# 2013 UNIT VECTORS IN THE PLANE

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Abstract. Given a norm in the plane and 2013 unit vectors in this norm, there is a signed sum of these vectors whose norm is at most one.

Let $B$ be the unit ball of a norm $\|.\|$ in $\mathbb{R}^d$, that is, $B$ is an 0-symmetric convex compact set with nonempty interior. Assume $V \subset B$ is a finite set. It is shown in [1] that, under these conditions, there are signs $\varepsilon(v) \in \{-1,+1\}$ for every $v \in V$ such that $\sum_{v\in V} \varepsilon(v)v \in dB$. That is, a suitable signed sum of the vectors in $V$ has norm at most $d$. This estimate is best possible: when $V = \{e_1, e_2, \dots, e_d\}$ and the norm is $\ell_1$, all signed sums have $\ell_1$ norm d.

In this short note we show that this result can be strengthened when $d = 2$, $|V| = 2013$ (or when $|V|$ is odd) and every $v \in V$ is a unit vector. So from now onwards we work in the plane $\mathbb{R}^2$.

**Theorem 1.** *Assume $V \subset \mathbb{R}^2$ consists of unit vectors in the norm $\|.\|$ and $|V|$ is odd. Then there are signs $\varepsilon(v) \in \{-1,+1\}$ ($\forall v \in V$) such that $\|\sum_{v\in V} \varepsilon(v)v\| \le 1$.*

This result is best possible (take the same unit vector $n$ times) and does not hold when $|V|$ is even.

Before the proof some remarks are in place here. Define the convex polygon $P = \text{conv}\{\pm v : v \in V\}$. Then $P \subset B$, and $P$ is again the unit ball of a norm, $V$ is a set of unit vectors of this norm. Thus it suffices to prove the theorem only in this case.

A vector $v \in V$ can be replaced by $-v$ without changing the conditions and the statement. So we assume that $V = \{v_1, v_2, \dots, v_n\}$ and the vectors $v_1, v_2, \dots, v_n, -v_1, -v_2, \dots, -v_n$ come in this order on the boundary of $P$. Note that $n$ is odd. We prove the theorem in the following stronger form.

**Theorem 2.** *With this notation $\|v_1 - v_2 + v_3 - \dots - v_{n-1} + v_n\| \le 1$.*

**Proof.** Note that this choice of signs is very symmetric as it corresponds to choosing every second vertex of $P$. So the vector $u = 2(v_1 - v_2 + v_3 - \dots - v_{n-1} + v_n)$ is the same (or its negative) when one starts with another vector instead of $v_1$. Define $a_i = v_{i+1} - v_i$ for

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$i = 1, \dots, n-1$ and $a_n = -v_1 - v_n$ and set $w = a_1 - a_2 + a_3 - \dots + a_n$. It simply follows from the definition of $a_i$ that

$$w = -2(v_1 - v_2 + v_3 - \dots - v_{n-1} + v_n) = -u.$$

Consequently $\|u\| = \|w\|$ and we have to show that $\|w\| \leq 2$.

Consider the line $L$ in direction $w$ passing through the origin. It intersects the boundary of $P$ at points $b$ and $-b$. Because of symmetry we may assume, without loss of generality, that $b$ lies on the edge $[v_1, -v_n]$ of $P$. Then $w$ is just the sum of the projections onto $L$, in direction parallel with $[v_1, -v_n]$, of the edge vectors $a_1, -a_2, a_3, -a_4, \dots, a_n$. These projections do not overlap (apart from the endpoints), and cover exactly the segment $[-b, b]$ from $L$. Thus $\|w\| \leq 2$, indeed. □

**Remark.** There is another proof based on the following fact. $P$ is a zonotope defined by the vectors $a_1, \dots, a_n$, translated by the vector $v_1$. Here the zonotope defined by $a_1, \dots, a_n$ is simply

$$Z = Z(a_1, \dots, a_n) = \left\{ \sum_1^n \alpha_i a_i : 0 \leq \alpha_i \leq 1 \ (\forall i) \right\}.$$

The polygon $P = v_1 + Z$ contains all sums of the form $v_1 + a_{i_1} + \dots + a_{i_k}$ where $1 \leq i_1 < i_2 < \dots < i_k \leq n$. In particular with $i_1 = 2, i_2 = 4, \dots, i_k = 2k$

$$v_1 + a_2 + a_4 + \dots a_{2k} = v_1 - v_2 + v_3 - \dots - v_{2k} + v_{2k+1} \in P.$$

This immediately implies a strengthening of Theorem 1 (which also follows from Theorem 2).

**Theorem 3.** *Assume $V \subset \mathbb{R}^2$ consists of $n$ unit vectors in the norm $\|.\|$. Then there is an ordering $\{w_1, \dots, w_n\}$ of $V$, together with signs $\varepsilon_i \in \{-1, +1\}$ $(\forall i)$ such that $\|\sum_1^k \varepsilon_i w_i\| \leq 1$ for every odd $k \in \{1, \dots, n\}$.*

Of course, for the same ordering, $\|\sum_1^k \varepsilon_i w_i\| \leq 2$ for every $k \in \{1, \dots, n\}$. We mention that similar results are proved by Banaszczyk [2] in higher dimension for some particular norms.

In [1] the following theorem is proved. Given a norm $\|.\|$ with unit ball $B$ in $R^d$ and a sequence of vectors $v_1, \dots, v_n \in B$, there are signs $\varepsilon_i \in \{-1, +1\}$ for all $i$ such that $\|\sum_1^k \varepsilon_i w_i\| \leq 2d - 1$ for every $k \in \{1, \dots, n\}$. Theorem 1 implies that this result can be strengthened when the $v_i$s are unit vectors in $\mathbb{R}^2$ and $k$ is odd.

**Theorem 4.** *Assume $v_1, \dots, v_n \in \mathbb{R}^2$ is a sequence of unit vectors in the norm $\|.\|$. Then there are signs $\varepsilon_i \in \{-1, +1\}$ for all $i$ such that $\|\sum_1^k \varepsilon_i w_i\| \leq 2$ for every odd $k \in \{1, \dots, n\}$.*

The bound 2 here is best possible as shown by the example of the max norm and the sequence $(-1, 1/2), (1, 1/2), (0, 1), (-1, 1), (1, 1)$.

The **proof** goes by induction on $k$. The case $k = 1$ is trivial. For the induction step $k \to k + 2$ let $s$ be the signed sum of the first $k$ vectors with $\|s\| \leq 2$. There are vectors $u$ and $w$ (parallel with $s$) such that $s = u + w$, $\|u\| = 1, \|w\| \leq 1$. Applying Theorem 1 to $u, v_{k+1}$and $v_{k+2}$ we have signs $\varepsilon(u), \varepsilon_{k+1}$ and $\varepsilon_{k+2}$ with $\|\varepsilon(u)u + \varepsilon_{k+1}v_{k+1} + \varepsilon_{k+2}v_{k+2}\| \leq 1$. Here we can clearly take $\varepsilon(u) = +1$. Then

$$\|s + \varepsilon_{k+1}v_{k+1} + \varepsilon_{k+2}v_{k+2}\| \leq \|u + \varepsilon_{k+1}v_{k+1} + \varepsilon_{k+2}v_{k+2}\| + \|w\| \leq 2$$

finishing the proof. □

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## References

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