Minimum order of graphs with given coloring parameters

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Abstract

A complete k-coloring of a graph $G = (V, E)$ is an assignment $\varphi : V \to \{1, \ldots, k\}$ of colors to the vertices such that no two vertices of the same color are adjacent, and the union of any two color classes contains at least one edge. Three extensively investigated graph invariants related to complete colorings are the minimum and maximum number of colors in a complete coloring (*chromatic number* $\chi(G)$ and *achromatic number* $\psi(G)$, respectively), and the *Grundy number* $\Gamma(G)$ defined as the largest k admitting a complete coloring φ with exactly

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k colors such that every vertex $v \in V$ of color $\varphi(v)$ has a neighbor of color i for all $1 \le i < \varphi(v)$. The inequality chain $\chi(G) \le \Gamma(G) \le \psi(G)$ obviously holds for all graphs G. A triple (f, g, h) of positive integers at least 2 is called *realizable* if there exists a *connected* graph G with $\chi(G) = f$, $\Gamma(G) = g$, and $\psi(G) = h$. In [8], the list of realizable triples has been found. In this paper we determine the minimum number of vertices in a connected graph with chromatic number f, Grundy number g, and achromatic number h, for all realizable triples (f, g, h) of integers. Furthermore, for $f = g = 3$ we describe the (two) extremal graphs for each $h \ge 6$. For $h = 4$ and 5, there are more extremal graphs, their description is contained as well.

Keywords: graph coloring, Grundy number, achromatic number, greedy algorithm, extremal graph, bipartite graph

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1 Introduction

A *complete coloring* of a graph is an assignment of colors to the vertices in such a way that adjacent vertices receive different colors, and there is at least one edge between any two color classes. In other words, the coloring is proper and the number of colors cannot be decreased by identifying two colors.

Let $G = (V, E)$ be any simple undirected graph. The minimum number of colors in a *proper* coloring is the *chromatic number* $\chi(G)$, and all proper χ-colorings are necessarily complete. The maximum number of colors in a *complete* coloring is the *achromatic number* $\psi(G)$. Every graph admits a complete coloring with exactly k colors for all $\chi \le k \le \psi$ (Harary *et al.*, [15]). An important variant of complete coloring, called *Grundy coloring* or Grundy numbering, requires a proper coloring $\varphi : V \to \{1, \dots, k\}$ such that every vertex $v \in V$ has a neighbor of color i for each $1 \le i < \varphi(v)$. The largest integer k for which there exists a Grundy coloring of G is denoted by $\Gamma(G)$ and is called the *Grundy number* of G. Certainly, $\Gamma(G)$ is sandwiched between $\chi(G)$ and $\psi(G)$. One should emphasize that $\chi(G)$ and $\psi(G)$ are defined in terms of unordered colorings, i.e., permutation of colors does not change the required property of a coloring. On the other hand, in a Grundy coloring the order of colors is substantial.

Proper colorings have found a huge amount of applications and hence, besides their high importance in graph theory, they are very well motivated from the practical side, too. The chromatic number occurs in lots of optimization problems. The achromatic number looks less practically motivated, nevertheless it expresses the worst case of a coloring algorithm which creates a proper color partition of a graph in an arbitrary way and then applies the improvement heuristic of identifying two colors as long as no monochromatic edge is created. Grundy colorings have strong motivation from game theory; moreover, $\Gamma(G)$ describes the worst case of First-Fit coloring algorithm when applied to a graph G if we do not know the graph in advance, the vertices arrive one by one, and we irrevocably assign the smallest feasible color to each new vertex as a best local choice. Then the number of colors required for a worst input order is exactly $\Gamma(G)$. For this reason, $\Gamma(G)$ is also called the *on-line First-Fit chromatic number* of G in the literature.

An overview of on-line colorings and a detailed analysis of the First-Fit version is given in [4]. A more extensive survey on the subject can be found in [19]. The performance of First-Fit is much better on the average than in the worst case. This is a good reason that it has numerous successful applications. This nicely shows from a practical point of view that the Grundy number is worth investigating.

The definition of Grundy number is usually attributed to Christen and Selkow [10], although its roots date back to the works of Grundy [12] four decades earlier; and in fact $\Gamma(G)$ of an undirected graph G is equal to that of the digraph in which each edge of G is replaced with two oppositely oriented arcs. In general, computing the Grundy number is NP-hard, and it remains so even when restricted to some very particular graph classes, e.g., to bipartite graphs or complements of bipartite graphs ([16] and [26], respectively). Actually, the situation is even worse: there does not exist any polynomial-time approximation scheme to estimate $\Gamma(G)$ unless $\mathsf{P} = \mathsf{NP}$ [20], and for every integer c it is coNP-complete to decide whether $\Gamma(G) \le c\,\chi(G)$, and also whether $\Gamma(G) \le c\,\omega(G)$, where $\omega(G)$ denotes the clique number of G (see [1]). Several bounds on $\Gamma(G)$ in terms of other graph invariants were given, e.g., in [5, 27, 28]. On the other hand, by the finite basis theorem of Gyárfás *et al.* [13] the problem of deciding whether $\Gamma(G) \ge k$ can be solved in polynomial time, when k is a fixed integer (see also [6] for results on Grundy critical graphs). Moreover, there are known efficient algorithms to determine the Grundy number of trees [17] and more generally of partial k-trees [24].

Concerning the achromatic number, on the positive side there exists a constant-approximation for trees [9] and a polynomial-time exact algorithm for complements of trees [25]. But in a sense, the computation of $\psi(G)$ is harder than that of $\Gamma(G)$. It is NP-complete to determine $\psi(G)$ on connected graphs that are simultaneously interval graphs and co-graphs [3], and even on trees [7, 23]. Moreover, no randomized polynomial-time algorithm can generate with high probability a complete coloring with $C\psi(G)/\sqrt{n}$ colors for arbitrarily large constant C, unless $\mathsf{NP} \subseteq \mathsf{RTime}(n^{\mathrm{poly}\log n})$, and under the same assumption $\psi(G)$ cannot be approximated deterministically within a multiplicative $\lg^{1/4-\varepsilon} n$, for any $\varepsilon > 0$ [22], although some $o(n)$-approximations are known [9, 21].

The strong negative results above concerning algorithmic complexity also mean a natural limitation on structural dependencies, for all the three graph invariants χ, Γ, ψ. On the other hand, quantitatively, the triple (χ, Γ, ψ) can take any non-decreasing sequence of integers at least 2. (The analogous assertion for (χ, ψ) without Γ appeared in [2].) For example, if $\chi = \Gamma = 2$, then properly choosing the size of a union of complete graphs on two vertices will do for any given ψ. Assuming connectivity, however, makes a difference. Let us call a triple (f, g, h) of integers with $2 \le f \le g \le h$ *realizable* if there exists a *connected* graph G such that $\chi(G) = f$, $\Gamma(G) = g$, and $\psi(G) = h$. It was proved by Chartrand *et al.* [8] that a triple is realizable if and only if either $g \geq 3$ or $f = g = h = 2$.

Here we address the naturally arising question of smallest connected graphs with the given coloring parameters. Namely, for a realizable triple (f, g, h), let us denote by $n(f, g, h)$ the minimum order of a connected graph G with $\chi(G) = f$, $\Gamma(G) = g$ and $\psi(G) = h$. The lower bound $n(f, g, h) \geq 2h - f$ was proved in [8, Theorem 2.10] in the stronger form $2\psi - \omega$ (where ω denotes clique number), and this estimate was also shown to be tight for $g = h$. On the other hand the order of graphs constructed there to verify that the triple (f, g, h) is realizable was rather large, and had a high growth rate. In particular, for every fixed f and g, the number of vertices in the graphs of [8] realizing (f, g, h) grows with h^2 as h gets large, while the lower bound is linear in h. For instance, the construction for $f = g$, described in [18], takes the complete graph K_f together with a pendant path P_k, having properly chosen number of vertices k, and applies the facts that very long paths make ψ arbitrarily large and that the removal of the endvertex of a path (or actually any vertex of any graph) decreases ψ by at most 1, as proved in [11].

In this paper we determine the exact value of $n(f, g, h)$ for every realizable triple (f, g, h), showing that the lower bound $2h-f$ is either tight or just one below optimum. It is easy to see that the complete graph K_f verifies $n(f, f, f) = f$ for all $f \geq 2$. For the other cases it will turn out that the formula depends on whether $f < g$. These facts are summarized in the following two theorems; the case $g = h$ was already discussed in [8].

Theorem 1 *For $2 \leq f < g$ and for $f = g = h$, $n(f, g, h) = 2h - f$.*

Theorem 2 *For $2 < f = g < h$, $n(f, g, h) = 2h - f + 1$.*

Remark 1 *As one can see, the minimum does not depend on g, apart from the distinction between $f = g$ and $f < g$.*

The two theorems above will be proved in the following two sections, while in the last section for each $h > 3$ we determine the number of extremal graphs (the graphs of minimum order) that realize the triple $(3, 3, h)$.

Theorem 3 *Let $f = g = 3$. If $h = 4$, then there are seven extremal graphs. If $h = 5$, then there are three such graphs, while for every $h \geq 6$ there are exactly two of them.*

Along the proof of Theorem 3 we will also determine the structure of all extremal graphs.

In what follows we need some additional terms and notation. Given a graph $G = (V, E)$ and a vertex set $X \subseteq V$, the subgraph induced by X will be denoted by $G[X]$. A set of vertices X is *dominated by* another one, say D, if every $x \in X \setminus D$ has at least one neighbor in D. A set D is *dominating* if it dominates the whole vertex set. We will also say that a set D dominates a subgraph H in the sense that D dominates $V(H)$. A set S is said to be *stable* if it does not contain any pair of adjacent vertices.

2 The case $\chi < \Gamma$ — Proof of Theorem 1

First, we consider the lower bound for $n(f, g, h)$. From [8, Theorem 2.10], it follows that for all f, g, h satisfying conditions of Theorem 1, it holds that $n(f, g, h) \geq 2h - f$. The idea is that h color classes on fewer than $2h - f$ vertices would yield at least $f + 1$ singleton classes, but then they should be mutually adjacent if the coloring is complete. This leads to the contradiction $f + 1 \leq \omega(G) \leq \chi(G)$.

Considering the tightness of the bound we construct an appropriate bipartite graph to prove the following lemma

Lemma 1 *The lower bound $2h - f$ is tight for the case $f = 2$.*

Construction I

For every $k \geq 2$ we define a *basic bipartite graph* B_k, with vertex set $X = U \cup W$, where $U = \{u_1, u_2, \ldots, u_{k-1}\}$ and $W = \{w_2, w_3, \ldots, w_k\}$, and edge set

$$C = \{u_i w_j \mid 1 \leq i < j \leq k\}.$$

We call U and W the *partite sets* of B_k. See Figure 1 for the graph B_7 that will be used several times.

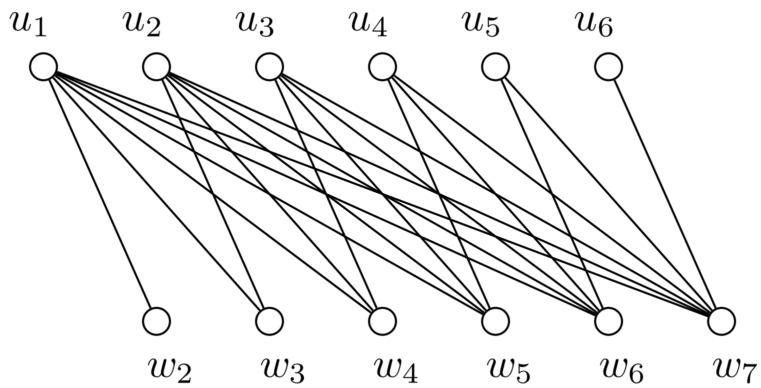


Figure 1: Graph B_7

For $g = 3$, we consider B_h itself. However, for $g \geq 4$, we denote $\gamma = g - 3$, and modify the graph B_h by inserting the following set of edges:

$$\{u_i w_j \mid 1 \leq i - j \leq \gamma,\ 2 \leq i \leq h - 1,\ 2 \leq j \leq h - 2\}.$$

We shall call them the *inserted edges*. □

For any $g \geq 3$, the graph with inserted edges will be denoted by $G(2, g, h)$, and in the following text, briefly by G^*. We show an example of G^* in Figure 2.

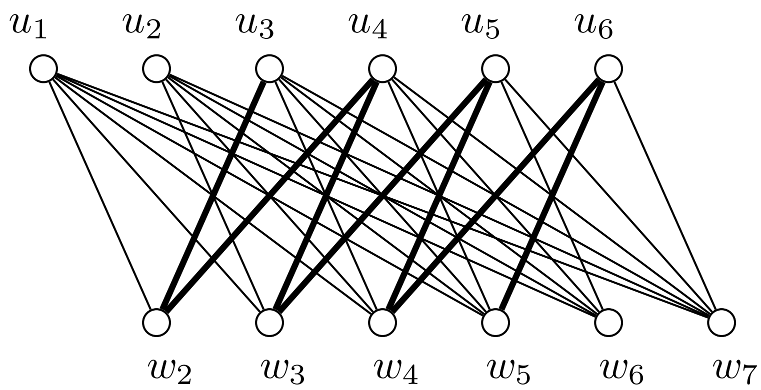


Figure 2: Graph $G(2,5,7)$

Proposition 1 *$\chi(G^*) = 2$, $\Gamma(G^*) = g$, and $\psi(G^*) = h$.*

Proof By definition, G^* is bipartite, i.e., $\chi(G^*) = 2$ and the color classes $\{u_1\}$, $\{u_2, w_2\}$, $\dots$, $\{u_{h-1}, w_{h-1}\}$, $\{w_h\}$ verify that $\psi(G^*) \geq h$ is valid, whereas $\psi(G^*) \leq h$ also holds because G^* has no more than $2h-2$ vertices. Hence, what remains to prove is that $\Gamma(G^*) = g$.

For $g < h$, the proof of the lower bound $\Gamma(G^*) \geq g$ is obtained by assigning color g to u_1, color $i-1$ to $\{u_i, w_i\}$ for $i = 2, \dots, g-1$, color 1 again to $u_g, \dots, u_{h-1}$, and finally color $g-1$ to w_j for $j = g, \dots, h$. Consequently, $\Gamma(G^*) \geq g$.

For $g = h$, the only difference is that in the set U, color 1 is assigned to exactly one vertex, namely to u_2.

To prove the upper bound on $\Gamma(G^*)$ is more difficult. We manage it as a separate statement.

Claim 1 $\Gamma(G^*) \leq g$

Throughout the argumentation, we assume that $\Gamma(G^*) > g$. Claim 1 will be a consequence of Claim 3 below. Before proving those claims we need some additional notation and a simple Claim 2.

Let us call a stable set S a *double set* if it meets both partite sets. Considering a *Grundy coloring* of G^* with $\Gamma(G^*)$ colors, a *double class* is a color class which is a double set. Graph G^* is bipartite and there must be an edge between any two color classes. Moreover, the classes containing u_1, w_h respectively, are non-double. Thus, the number of non-double classes is exactly 2. We denote the double classes by $D_1, D_2, \dots, D_\Delta$, indexed with their colors; here $\Delta = \Gamma(G^*) - 2$.

Let S be any double set, let $I = I(S)$ be the minimum of $\{i \mid u_i \in S\}$ and let $J = J(S)$ be the maximum of $\{j \mid w_j \in S\}$. For $S = D_k$ we denote $I(S)$ also by I_k and similarly J_k for $J(S)$.

The following claim is straightforward, so the proof is omitted.

Claim 2 *$I(S) \geq J(S)$ for any double set S and, in particular, for any double class.* □

We shall use the term *reduced graph* and notation R_t for the bipartite graph $K_{t,t} - tK_2$ obtained from the complete bipartite graph $K_{t,t}$ by omitting a 1-factor. (This graph was taken in [8] for $f = 2$ and $g = h > 2$.) The essence of the proof is given in the following claim. Let us recall that Δ is the number of double classes.

Claim 3

(i) *The subgraph H induced by the vertex set*

$$\{u_I \mid I = I(D),\ D \text{ is a double class}\} \cup \{w_J \mid J = J(D),\ D \text{ is a double class}\}$$

is isomorphic to R_Δ.

(ii) *The graph G^* does not contain any induced subgraph isomorphic to R_Θ with $\Theta \geq \gamma + 2$.*

Proof We start with an observation that nothing has been stated concerning the position of $\{I_k, J_k\}$ in the "omitted 1-factor".

In order to prove (i) we take two arbitrary double classes D_k and D_K with $k < K$ and consider the vertices u_i, u_I, w_j, w_J where $i = I_k,\ I = I_K,\ j = J_k,\ J = J_K$. We will prove that $u_I w_j \in E(G^*)$ and $u_i w_J \in E(G^*)$, this will yield the assertion since $\{I_k, J_k \mid 1 \leq k \leq \Delta\}$ will play the role of the "omitted 1-factor".

Let us consider the first statement. Suppose for a contradiction that $u_I w_j \notin E(G^*)$. From the properties of Grundy coloring, u_I has some neighbor w_λ in D_k. By the maximality of j, $j \geq \lambda$ holds and w_j is adjacent to u_I for every $I \neq j$. We may suppose the latter case since otherwise the proof is done. Similarly, we obtain that w_J has a neighbor u_μ in D_k.

Since all of $w_{j-\gamma}, w_{j-\gamma+1}, \ldots, w_h$ except w_j are adjacent to u_I, the non-edge $u_I w_J$ implies $J < I - \gamma$, and $i \leq \mu$ also holds. Hence, the following chain of inequalities is valid:

$$J < I - \gamma = j - \gamma \leq i - \gamma \leq \mu - \gamma.$$

Consequently, the vertices u_I and w_j are adjacent, as claimed.

By the central symmetry of G^*, the relation $u_i w_J \in E(G^*)$ is established in the same way.

In order to prove (ii) suppose that G^* has an induced subgraph R isomorphic to R_Δ with $\Delta \geq \gamma + 2$. Let Z be the vertex set of R. If $u \in Z \cap U$, $w \in Z \cap W$, and $uw \notin E(G^*)$, then we call w the *match* of u and vice versa. Let us denote by $a = u_M$ the vertex of largest subscript in $Z \cap U$ and by $b = w_m$ the vertex of smallest subscript in $Z \cap W$.

First, suppose $ab \in E(G^*)$. In case $M < m$, the graph R would be a complete bipartite graph. So $M \le m+\gamma$. Let us take any vertex u_ι in $(Z \cap U) \setminus \{a\}$. We observe that $m \le \iota < M$. Indeed, the second inequality is trivial; and if the first one was not true then u_ι would have no match. This means that all the subscripts of the vertices in $Z \cap U$ are in the interval $[m, M] \subseteq [m, m+\gamma]$ and thus $\Delta \le \gamma + 1$, proving the assertion if $ab \in E(G^*)$.

Second, suppose $ab \notin E(G^*)$. Then the match of $a = u_M$ is $b = w_m$. Surely, for every vertex u_ι in $Z \cap U$ with $\iota \ne M$, we have $\iota \le m + \gamma$ because $u_\iota w_m \in E(G^*)$, and $m < \iota$, for otherwise u_ι would have no match. Consequently, $\Delta \le \gamma + 1$.

Thus we have proved Claim 3. □

As a consequence of the above claim, $\Delta \le \gamma + 1 = g - 2$ and $\Gamma(G^*) \le g$ follow. Thus Claim 1 is established; moreover Proposition 1 and Lemma 1 are proved. □□□

Note that we have also proved Theorem 1 for $f = 2$. It has been observed in [8, Proposition 2.8] that if we take the join of the current graph with a new vertex, then each of χ, Γ, ψ increases by exactly 1. Thus, for a given triple (f, g, h) with $f \ge 3$ we can start from $G(2, g', h')$, where

$$g' = g - f + 2, \quad h' = h - f + 2$$

and join it with K_{f-2}. In this way we obtain a connected graph that realizes (f, g, h) and has exactly $(2h' - 2) + (f - 2) = 2h - f$ vertices. This completes the proof of Theorem 1. □

3 The case $\chi = \Gamma$ — Proof of Theorem 2

Similarly as above, we shall give the proof in two parts. Before proving the lower bound, we show an auxiliary statement which will be applied many times.

Claim 4 *Given a graph G, suppose there exists an induced subgraph H of G, with $\Gamma(H) \ge k$ and a stable set, disjoint from $V(H)$ and dominating H. Then the Grundy number of G is strictly larger than k.*

Proof For both G and H, the stable set can get color 1, and the vertices of H can be colored with numbers one larger than in the original Grundy numbering of H. This induces a subgraph of Grundy number larger than k, and we can use the fact that Γ is monotone with respect to taking induced subgraphs. □

Remark 2 *In most of the applications, H will be a K_3 or a P_4 and k will be 3. As another special case, we shall often find a maximal stable set of G, disjoint from some subgraph H of G.*

Now we are in a position to establish the first part of the theorem.

Lemma 2 *If $2 < f = g < h$, then $n(f, g, h) \geq 2h - f + 1$.*

Proof Suppose for a contradiction that there exists a graph G with the given coloring parameters on $n < 2h - f + 1$ vertices. By what has been said at the beginning of Section 2, this implies $n = 2h - f$. From the argument sketched there we also see that the conditions $n = 2h - f$ and $\psi(G) = h > f$ imply that in any complete h-coloring of G there exist f singleton color classes, say $S_1 = \{y_1\}, S_2 = \{y_2\}, \ldots, S_f = \{y_f\}$, inducing a complete subgraph in G. Moreover, by the condition $h > f$, there is at least one further color class, say S_{f+1}.

Since the coloring is assumed to be complete, each y_i $(1 \leq i \leq f)$ is adjacent to at least one vertex of S_{f+1}. Hence, by Claim 4 we obtain $\Gamma(G) > f = g$, a contradiction. Lemma 2 is established. □

The proof of the following lemma, which establishes the second part of the theorem, will be split into several claims.

Lemma 3 *If $2 < f = g < h$, then $n(f, g, h) \leq 2h - f + 1$.*

Proof Two graphs will be constructed with the appropriate number of vertices, for $f = g = 3$. This will be enough since the simple extension adjoining K_{h-f+3} will work like earlier (see end of the proof of Theorem 1).

First, we recall the simple fact from the proof of [8, Proposition 2.5] that a connected graph G has $\Gamma(G) = 2$ if and only if G is a complete bipartite graph. This can be extended also to disconnected graphs:

Claim 5 *For a graph G, $\Gamma(G) \leq 2$ is equivalent to the following property:*

(II) *Each component of G is either an isolated vertex or a complete bipartite graph.* □

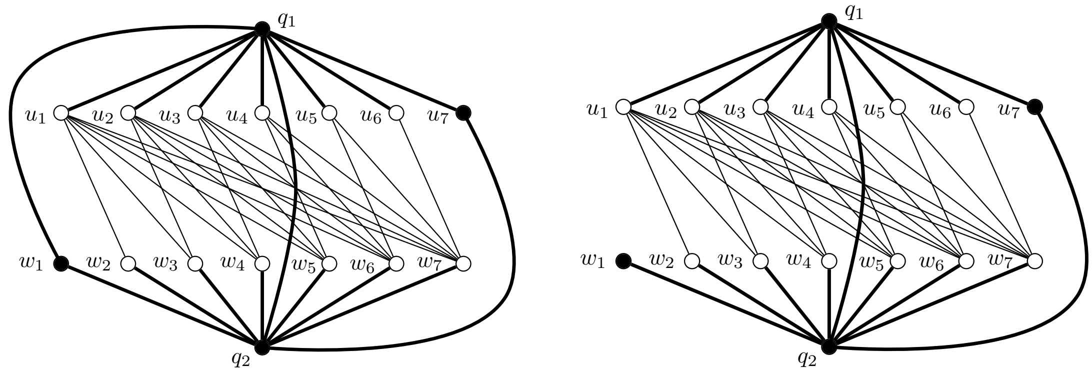
Note that under the present conditions we have $h \geq 4$. Starting from the basic bipartite graph B_{h-2} introduced in the previous section, we are going to construct two graphs L_1 and L_2 with larger parameters by inserting vertices and edges, different from Construction I.

Construction II

Let $\ell = h - 2$ and consider the bipartite graph $B_\ell = (X, C)$ of Section 2. First, we extend B_ℓ with two isolated vertices u_ℓ and w_1 to obtain the *extended graph* B'_h. We also introduce the notation $U' = \{u_1, \ldots, u_\ell\}$, $W' = \{w_1, \ldots, w_\ell\}$. Let $L_i = (Y, D_i)$, $i \in \{1, 2\}$ be the graph with the vertex set $Y = U' \cup W' \cup \{q_1, q_2\}$, and the edge set $D_1 = D_0 \cup \{q_2 u_\ell, q_1 w_1, q_1 q_2\}$ or $D_2 = D_0 \cup \{q_2 u_\ell, q_1 q_2\}$, respectively, where $D_0 = C \cup \{q_1 u \mid u \in U'\} \cup \{q_2 w \mid w \in W'\}$. □

In Figure 3 we present an example of L_1 and L_2, when $\ell = 7$ (white vertices induce B_ℓ). From now on, we shall also refer to both graphs shortly as L.

The following claim contains an easier part of the proof of Lemma 3. Later we shall deal with a more difficult one.



(a) Graph L_1 (b) Graph L_2

Figure 3: Two extremal graphs

Claim 6

(i) $\chi(L) = 3$,

(ii) $\psi(L) = h$.

Proof (i) The graph L contains a triangle, and we can easily find a coloring with three colors, shown in Figure 4.

(ii) The vertex partition of L into the h stable sets $\{q_1\}$, $\{q_2\}$, and $\{u_i, w_i\}$, $i \in \{1, \ldots, \ell\}$ is a complete coloring with exactly $h > 3$ colors. Thus, $\psi(L) \geq h$. Conversely, the upper bound h for $\psi(L)$ comes from the arguments used in the proof of Lemma 2. The fact that $|V(L)| = 2h - 2$ is also important but it follows easily from the construction. □

We continue with the harder part of the lemma. First we state a property similar to that of Claim 2.

Claim 7 *The maximal stable sets S of graph B'_h are of the following form:*

$$S = S_N = \{w_1, \ldots, w_N\} \cup \{u_N, \ldots, u_\ell\}$$

for some $1 \leq N \leq \ell$. □

We note that the extremal cases $N = 1$ and $N = \ell$ correspond to $U' \cup \{w_1\}$ and $W' \cup \{u_\ell\}$, respectively. Moreover $\{u_\ell, w_1\}$ is contained in all S, hence every S meets both U' and W'.

Concerning stable sets of the graph L, we have the following:

Claim 8 *For any maximal stable set S of L, the induced subgraph $L - S$ has property* (Π) *stated in Claim 5.*

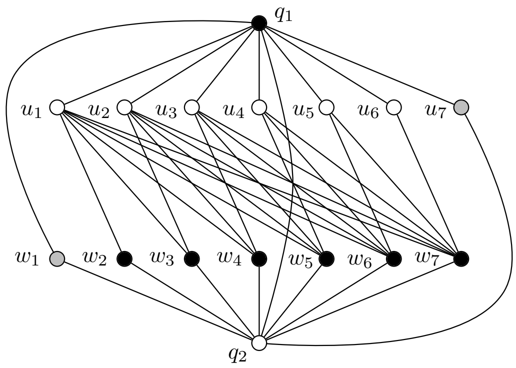


Figure 4: Proper 3-coloring of L_1

Proof Take an arbitrary S. There are three cases:

(a) $S \subseteq V(B'_h)$,

(b) $q_1 \in S$,

(c) $q_2 \in S$.

In the first case, S is a maximal stable set of the graph B'_h, and by Claim 7 we have $S = S_N$ for some $1 \le N \le \ell$. The complement of S with respect to the vertex set of L is the set $\{u_1, ..., u_{N-1}\} \cup \{w_{N+1}, ..., w_\ell\} \cup \{q_1, q_2\}$ and induces a complete bipartite graph with vertex classes $\{u_1, ..., u_{N-1}\} \cup \{q_2\}$ and $\{w_{N+1}, ..., w_l\} \cup \{q_1\}$ since $u_\ell \in S$ necessarily holds. This verifies property (Π).

In the second case, $S \subseteq W' \cup \{q_1\}$ and, in fact, S is equal to this set, because of maximality. The complement of S with respect to the vertex set of L, namely $U' \cup \{q_2\}$, induces a subgraph consisting of U (see Construction I to recall the definition) as $\ell - 1$ isolated vertices, together with the isolated edge $u_\ell q_2$. So it does have property (Π).

The third case is similar and yields that $L - S$ is induced by the stable set W plus the isolated edge $u_\ell q_1$. □

Finally, we prove

Claim 9 $\Gamma(L) = 3$.

Proof Let φ be an arbitrary Grundy coloring of L, and consider the stable set S formed by the vertices of color 1 under φ. Since every vertex of higher color has a neighbor of color 1, the set S is a maximal stable set of L. Thus, by Claim 8, the subgraph $L - S$ satisfies property (Π). Now Claim 5 implies that every Grundy coloring of $L - S$ uses at most 2 colors. This implies $\Gamma(G) \le 3$, whereas the presence of a subgraph K_3 induced by $\{u_\ell, q_1, q_2\}$ yields equality. □

To get a general construction of a graph that realizes (f, g, h) with $2h - f + 1$ vertices, it is enough to repeat the process of adding a clique of order $f - 3$, as we did at the end of the proof of Theorem 1. This completes the proof of Lemma 3. □

In this way Theorem 2 has been proved as well. □

4 The lists of extremal graphs

Our goal in this section is to prove Theorem 3. Along the proof we shall also describe the structure of extremal graphs that we call here h-optimal graphs.

Definition 1 Suppose $h \geq 4$. We say that a graph $G = (V, E)$ is *h-optimal* if $\chi(G) = \Gamma(G) = 3$, $\psi(G) = h$ and $|V(G)| = 2h - 2$.

In the following statements G will be an h-optimal graph and $\mathcal{H}$ will be an arbitrary, fixed complete h-coloring of G. Let us also recall that $\ell = h - 2$. The results in Section 3 directly imply the following

Claim 10 *In an h-optimal graph, there are exactly two color classes of one element and exactly ℓ classes of two elements in $\mathcal{H}$.* □

Definition 2 The color classes consisting of two elements will be called *pairs*. For a vertex x in the pair, the other vertex will be denoted mostly by x'; they will be the *couples* of each other.

Notation We denote by ϕ_1 and ϕ_2 the two vertices of the singleton color classes, and we set $\Phi = \{\phi_1, \phi_2\}$. The pairs will be denoted by $M_1, M_2, \ldots, M_\ell$ and M will be their union. Moreover, F_i is the subgraph induced by the vertex set $M_i \cup \Phi$.

Claim 11 *For any $1 \leq i \leq \ell$, the subgraph F_i is isomorphic to a P_4 or it contains a triangle as a subgraph. Consequently, $\Gamma(F_i) = 3$.*

Proof Simply we use the fact that ϕ_1 and ϕ_2 are adjacent and because of the completeness of the coloring $\mathcal{H}$, both of them have at least one neighbor in M_i. The last statement follows from the monotonicity of Γ, and simple facts that $\Gamma(P_4) = \Gamma(K_3) = 3$ and that the only 4-vertex graph having Grundy number greater than 3 is K_4. □

Definition 3 If F_i is isomorphic to P_4, then M_i is called a *pair of P_4-type*, otherwise we call it a *pair of K_3-type*.

Notation Let S_i be any maximal stable set of G containing M_i.

Claim 12 *For $1 \leq i, j \leq \ell$, the pairs M_i and M_j are joined by exactly one edge.*

Proof The completeness of $\mathcal{H}$ implies that S_i is disjoint from Φ. Furthermore, because of $\Gamma(F_j) = 3$, S_i has to intersect F_j in some vertex, by Claim 4. Consequently, $S_i \cap M_j \neq \emptyset$. Similarly we obtain $S_j \cap M_i \neq \emptyset$. These conditions (two empty triples inside $M_i \cup M_j$) leave room for just one edge between M_i and M_j. □

Claim 13 *If $u \in M_i$ and $\{\phi_1, \phi_2, u\}$ induces K_3, then u is an isolated vertex in $G[M]$.*

Proof Let H be a triangle induced by $\{\phi_1, \phi_2, u\}$. Assume on the contrary that u has a neighbor in M_j for some $j \neq i$. Then the set S_j is disjoint from $V(H)$ and dominates H, which implies $\Gamma(G) > 3$, by Claim 4. A contradiction. □

From Claims 11 and 13 we obtain

Corollary 1 *If a pair has no isolated vertex in $G[M]$, then it is of P_4-type.* □

Definition and notation Let J be the union of the pairs containing some isolated vertex of the graph $G[M]$. In what follows we say that a vertex is *non-isolated* if it is not isolated in $G[M]$. Let $T = M \setminus J$.

In a series of the three subsequent claims we reveal the structure of the graph induced by T. Next, in Lemma 4 we analyze the number of pairs in the set J.

Claim 14 *T induces a bipartite graph.*

Before we prove Claim 14, we give some definitions and state some facts.

Definition 4 An induced P_4 of G with the middle edge $\phi_1\phi_2$ will be called an *emphasized P_4*.

Proposition 2 *If a maximal stable set S contains some pair then it intersects each emphasized P_4 in at least one endvertex.*

Proof As we know from Claim 4, the set S intersects every P_4. By assumption, S contains some pair M_i. Using the properties of the complete coloring, both ϕ_1 and ϕ_2 have some neighbor in M_i. Therefore $\phi_1, \phi_2 \notin S$ and hence $S \cap P_4$ must be an endvertex. □

Now, let us define the following sets of vertices:

$$X = \{x \mid x \in T \text{ and } x \text{ is adjacent to } \phi_2\},$$

$$Y = \{y \mid y \in T \text{ and } y \text{ is adjacent to } \phi_1\}.$$

By Claim 13, T is the disjoint union of X and Y, moreover, $|X| = |Y|$.

Proposition 3 *For any vertex $x \in X$, there exists a maximal stable set S containing some pair but not containing x. The same is true for any $y \in Y$, too.*

Proof By definition, every vertex in T is non-isolated in $G[M]$. Since $x \in T$, there exists a vertex z adjacent to x such that $z \in M_i$ for some i. The set S_i above can play the role of S in the proposition. □

Proof of Claim 14 We show that Y does not induce any edges. For X, the proof is analogous.

Suppose $\eta_1, \eta_2 \in Y$ and η_1, η_2 are adjacent. Then $\{\eta_1', \phi_2, \phi_1, \eta_2\}$ induces a P_4 because η_2 has exactly one neighbor in $\{\eta_1', \eta_1\}$. By Proposition 3, we have a maximal stable set S containing some pair with $\eta_1' \notin S$. By Proposition 2, S intersects both induced 4-paths $\eta_1'\phi_1\phi_2\eta_1$ and $\eta_1'\phi_1\phi_2\eta_2$, in one of their endvertices. It does not contain η_1', thus it must contain both η_1 and η_2, a contradiction. □

Claim 15 *T induces a $2K_2$-free graph.*

Proof Assume on the contrary that we have a $2K_2$ in $G[T]$. We denote its edges by xy and $\overline{x}\,\overline{y}$. Take the 6-vertex subgraph H of G induced by $\{x, \overline{x}, y, \overline{y}, \phi_1, \phi_2\}$. The subgraph H has a maximal stable set $\{x, \overline{y}\}$ and the remaining graph is a P_4, because of the definition of the sets X and Y. By Claim 4, we get $\Gamma(H) > 3$, a contradiction. □

Let τ be the number of pairs in T. It is a well-known fact that for a bipartite graph with partite sets X, Y of the same cardinality, $2K_2$-freeness is equivalent to the following.

Property (∗): X and Y can be ordered in such a way that $X = (x_1, x_2, \ldots, x_\tau)$, $Y = (y_1, y_2, \ldots, y_\tau)$, and $N(x_i) \subseteq N(x_j)$ for every $i < j$ and $N(y_i) \supseteq N(y_j)$ for every $i < j$.

In the next claim we use the extended graphs B' defined in Section 3.

Claim 16 *The set T induces a graph isomorphic to the graph B'_τ.*

Proof Let us pick a counterexample T of smallest size. We state that in T the couple of x_1 is y_1. Suppose, for a contradiction, that the couple of x_1 is y_j, for some $j > 1$. The pair x_1y_1 cannot be an edge since otherwise, by Property (∗), y_1 would be adjacent to everything in X and it would not have any couple. From the nonadjacency of x_1 and y_1 and Property (∗), x_1 is isolated in T. Consequently, y_j has some neighbor in every class of the complete coloring, except its own class $\{x_1, y_j\}$. Since $G[T]$ is bipartite, y_j is adjacent to all the vertices in $X \setminus x_1$. By Property (∗), y_1 is also adjacent to these vertices. Consequently, the only couple of y_1 could be x_1, a contradiction. Thus, the couple of x_1 is y_1 indeed.

Taking the graph induced by $T' = T \setminus \{x_1, y_1\}$, would be a smaller counterexample. Claim 16 is proved. □

The next step is to manage the isolated vertices of $G[M]$.

Lemma 4 *The set J contains exactly two pairs.*

Proof Let ξ be the number of pairs in J.

Claim 17 $\xi \leq 2$

Proof The assertion obviously holds for $\ell = 2$. Hence assume that $\ell \geq 3$.

Suppose for a contradiction that $\xi \geq 3$. Without loss of generality, we may assume that $M_i = \{u_i, u'_i\}$, $i \in \{1, 2, 3\}$ are arbitrary pairs in J such that u_i is isolated in $G[M]$. By the completeness of the coloring $\mathcal{H}$, the non-isolated vertices u'_i of pairs M_i are mutually adjacent. In what follows we use Q to denote the complete subgraph induced by $\{u'_1, u'_2, u'_3\}$.

If $\ell \geq 4$, then there exists a pair $M_j = \{r, r'\}$, $j > 3$. For each $i \in \{1, 2, 3\}$, consider the edge e_i between M_i and M_j. Obviously, e_i contains u'_i. Consequently, the stable set $\{r, r'\}$ dominates Q. Using Claim 4, we obtain a contradiction. Hence it remains to consider the case when $\ell = 3$.

Let $\ell = 3$ and assume for a contradiction that $\xi \geq 3$, which in this case, by $\xi \leq \ell$, means $\xi = 3$. Also recall that under such assumptions we consider only 8-vertex graphs. Now, observe that for $i \in \{1, 2\}$ the vertex sets $N_i = V(Q) \setminus N(\phi_i)$ have the following properties:

(Π_1) $N_1 \cup N_2 = Q$.

Suppose not. Then, considering some uncovered vertex, it would be isolated in $G[M]$ (by Claim 13), contradicting the completeness of the coloring $\mathcal{H}$.

(Π_2) $N_1 \cap N_2 \neq \emptyset$.

For a contradiction, using (Π_1), we may assume that $N_1 = \{u'_2, u'_3\}$, $N_2 = \{u'_1\}$. Taking the triangle induced by $\{\phi_2, u'_2, u'_3\}$ and the stable set M_1 which dominates this triangle, we get a contradiction by Claim 4.

Thus we may assume that N_1 and N_2 have some common vertex, say u'_1. Hence, by the completeness of the coloring $\mathcal{H}$, it holds that $\phi_1 u_1 \in E$ and $\phi_2 u_1 \in E$.

Between the two sets $\{\phi_1, \phi_2\}$ and $\{u'_2, u'_3\}$ we have some edge, because of the connectedness condition, say $\phi_2 u'_2 \in E$. This implies $\phi_1 u'_2 \notin E$, since otherwise we would contradict (Π_1).

In order to avoid the P_4 induced by $\{\phi_1, \phi_2, u'_2, u'_3\}$ and dominated by the stable set M_1, we state that, by (Π_1), either $\phi_1 u'_3$ or $\phi_2 u'_3$ is an edge. If $\phi_2 u'_3 \in E$, then the triangle induced by $\{\phi_2, u'_2, u'_3\}$ is dominated by M_1, a contradiction by Claim 4. Hence $\phi_2 u'_3 \notin E$, and consequently $\phi_1 u'_3 \in E$. However, in this case the P_4 induced by $\{u_1, \phi_1, u'_3, u'_2\}$ is dominated by the stable set $\{\phi_2, u'_1\}$. Since this cannot be affected by any further edges, we get a contradiction by Claim 4. □

Claim 18 $\xi \geq 2$.

Proof If $h = 4$, then the assertion holds, since by Claim 12, each of the two pairs contains exactly one vertex that is isolated in $G[M]$. In what follows we assume that $h \geq 5$.

Case 1 We prove $\xi \neq 0$.

Suppose that $\xi = 0$. By Claim 16, it holds that $G[T]$ is isomorphic to B'_τ. There are two isolated vertices in this graph but $M = T$, by the assumption of the claim, which means that there is no isolated vertex in M, a contradiction.

Case 2 We prove $\xi \neq 1$.

Let $M_1 = \{u_1, u'_1\}$ be a pair with u_1 being isolated in $G[M]$, and let $M_i = \{u_i, v_i\}$, $i \in \{2, \ldots, \ell\}$ be the pairs that have no vertices that are isolated in $G[M]$, consequently, the pairs of P_4-type. Assume that $\{u_2, \ldots, u_\ell\}$ are adjacent to ϕ_1, while $\{v_2, \ldots, v_\ell\}$ to ϕ_2, and consider M_1, two distinct pairs M_i, M_j chosen *arbitrarily* from $M \setminus J$. Recall, that by Claim 12 there are exactly three edges between the vertices of M_1, M_i and M_j, while by Claim 16 it holds that $u_iu_j, v_iv_j \notin E$. Consequently, we have three possibilities:

(a) $u'_1v_i, u'_1u_j, u_iv_j \in E$,

(b) $u'_1v_i, u'_1u_j, v_iu_j \in E$,

(c) $u'_1v_i, u'_1v_j, u_iv_j \in E$.

By symmetry, and by Claim 14, there are no further possibilities. Also note, that besides the above-mentioned edges it will be enough to consider the edges between Φ and M_1 and that by Claim 13, the vertex u'_1 cannot be a common neighbor of ϕ_1 and ϕ_2.

We start with a simultaneous analysis of (a) and (b). Assume that $\phi_2u'_1 \in E$. Then (a) implies that a path P_4 induced by $\{u_j, \phi_1, \phi_2, v_j\}$ is dominated by $\{u_i, u'_1\}$, while from (b) we obtain a triangle induced by $\{\phi_2, u'_1, v_i\}$ and dominated by M_j. Since this cannot be affected by adding any further edges, $\phi_2u'_1 \notin E$ and hence, by completeness, ϕ_2u_1 must be an edge. If so, then for (a), independently of whether $\phi_1u_1 \in E$ or $\phi_1u'_1 \in E$, a path P_4 induced by $\{u_j, \phi_1, \phi_2, v_i\}$ is dominated by M_1. Now, for (b), if $\phi_1u'_1 \notin E$, then $\{u_1, u_j\}$ dominates a path P_4 induced by $\{u'_1, v_i, \phi_2, \phi_1\}$. On the other hand, if $\phi_1u'_1 \in E$, then a triangle induced by $\{\phi_1, u_j, u'_1\}$ is dominated by M_i. Note that the analysis in case (b) is independent of whether ϕ_1u_1 is an edge.

Hence $\phi_2u_1 \notin E$ and it finally follows that neither $\phi_2u'_1$ nor ϕ_2u_1 is an edge, which contradicts the completeness of the coloring $\mathcal{H}$.

It remains to consider case (c). First, observe that whenever all pairs M_i, M_j, $i, j \in \{2, \ldots, \ell\}$ satisfy $u'_1v_i, u'_1v_j, u_iv_j \in E$, then u'_1 is adjacent to each vertex in $\{v_2, \ldots, v_\ell\}$. Now, considering adjacencies between the pairs in $M \setminus J$, by Claim 16

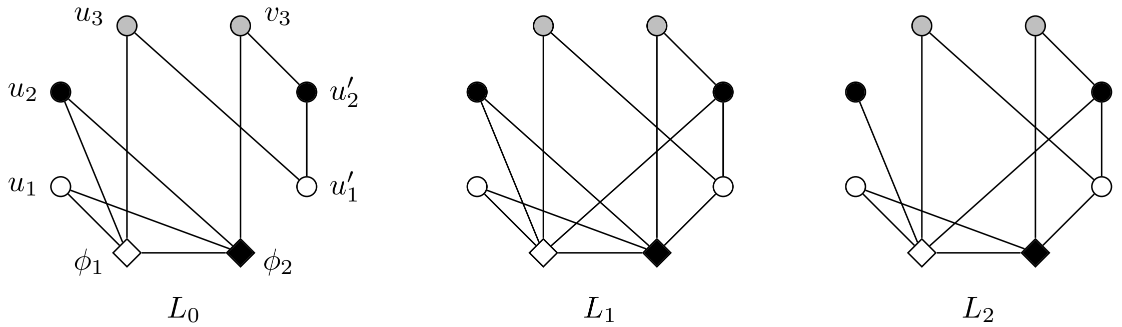


Figure 5: All h-optimal graphs for $h = 5$ and their achromatic colorings ($\chi = \Gamma = 3$, $\psi = 5$, $n = 2\psi - \chi + 1 = 8$)

either v_2, u_ℓ or v_ℓ, u_2 are isolated in $G[M \setminus J]$. Extending the scope to $G[M]$, both v_2 and v_ℓ become neighbors of u'_1, but either u_ℓ or u_2 remains isolated. This clearly contradicts our assumption that $\xi = 1$. □

We have shown Claims 17 and 18, and thus Lemma 4 as well. □

Let L_0, L_1 and L_2 be the graphs presented in Figure 5.

Lemma 5 *If $h = 5$, then a graph G is h-optimal if and only if it is isomorphic to one of the graphs in $\{L_0, L_1, L_2\}$.*

Proof By Lemma 4 a graph G contains exactly one pair of P_4-type that consists of non-isolated vertices. Let $M_3 = \{u_3, v_3\}$ be such a pair. For $i \in \{1, 2\}$ let $M_i = \{u_i, u'_i\}$ be the pairs having u_i isolated in $G[M]$.

Observe that u'_1 and u'_2 are adjacent and each of them must have some neighbor in M_3. Moreover, since vertices in M_3 are non-isolated, by Claim 12 the neighbors of u'_1, u'_2 in M_3 must be disjoint. Without loss of generality we may assume that $u'_1u_3, u'_2v_3 \in E$. By Claim 12, there are no other edges between $M_1, \ldots, M_3$, so it remains to consider the edges incident to ϕ_1 or ϕ_2.

Since M_3 is of P_4-type, assume that $\phi_1u_3, \phi_2v_3 \in E$ and consequently $\phi_1v_3, \phi_2u_3 \notin E$. Now, if both $\phi_2u'_1 \in E$ and $\phi_2u'_2 \in E$, then M_3 dominates a triangle induced by $\{\phi_2, u'_1, u'_2\}$, a contradiction. This cannot be affected by any further edges and hence ϕ_2 cannot be simultaneously adjacent to u'_1 and u'_2. Consequently, by completeness, ϕ_2 must be adjacent to at least one vertex in $\{u_1, u_2\}$.

Case 1 Assume that $\phi_2u'_1 \in E$ and $\phi_2u'_2 \notin E$.

Consequently, by completeness $\phi_2u_2 \in E$. By Claim 13 we have $\phi_1u'_1 \notin E$ and hence $\phi_1u_1 \in E$. However, since the current graph is bipartite, we need to consider further edges. If $\phi_1u'_2 \notin E$, then M_3 dominates a path P_4 induced by $\{\phi_1, \phi_2, u'_1, u'_2\}$, and this cannot be altered neither by $\phi_2 u_1$ nor by $\phi_1 u_2$. Hence $\phi_1 u_2' \in E$. The graph is still bipartite. Now, adding either $\phi_2 u_1$ or $\phi_1 u_2$ results in the graph L_1, while adding both edges gives the graph L_2.

Case 2 Assume that $\phi_2 u_2' \in E$ and $\phi_2 u_1' \notin E$.

From Claim 13 it follows that $\phi_1 u_2' \notin E$, while by completeness $\phi_1 u_1 \in E$ or $\phi_1 u_1' \in E$. Assume that $\phi_1 u_1' \in E$ and consider a subgraph H induced by $\Phi \cup M \setminus \{u_1, u_2\}$. Then a path P_4 induced by $\{\phi_1, u_1', u_2', v_3\}$ is dominated by $\{\phi_2, u_3\}$. Hence $\phi_1 u_1' \notin E$. This in turn results in a graph containing a subgraph P_4 induced by $\{\phi_1, \phi_2, u_1', u_2'\}$ and dominated by M_3. Since there are no further edges that could be added between vertices of H, we get a contradiction by Claim 4.

Case 3 Assume that $\phi_2 u_1', \phi_2 u_2' \notin E$.

By completeness, $\phi_2 u_1, \phi_2 u_2 \in E$. Note that it remains to consider the edges incident to ϕ_1. Consider a subgraph H induced by $\Phi \cup M \setminus \{u_1, u_2\}$. To argue that either ϕ_1 is adjacent to both u_1' and u_2' or to none of them observe that whenever only one of the edges is present, then M_3 dominates a path P_4 induced by $\{\phi_1, \phi_2, u_1', u_2'\}$. On the other hand, if we assume that both edges are present, then M_3 dominates a triangle induced by $\{\phi_1, u_1', u_2'\}$. If both edges are missing, then by completeness $\phi_1 u_1, \phi_2 u_2 \in E$, and we get the graph L_0, that obviously realizes a triple $(3, 3, 5)$. Thus, either we get a h-optimal graph L_0 or a contradiction by Claim 4. □

It is not hard to see that the arguments analogous to those in the proof of Lemma 5 can be used to obtain the list of all graphs that are h-optimal for $h = 4$. Let $L_1, \ldots, L_7$ be the graphs presented in Figure 6.

Lemma 6 *If $h = 4$, then a graph G is h-optimal if and only if it is isomorphic to one of the graphs in $\{L_1, \ldots, L_7\}$.*

4.1 End of the proof of Theorem 3 for $h \geq 6$

By the analysis above, to complete the proof of Theorem 3 it remains to focus on J and Φ when $h \geq 6$.

Notation Let ι_1, ι_2 be the vertices that are isolated in $G[M]$, and let ν_1, ν_2 be their couples, respectively. Let $M_i = \{u_i, v_i\}$, $i \in \{1, \ldots, \tau\}$ be the pairs in T, that is, the pairs without isolated vertices. Recall that $\tau = \ell - 2 = h - 4$.

The results on $2K_2$-free bipartite graphs entitle us to suppose that the set of edges in $G[T]$ is $\{u_i v_I \mid I > i\}$.

Claim 19 *$\{j \mid v_j \nu_1 \in E\}$ and $\{k \mid u_k \nu_1 \in E\}$ are intervals. (Here the empty set is also considered as an interval.)*

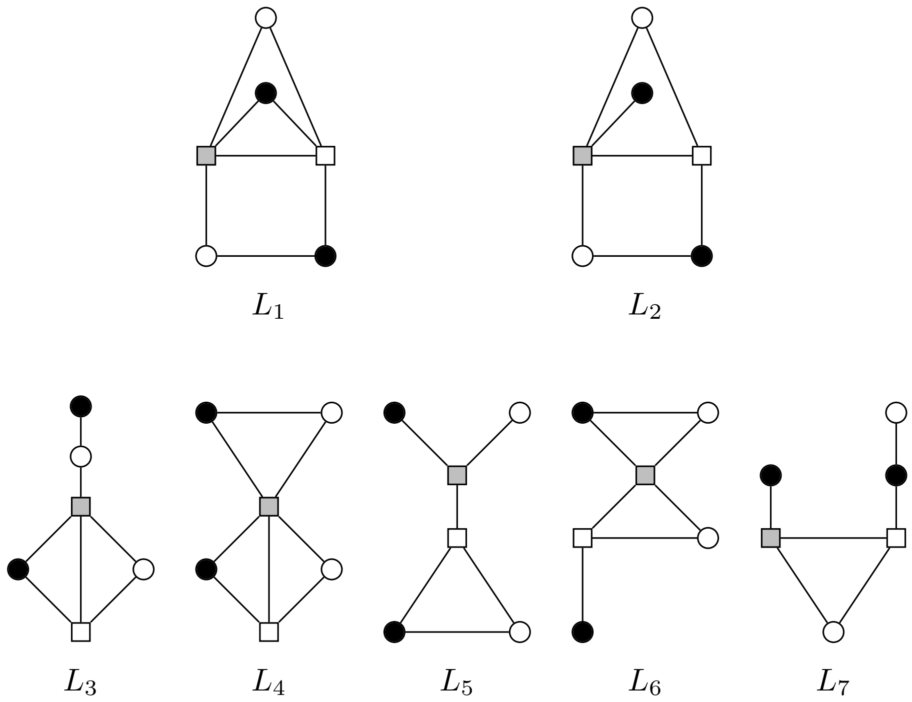


Figure 6: The seven 4-optimal graphs and their achromatic colorings

Proof Suppose first that there are subscripts j and $k > j$ such that $\nu_1 v_j \in E$ and $\nu_1 u_k \in E$. Then we would have a stable set $\{\iota_1, \nu_1\}$ dominating the P_4 induced by $\{u_k, \phi_2, \phi_1, v_j\}$, a contradiction. (The relation of ϕ_1 and ϕ_2 to the other vertices can be read out from the proof of Claim 14.)

The vertex ν_1 is adjacent to exactly one of u_i, v_i for every i. Thus, denoting the neighbor of ν_1 in Y with the smallest subscript by v_j, $N(\nu_1) \cap Y = \{v_j, v_{j+1}, \ldots, v_\tau\}$ and $N(\nu_1) \cap X = \{u_1, u_2, \ldots, u_{j-1}\}$. □

Claim 20 *One of* $N(\nu_i) \cap X$, $N(\nu_i) \cap Y$ $(i \in \{1, 2\})$ *is empty.*

Proof Otherwise $\{\nu_1, \nu_2, u_1, v_\tau\}$ would induce a K_4 and the Grundy number would be greater than 3. □

We may assume $N(\nu_2) \cap X = \emptyset$. Then

Claim 21 (a) $N(\nu_2) \cap Y = Y$, (b) $N(\nu_1) \cap Y = \emptyset$ *and* (c) $N(\nu_1) \cap X = X$.

Proof (a) is an obvious consequence of the assumption.

Suppose $N(\nu_1) \cap Y$ is a nonempty proper subset of Y. If there was a j such that $v_j \nu_1 \in E$ and $v_{j-1}\nu_1 \notin E$, then the stable set $\{u_{j-1} v_{j-1}\}$ would dominate the triangle induced by $\{\nu_1, \nu_2, v_j\}$, a contradiction. Consequently, if the Claim was not true, then such a j would not exist.

If $N(\nu_1) \cap Y = Y$ then one can find the stable set $\{u_1, v_1\}$ that dominates the triangle induced by $\{\nu_1, \nu_2, v_2\}$, a contradiction again. (Note that we used $h \geq 6$.)

Otherwise, $N(\nu_1) \cap Y = \{v_1, v_2, \ldots, v_j\}$ for some $j < \tau$. Thus $\nu_1 v_{j+1} \notin E$ and $\nu_1 u_{j+1} \in E$, contradicting the arguments in the proof of Claim 19. This implies the validity of (b), from which (c) follows, too. □

Now we can concentrate on the 8 pairs of vertices between the sets $\{\phi_1, \phi_2\}$ and $\{\nu_1, \nu_2, \iota_1, \iota_2\}$ since these are the remaining undetermined ones. First we prove

Claim 22 *$\nu_2\phi_1 \notin E$ (and $\nu_1\phi_2 \notin E$).*

Proof Otherwise the stable set $\{u_1, v_1\}$ would dominate the triangle induced by $\{\nu_2, \phi_1, v_\tau\}$. (The same works for the second statement.) □

Claim 23 *$\nu_1\phi_1 \in E$ (and $\nu_2\phi_2 \in E$).*

Proof Suppose $\nu_1\phi_2 \notin E$. Then the stable set $\{\nu_1, \phi_1\}$ dominates the P_4 induced by $\{v_1, \nu_2, v_2, u_1\}$. (The same works for the second statement.) □

The completeness of the coloring implies $\iota_2\phi_1 \in E$ and $\iota_1\phi_2 \in E$. The last fact we need is

Claim 24 *It is impossible that both $\iota_1\phi_1$ and $\iota_2\phi_2$ are non-edges.*

Proof In this case the whole graph would be bipartite (with partite sets $X \cup \{\phi_1, \nu_2, \iota_1\}$ and $Y \cup \{\phi_2, \nu_1, \iota_2\}$), a contradiction with the assumptions. □

Now we identify the notation above with that of the examples in Figure 3 in the following way: $\phi_1 \longrightarrow q_2$, $\phi_2 \longrightarrow q_1$, $\iota_1 \longrightarrow u_{\tau+1}$, $\iota_2 \longrightarrow w_1$.

Let us look now at the graphs L_1 and L_2. In both graphs $\iota_2 q_2 \in E$, moreover ι_2 and ϕ_2 are adjacent in L_1, while they are non-adjacent in L_2. The only further possible situation would be the converse but this would yield a graph isomorphic to L_2, L_1 respectively.

This completes the proof of Theorem 3. □

5 Concluding remarks

In this paper, for all realizable triples (f, g, h) of integers, we determined the minimum order of connected graphs G such that $\chi(G) = f$, $\Gamma(G) = g$, and $\psi(G) = h$. We completely described also the list of graphs attaining the minimum in all cases where $f < g \leq h$ or $f = g = 3$. For the other triples the corresponding characterization of graphs remains unsolved:

Problem 1 *For larger common values* $f = g > 3$*, and* $h > f$*, determine the list of* h*-optimal graphs.*

Since the clique number is a universal lower bound on the chromatic number, one can study the extended chain of inequalities $\omega(G) \leq \chi(G) \leq \Gamma(G) \leq \psi(G)$. In this context the following problem arises in a natural way:

Problem 2 *Let* $2 \leq a \leq b \leq c \leq d$ *be integers.*

(i) *Give necessary and sufficient conditions for the existence of connected graphs* G *with* $\omega(G) = a$, $\chi(G) = b$, $\Gamma(G) = c$, $\psi(G) = d$.

(ii) *If such graphs exist, determine their minimum order* $n_0 = n_0(a, b, c, d)$*, and characterize the graphs whose number of vertices attains this minimum.*

Probably, already some particular cases are quite hard:

Problem 3 *Solve the analogous problems for three-element subsets of* $\{\omega, \chi, \Gamma, \psi\}$.

Similar characterizations for graphs with restricted structural properties would also be of interest.

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