y_j^2t_it_ju_j \\
&+2x_iy_it_iu_it_j +2x_iy_it_iu_iu_j +2x_iy_it_it_ju_j +2x_iy_iu_it_ju_j -2x_iy_jt_iu_it_j \\
&-2x_iy_jt_iu_iu_j -2x_iy_jt_it_ju_j -2x_iy_ju_it_ju_j -2y_ix_jt_iu_it_j -2y_ix_jt_iu_iu_j \\
&-2y_ix_jt_it_ju_j -2y_ix_ju_it_ju_j +2x_jy_jt_iu_it_j +2x_jy_jt_iu_iu_j +2x_jy_jt_it_ju_j \\
&+2x_jy_ju_it_ju_j -2x_i^2u_iu_j -2y_i^2t_it_j -2x_j^2u_iu_j -2y_j^2t_it_j -2x_iy_it_iu_i +2x_iy_it_iu_j \\
&+2x_iy_iu_it_j -2x_iy_it_ju_j +2x_iy_jt_iu_i -2x_iy_jt_iu_j -2x_iy_ju_it_j +2x_iy_jt_ju_j \\
&+2y_ix_jt_iu_i -2y_ix_jt_iu_j -2y_ix_ju_it_j +2y_ix_jt_ju_j -2x_jy_jt_iu_i +2x_jy_jt_iu_j \\
&+2x_jy_ju_it_j -2x_jy_jt_ju_j -2x_ix_ju_i^2 -2x_ix_ju_j^2 -2y_iy_jt_j^2 -2y_iy_jt_i^2 +24t_iu_it_ju_j \\
&+x_i^2u_i^2 +x_i^2u_j^2 +y_i^2t_i^2 +y_i^2t_j^2 +x_j^2u_i^2 +x_j^2u_j^2 +y_j^2t_i^2 +y_j^2t_j^2 -12t_i^2u_j^2 -12u_i^2t_j^2 \\
&-4t_i^2 -4u_i^2 -4t_j^2 -4u_j^2 -8t_iu_it_j -8t_iu_iu_j -8t_it_ju_j +8t_iu_j^2 +8t_i^2u_j +8u_i^2t_j \\
&+8u_it_j^2 -8u_it_ju_j +8t_it_j +8u_iu_j = 0, \qquad i = 3, \ldots, 6,\ j = i+1, \ldots, 7.
\end{aligned} \tag{12}$$

As the first two lines ℓ_1, ℓ_2 are fixed, the aim is to find five lines $\ell_3, \ldots, \ell_7$ such that the distance of each pair of lines is 2. System (10)-(12) has 20 equations and 20 variables ($x_i, y_i, t_i, u_i,\ i = 3, \ldots, 7$). Each equation is a multivariate polynomial equation. Note that without fixing the angle of lines ℓ_1 and ℓ_2 at a given value, 90 degrees by our choice, we would have a system of 20 equations and 21 variables and we would lose the chance of finding isolated roots.

The numerical solution is presented in the next section. We emphasize here that, via methods such as alphaCertified discussed in subsection 3.1, the numbers themselves given in Table 1 prove the existence of real solutions of system (10)-(12) and hence prove Littlewood's conjecture. Nevertheless, we also outline the method of computing the approximate solutions in Table 1.

2 Solving the polynomial system by the polyhedral homotopy continuation method

The polyhedral homotopy continuation method is developed in [8] to approximate all isolated zeros of a polynomial system and is well implemented in software HOM4PS-2.0 [10]. The numerical experiments show that the method is efficient and reliable. More importantly, it can handle the large scale polynomial systems such as the system (10)-(12).

For a system of polynomials $P(\mathbf{x}) = (p_1(\mathbf{x}), \ldots, p_n(\mathbf{x}))$ with $\mathbf{x} = (x_1, \ldots, x_n)$, write

$$p_j(\mathbf{x}) = \sum_{\mathbf{a}\in\mathcal{S}_j} c_{j,\mathbf{a}}\mathbf{x}^{\mathbf{a}}, \qquad j = 1, \ldots, n,$$

where $\mathbf{a} = (a_1, \ldots, a_n) \in (\mathbb{N} \cup \{0\})^n$, $c_{j,\mathbf{a}} \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$, $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$, and $\mathcal{S}_j \subset (\mathbb{N} \cup \{0\})^n$ is finite.

Let $\omega_j : \mathcal{S}_j \to \mathbb{R}$ be a random lifting function on $\mathcal{S}_j$ which lifts $\mathcal{S}_j$ to its graph $\hat{\mathcal{S}}_j = \{\hat{\mathbf{a}} = (\mathbf{a}, \omega_j(\mathbf{a}))|\mathbf{a}\in\mathcal{S}_j\} \subset \mathbb{R}^{n+1}$. A collection of pairs $(\{\mathbf{a}_1, \mathbf{a}'_1\}, \ldots, \{\mathbf{a}_n, \mathbf{a}'_n\})$ where $\{\mathbf{a}_j, \mathbf{a}'_j\} \subseteq \mathcal{S}_j$ is called a *mixed cell* if there exists $\hat{\boldsymbol{\alpha}} = (\boldsymbol{\alpha}, 1) \in \mathbb{R}^{n+1}$ such that

$$\langle \hat{\mathbf{a}}_j, \hat{\boldsymbol{\alpha}}\rangle = \langle \hat{\mathbf{a}}'_j, \hat{\boldsymbol{\alpha}}\rangle < \langle \hat{\mathbf{a}}, \hat{\boldsymbol{\alpha}}\rangle \quad \text{for all } \mathbf{a} \in \mathcal{S}_j\backslash\{\mathbf{a}_j, \mathbf{a}'_j\}, \ \ j = 1, \ldots, n.$$

Here, $\langle \, , \, \rangle$ stands for the usual inner product in the Euclidean space $\mathbb{R}^{n+1}$. It is well-known that the number of mixed cells of a polynomial system is finite [8]. Those mixed cells play an important role in constructing the polyhedral homotopy.

Consider a given mixed cell $C = (\{\mathbf{a}_{11}, \mathbf{a}_{12}\}, \ldots, \{\mathbf{a}_{n1}, \mathbf{a}_{n2}\})$ with inner normal $\boldsymbol{\alpha} \in \mathbb{R}^n$, where $\{\mathbf{a}_{j1}, \mathbf{a}_{j2}\} \subseteq \mathcal{S}_j$ for each $j = 1, \ldots, n$. Let $\bar{c}_{j,\mathbf{a}}$ be a randomly chosen number in $\mathbb{C}$, and denote

$$\beta_j = \min_{\mathbf{a}\in\mathcal{S}_j} < \hat{\mathbf{a}}, \hat{\boldsymbol{\alpha}} >=< \hat{\mathbf{a}}_{j1}, \hat{\boldsymbol{\alpha}} >=< \hat{\mathbf{a}}_{j2}, \hat{\boldsymbol{\alpha}} > .$$

HOM4PS-2.0 constructs the homotopy to be $H(\mathbf{x}, t) = (h_1(\mathbf{x}, t), \ldots, h_n(\mathbf{x}, t))$, $t \in (-\infty, 0]$, where

$$h_j(\mathbf{x}, t) = \sum_{\mathbf{a} \in \mathcal{S}_j} [(1 - e^t)\bar{c}_{j,\mathbf{a}} + e^t c_{j,\mathbf{a}}]\mathbf{x}^{\mathbf{a}} e^{t*(<\hat{\mathbf{a}},\hat{\boldsymbol{\alpha}}>-\beta_j)} \quad \text{for} \quad j = 1, \ldots, n.$$

Note that $H(\mathbf{x}, 0) = P(\mathbf{x})$. When t goes to $-\infty$, $H(\mathbf{x}, t)$ becomes a binomial system

$$\begin{cases} \bar{c}_{11}\mathbf{x}^{\mathbf{a}_{11}} + \bar{c}_{12}\mathbf{x}^{\mathbf{a}_{12}} = 0 \\ \vdots \\ \bar{c}_{n1}\mathbf{x}^{\mathbf{a}_{n1}} + \bar{c}_{n2}\mathbf{x}^{\mathbf{a}_{n2}} = 0 \end{cases}$$

having $|\det(\mathbf{a}_{11} - \mathbf{a}_{12}, \ldots, \mathbf{a}_{n1} - \mathbf{a}_{n2})|$ nonsingular isolated solutions which provide the starting points for tracking the solution paths of $H(\mathbf{x}, t) = 0$ from $t = -\infty$ to 0. For the details of the algorithm for tracking the solution paths, see [10].

The polynomial system (10)-(12) consists of 20 equations in 20 variables. We obtain $180,734$ mixed cells of the system by software MixedVol-2.0 [4], which provide $121,098,993,664$ homotopy curves to be tracked. In order to track so many curves efficiently, we use the subroutines in the TBB library (Thread Building Blocks) to distribute data over multiple cores for parallel computation. Employing total 12 cores in 2 Intel Xeon X5650 2.66 GHz CPUs, 20 million curves are completed in a week. The first real solution is found after tracking 25 million paths, and the second one is found after tracking 80 million paths.

	first solution	second solution
x_3	11.675771704477	2.075088491891
y_3	-4.124414157636	-2.036516392124
t_3	0.704116159640	-0.030209763440
u_3	0.235129952793	0.599691085438
x_4	3.802878122730	-2.688893665930
y_4	-2.910611127075	4.070505903499
t_4	0.895623427074	0.184499043058
u_4	-0.149726023342	0.426965115851
x_5	8.311818491659	-4.033142850644
y_5	-1.732276613733	-2.655943449984
t_5	2.515897624878	0.251380280590
u_5	-0.566129665502	0.516678258430
x_6	-6.487945444917	6.311134419772
y_6	-8.537495065091	-5.229892181735
t_6	0.785632006191	-0.474742889365
u_6	0.338461562103	1.230302197822
x_7	-3.168475045360	3.914613907006
y_7	-2.459640638529	-7.881492743224
t_7	0.192767499267	1.698198197367
u_7	0.536724141124	-1.164062857743

Table 1. Two solutions of system (10)-(12) by HOM4PS-2.0

Since system (10)-(12) is symmetric in the five 4-tuples (x_j, y_j, t_j, u_j), $j = 3, \ldots, 7$, each solution represents a family of $5! = 120$ solutions, all of them resulting in the same arrangements of the cylinders. The two solutions in Table 1 are obviously not permutations of each other. However, due to the rotational and reflectional symmetries of the orthogonally fixed pair of cylinders C_1, C_2, any arrangement represents a family of 8 congruent arrangements. In order to show that the two solutions in Table 1 are non-congruent arrangements, we have computed the angles between the pairs of cylinders. The two sets of pairwise angles are disjoint except for the right angle of C_1, C_2. Consequently, the two arrangements in Figure 1 and 2 are not congruent.

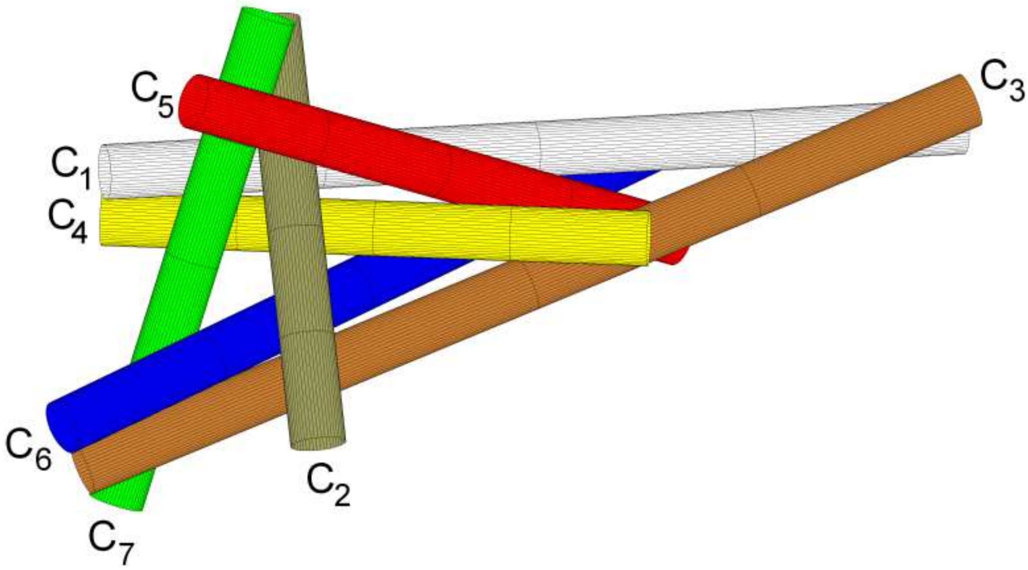

Figure 1. The first set of seven mutually touching infinite cylinders

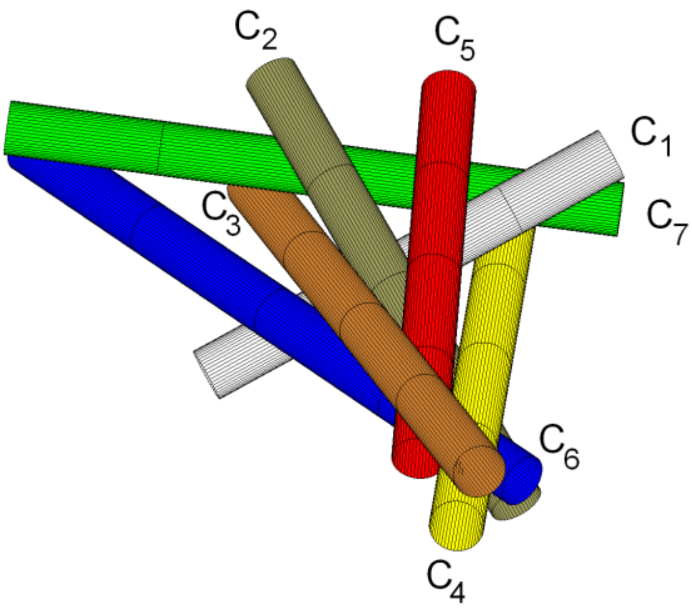

Figure 2. The second set of seven mutually touching infinite cylinders

3 Verification of the roots

HOM4PS-2.0 provides the solution up to 50 digits (the first 12 of which being correct), that can be used as a starting point of a solver using floating-point arithmetic like `fsolve` in Maple 13. With several accuracy levels adjusted previously by `Digits:=`10^r ($r = 2, 3, 4$), CPU times of running `fsolve` without any further specification on a personal computer with Pentium(R) 4 CPU 3.4GHz and 2GB of RAM is listed in Table 2.

`Digits`	10^2	10^3	10^4
CPU time	0.6 seconds	4 seconds	130 seconds

Table 2. CPU time of `fsolve` for 10^r ($r = 2, 3, 4$) correct digits

It's worth noting that `fsolve` recovers the solutions, within approximately the same CPU time as in Table 2, even if the starting values are truncated at 2 digits. Moreover, truncation at 1 digit still works for the first solution. Truncation at 1 digit, except for t_3, which is truncated at 2 digits (-0.03), works for the second solution.

However, a large number of correct digits is still not mathematical correctness. Two exact verification methods, alphaCertified and the interval Krawczyk method are applied. Any of them would be sufficient of its own, nevertheless, two is at least not worse than one.

3.1 alphaCertified

Smale's α-theory [14] provides a positive, effectively computable constant $\alpha(F, \mathbf{x})$ for a polynomial system $F : \mathbb{C}^n \to \mathbb{C}^n$ and a point $\mathbf{x} \in \mathbb{C}^n$ with the property that if

$$\alpha(F, \mathbf{x}) \leq \frac{13 - 3\sqrt{17}}{4} \approx 0.1576,$$

then Newton's iteration starting from $\mathbf{x}$ converges quadratically to a solution $\boldsymbol{\xi}$ close to $\mathbf{x}$ of the system $F = \mathbf{0}$. Based on Smale's theory Hauenstein and Sottile [7] developed algorithms which, for given F and $\mathbf{x}$ compute an upper bound on $\alpha(F, \mathbf{x})$ and on some related quantities. On that basis they have built a multipurpose verification software called alphaCertified. It can produce a certificate that

(i) $\mathbf{x}$ is an approximate solution of $F = \mathbf{0}$ in the above sense;

(ii) an approximate solution corresponds to an isolated solution;

(iii) the solution $\boldsymbol{\xi}$ corresponding to $\mathbf{x}$ is real (for real F).

We have used alphaCertified v1.2.0 (August 15, 2011, GMP v4.3.1 & MPFR v2.4.1-p5) with Maple 13 interface. The input of alphaCertified is system (10)-(12) and the approximate solutions in Table 1. We need to write the first solution up to at least 12 digits, otherwise algorithm alphaCertified does not certify it. The output of alphaCertified with the first solution as in Table 1 consists of $\alpha = 4.4333 \cdot 10^{-2}$, $\beta = 3.1668 \cdot 10^{-12}$, $\gamma = 1.3999 \cdot 10^{10}$ (see [7] for the details of α, β, γ). The second solution has to be written up to at least 11 digits in order to be certified. The output of alphaCertified with the second solution (truncated at 11 digits) consists of $\alpha = 6.578 \cdot 10^{-2}$, $\beta = 2.2387 \cdot 10^{-11}$, $\gamma = 2.9392 \cdot 10^{9}$. Both solutions have been certified to be real and isolated solutions.

3.2 The interval Krawczyk method

We seek for real solutions among the numerical solutions with imaginary parts less than the heuristic threshold $\theta = 10^{-8}$. The residuals of the real solutions are less than $5 \cdot 10^{-14}$ and their condition numbers are at most $4.8 \cdot 10^{4}$, which show that these solutions are numerically reliable.

To guarantee that in a small neighborhood of each numerical solution there is a unique exact physical solution, the interval Krawczyk method [9] is applied for verification. The method is based on the following fact: for a smooth function $F : \mathbb{R}^n \to \mathbb{R}^n$ and a point $\mathbf{x} \in \mathbb{R}^n$, let $[\,\mathbf{x}\,]_r \subset \mathbb{R}^n$ be the ball centered at $\mathbf{x}$ with radius $r > 0$. Namely,

$$[\,\mathbf{x}\,]_r \;=\; \{\mathbf{y} \in \mathbb{R}^n : \|\mathbf{y} - \mathbf{x}\|_\infty \leq r\}\,,$$

where $\|\;\|_\infty$ is the infinity norm. Assuming that the derivative of F at $\mathbf{x}$, denoted by $DF(\mathbf{x})$, is nonsingular, the Krawczyk set of F associated with $[\,\mathbf{x}\,]_r$ is defined as

$$K(F, [\,\mathbf{x}\,]_r) \;=\; \mathbf{x} - DF(\mathbf{x})^{-1}F(\mathbf{x}) + \left[I - DF(\mathbf{x})^{-1}DF([\,\mathbf{x}\,]_r)\right]([\,\mathbf{x}\,]_r - \mathbf{x}).$$

If the Krawczyk set is contained in the interior of $[\,\mathbf{x}\,]_r$, then there exists a unique zero of F in $[\,\mathbf{x}\,]_r$.

The task of verification is implemented by using the interval arithmetic in INTLAB (INTerval LABoratory) [13]. In this implementation each numerical solution $\mathbf{x}$ is taken as the center of the ball $[\,\mathbf{x}\,]_r$ with radius $r = 10^{-8}$. Again, both solutions have been certified to be real and isolated solutions.

4 Conclusions and open questions

It remains an open question whether seven is the maximal number of mutually touching infinite cylinders. Following the same idea for eight cylinders, a polynomial system of 25 variables and 27 equations is resulted in. It is not yet dis/proven whether it has a solution. In case of seven cylinders, alternative choices instead of that the first two cylinders are orthogonal need to be analyzed. The maximal number of lines in $\mathbb{R}^n$ ($n > 3$) having the same pairwise nonzero distance is also unknown. The authors believe that the method proposed can be applied for a wide class of similar geometrical problems.

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