Given a sequence $A = (a_1, \ldots, a_n)$ of real numbers, a block $B$ of the sequence is either the empty set or it is $\{a_i, a_{i+1}, \ldots, a_j\}$ with $i \leq j$. The size $b$ of a block $B$ is the sum of its elements.

We show that when each $a_i \in [0, 1]$ and $k$ is a positive integer, there is a partition of $A$ into $k$ blocks $B_1, \ldots, B_k$ with $|b_i - b_j| \leq 1$ for every $i, j$. We extend this result in several directions.

1. Introduction

Assume $A = (a_1, \ldots, a_n)$ is a sequence of real numbers $a_i \in [0, 1]$. A block $B$ of the sequence is either the empty set or it is $\{a_i, a_{i+1}, \ldots, a_j\}$ with $i \leq j$. The size of the block $B$, to be denoted by $b$, is just the sum of the elements in $B$. Blocks $B_1, \ldots, B_k$ form a partition of $A$ if every element of $A$ belongs to exactly one block. We always assume that if $a_h$ is the last element of a non-empty block, then $a_{h+1}$ is the first element of the next non-empty block.

It is easy to see that, for a given $k \in \mathbb{N}$, there is a $k$-partition of $A$ into blocks $B_1, \ldots, B_k$, of sizes $b_1, \ldots, b_k$, such that

$$|b_i - b_j| \leq 2 \quad \text{for all } i, j \in [k].$$

(1.1)

Here and later, $[k]$ stands for the set $\{1, 2, \ldots, k\}$. To see this define $S_j = \sum_{i=1}^j a_i$ and set $S = S_n$. The condition $a_i \in [0, 1]$ implies that for every $h \in [k - 1]$ there is a subscript $m(h)$ such that $hS/k - 1/2 \leq S_{m(h)} \leq hS/k + 1/2$ and $m(h)$ is a non-decreasing function of $h$. The partial sums $S_0, S_{m(1)}, \ldots, S_{m(k-1)}, S_n$ split $A$ into $k$ blocks $B_1, \ldots, B_k$ that satisfy $S/k - 1 \leq b_i \leq S/k + 1$ for all $i$ and consequently $|b_i - b_j| \leq 2$ for all $i, j$.

In the first part of this paper we show the existence of a $k$-partition with $|b_i - b_j| \leq 1$. Then we extend this result to infinite sequences. Finally we show that the bound in (1.1) holds under much weaker conditions. Related problems are treated in [1] and [2].
2. Finite sequences

Our starting result is the following theorem.

**Theorem 2.1.** Given a sequence \( A = (a_1, \ldots, a_n) \) of real numbers \( a_i \in [0, 1] \) and a positive integer \( k \), there is a partition of \( A \) into \( k \) blocks \( B_1, \ldots, B_k \) with \( \max b_i \leq \min b_i + 1 \).

**Remarks.** This result is best possible in the sense that, in general, \( \max b_i - \min b_i \) cannot be made smaller than 1. There are many examples showing this, for instance \( A = (1/2, 1, \ldots, 1, 1/2) \) with \( k - 1 \) ones in the middle, or when every \( a_i = 1 \) and \( k \) does not divide \( n \).

When \( k > n \), the last example shows also that empty blocks have to be allowed. But no empty block can be present when \( \sum_1^n a_i > k \), and actually even when \( \sum_1^n a_i > k - 1 \).

**Proof.** Given a \( k \)-partition \( P \) of \( A \) with blocks \( (B_1, \ldots, B_k) \) let \( M(P) = \max_{i \in [k]} b_i \) and \( m(P) = \min_{i \in [k]} b_i \). We give an algorithm that finds the required partition. It starts with an arbitrary \( k \)-partition \( P \).

On each iteration, the current partition \( P \) is changed to another one, \( P^* \), and the only difference is that either the last element of \( B_h \) is moved to \( B_{h+1} \) or the first element of \( B_h \) is moved to \( B_{h-1} \) for a unique \( h \in [k] \).

Here comes the algorithm plus some comments.

1. Fix \( p \in [k] \) with \( M(P) = b_p \). So \( B_p \) is a maximal block of \( P \).
2. If \( M(P) \leq m(P) + 1 \), then stop.
3. If \( M(P) > m(P) + 1 \), then let \( m(P) = b_q \) for the \( q \in [k] \) which is closest to \( p \) (ties broken arbitrarily). Thus \( B_q \) is a minimal block of \( P \). Let \( B_h \) be the block next to \( B_q \) between \( B_p \) and \( B_q \). (Note that \( B_h \) is a non-empty block: if it were, then \( m(P) = 0 \) and we should have chosen \( B_h \) instead of \( B_q \).) So either \( p < q \) and then \( h = q - 1 \) and we define \( P^* \) by moving the last element from \( B_h = B_{q-1} \) to \( B_q \), or \( q < p \), and then \( h = q + 1 \) and \( P^* \) is obtained by moving the first element of \( B_h = B_{q+1} \) to \( B_q \). Set \( P = P^* \). If \( p = h \), then go to (1), else go to (2).

We prove next that this algorithm terminates with the required partition. Note first that in step (3) the size of every block in \( P^* \) is at most \( M(P) \). Indeed, the size of \( B_h \) does not increase, and the size of \( B_q \) increases by some \( a_i \leq 1 \) and since we have \( m(P) + 1 < M(P) \), \( m(P) + a_i < M(P) \) follows. This shows that \( M(P^*) \leq M(P) \), that is, the size of maximal block does not increase during the algorithm. Note also that no new block of size \( M(P) \) is created in step (3).

**Claim 2.2.** Step (3) is repeated at most \( kn \) times with \( B_p \) being the same block in \( P^* \) and in \( P \).
For the proof, let us define \( f(P, p) = \sum_{i=1}^{k} |i-p||B_i| \) where, as usual, \( |B_i| \) denotes the number of elements in \( B_i \). It is clear that \( f(P, p) \) takes positive integral values and is always less than \( kn \). It is also evident that \( f(P, p) < f(P^*, p) \), which proves the claim.

Thus after at most \( kn \) iteration of (3), the algorithm decreases the size of \( B_p \) and so goes to (1). Consequently it decreases either the number of maximal blocks or \( M(P) \). As there are only finitely many block partitions, the algorithm eventually terminates with (2).

\[ \square \]

\textbf{Proposition 2.3.} The above algorithm takes at most \( O(kn^3) \) steps.

The proof follows from three simple facts. Note that a block is just a set of consecutive elements of the sequence.

(a) No block can be fixed at step (1) more than once.
(b) No block can serve as maximal block for more than \( kn \) iterations of the loop “go to (2)” (and due to (a), in total, as well).
(c) There are no more than \( O(n^3) \) blocks.

\[ \square \]

Theorem \ref{thm:main} can be strengthened by removing the condition \( a_i \geq 0 \):

\textbf{Theorem 2.4.} Given a sequence \( A = (a_1, \ldots, a_n) \) of real numbers \( a_i \leq 1 \) for every \( i \in [n] \) with \( S = \sum_{i=1}^{n} a_i \geq 0 \) and a positive integer \( k \), there is a partition of \( A \) into \( k \) blocks \( B_1, \ldots, B_k \) with \( \max b_i \leq \min b_i + 1 \).

The proof is based on a lemma that can suitably preprocess the sequence \( A \).

\textbf{Lemma 2.5.} Given a sequence \( A = (a_1, \ldots, a_n) \) of real numbers \( a_i \leq 1 \) for every \( i \in [n] \) with \( \sum_{i=1}^{n} a_i \geq 0 \), there is a partition of \( A \) into blocks \( (C_1, \ldots, C_m) \) such that \( c_i = \sum_{j \in C_i} a_j \in [0, 1] \) for every \( i \in [m] \).

The proof is by induction on \( n \) and the case \( n = 1 \) is trivial. Assume the statement holds for sequences with fewer than \( n \) entries \((n \geq 2)\). We show that the statement holds for \( A = (a_1, \ldots, a_n) \). If \( S \leq 1 \), then we can choose a single block \( C_1 = A \). Otherwise \( S > 1 \) and we choose the smallest subscript \( h \in [n] \) such that the size, \( c_1 \), of the block \( C_1 = (a_1, \ldots, a_h) \) is positive. It is clear that \( c_1 \in (0, 1] \). The sequence \( A^* = (a_{h+1}, \ldots, a_n) \) has fewer than \( n \) elements, every \( a_i \leq 1 \), and the sum of its elements is \( S - c_1 > 0 \). So induction gives a partition of \( A^* \) into blocks \( (C_2, \ldots, C_m) \) with all \( c_i \in [0, 1] \). They, together with \( C_1 \) form the required partition of \( A \).

\[ \square \]

\textbf{Remark.} There is a simple algorithm that produces the partition \((C_1, \ldots, C_m)\). Namely, starting with \( A = (a_1, \ldots, a_n) \), check if there is an \( i \in [n-1] \) with \( a_ia_{i+1} \leq 0 \). If there is no such \( i \), then the partition with blocks \( C_i = (a_i) \) satisfies the requirements. If there is such an \( i \) replace \( A \) by the sequence \((a_1, \ldots, a_{i-1}, a_i + a_{i+1}, a_{i+2}, \ldots, a_n)\) of length
$n - 1$ and continue. The algorithm terminates either with the sequence 
$(0)$ consisting a single zero, or with a sequence $(c_1, \ldots, c_m)$ where each 
$c_i \in (0, 1]$. Note that preprocessing takes $O(n)$ iterations with this 
algorithm.

The proof of Theorem 2.4 is quite easy now. Just apply the prepro-
cessing lemma to $A$ to obtain the partition into blocks $(C_1, \ldots, C_m)$.
The sequence $C = (c_1, \ldots, c_m)$ satisfies the conditions $c_i \in [0, 1]$ so 
Theorem 2.1 applies and gives the suitable partition of $C$ which is, in 
fact, a suitable partition of $A$ as well.

Corollary 2.6. Given a sequence $A = (a_1, \ldots, a_n)$ of real numbers with 
a_i \in [-1, 1] for all $i$ and a positive integer $k$, there is a partition of $A$ 
into $k$ blocks $B_1, \ldots, B_k$ such that $\max b_i - \min b_i \leq 1$.

The proof follows immediately from Theorem 2.4 if $a_1 + \ldots + a_n \geq 0$. When 
a_1 + \ldots + a_n < 0, replace each $a_i$ by $-a_i$, and apply the 
same theorem. The resulting block partition is a block partition of the 
original sequence which satisfies $\max b_i - \min b_i \leq 1$. □

3. Infinite sequences

Assume now that $A = (a_1, a_2, \ldots)$ is an infinite sequence of real 
numbers $a_n \in [0, 1]$ and $a$ is a non-negative real. To extend our main 
theorem, we wish to find a partition of $A$ into blocks $(B_1, B_2, \ldots)$ such 
that $\inf b_n \leq a \leq \sup b_n \leq \inf b_n + 1$. This may not be possible if $\sum a_n$ 
is finite: for instance with $\sum a_n = 1000$ and $a = 400$ the size of the 
blocks must lie in $[399, 401]$. No set of blocks of this type can partition 
$A$, clearly. The case is different when $\sum a_n = \infty$.

Theorem 3.1. Given a sequence $A = (a_1, a_2, \ldots)$ of real numbers 
a_i \in [0, 1] with $\sum a_n = \infty$ and a real number $a \geq 0$, there is a partition 
of $A$ into blocks $B_1, B_2, \ldots$ with $\inf b_i \leq a \leq \sup b_i \leq \inf b_i + 1$.

Proof. The case $a = 0$ is easy: just choose $B_1$ to be the empty block 
and $B_i = (a_{i-1}, 1]$, a singleton, for $i = 2, 3, \ldots$. So assume $a > 0$. Recall 
that $S_n = \sum_{i=1}^n a_j$.

For every $k \in \mathbb{N}$ let $n(k)$ be the smallest subscript with

$$
(3.1) \quad ka \leq S_{n(k)} < ka + 1, \quad \text{so } S_{n(k)} = ka + \varepsilon(k),
$$

where $\varepsilon(k) \in [0, 1)$. We can apply Theorem 2.4 to the finite sequence 
$A_k = (a_1, \ldots, a_{n(k)})$ giving a $k$-partition of $A_k$ into blocks $(B^k_1, \ldots, B^k_k)$ 
satisfying

$$
\min_{i \in [k]} b^k_i \leq a + \frac{\varepsilon(k)}{k} \leq \max_{i \in [k]} b^k_i \leq \min_{i \in [k]} b^k_i + 1,
$$

where the middle inequality expresses the fact that the average is be-
tween the maximum and the minimum.

Assume now that, for some $k \in \mathbb{N}$, $\min_{i \in [k]} b^k_i \leq a$. Then we can 
produce the required partition as $(B^k_1, \ldots, B^k_k, B_{k+1}\ldots)$ by defining
$B_n$ for $n > k$ recursively as follows. If $B_n$ has been constructed, then let $B_{n+1}$ be the next block with $b_{n+1} \leq \min_{i \in [k]} b_i^k + 1$.

Assume now that $\min_{i \in [k]} b_i^k > a$ for every $k \in \mathbb{N}$. Then $b_i^k = a + \varepsilon_i^k$ for all $i \in [k]$ and $\sum_i \varepsilon_i^k = \varepsilon(k) < 1$. This implies that $\max_{i \in [k]} b_i^k = a + \varepsilon_i^k < a + 1$ (for some suitable $i \in [k]$).

It follows that there is an infinite subset $I$ of $\mathbb{N}$ such that $B_1^k$ is the same block for all $k \in I$. Call this block $B_1$. Further, there is an infinite $I_2 \subset I_1$ such that $B_2^k$ is the same block for all $k \in I_2$, call this block $B_2$, etc. We get a partition $(B_1, B_2, \ldots)$. Here $\sup b_i \leq a + 1$ follows from the inequality at the end of the last paragraph.

We show finally that $\inf b_j = a$. Observe first that $b_j = a + \varepsilon_j^k$ for all $k \in I_j$ so we may write $b_j = a + \varepsilon_j$. Then, for $k \in I_j$,

$$
\varepsilon_1 + \cdots + \varepsilon_j = \varepsilon_j^k + \cdots + \varepsilon_j^k \leq \varepsilon(k) < 1,
$$

showing that $\lim \varepsilon_j = 0$. This proves that, indeed, $\inf b_j = a$. \hfill \Box

**Remark.** We could have chosen, instead of (3.1), $n(k)$ as the minimal subscript with

$$
ka - 1 < S_{n(k)} \leq ka, \text{ so } S_{n(k)} = ka - \varepsilon(k),
$$

starting only for $k > 1/a$, say. Essentially the same proof works with this choice.

Again one can get rid of the condition $a_n \geq 0$.

**Theorem 3.2.** Let $A = (a_1, a_2, \ldots)$ be a sequence of real numbers $a_i \leq 1$ satisfying the condition that for every $k \in \mathbb{N}$ there is $S_n > k$. Then for every real number $a \geq 0$, there is a partition of $A$ into blocks $B_1, B_2, \ldots$ with $\inf b_i \leq a \leq \sup b_i \leq \inf b_i + 1$.

**Proof.** We prove the theorem by reducing it to Theorem 3.1. We construct a block partition of $A$ so that the size of every block lies in $(0, 1]$. The construction is straightforward. Find the smallest $i$ such that $S_i > 0$. Such an $i$ exists because the sequence $S_n$ is not bounded from above. Clearly, $S_i \leq 1$; let $(a_1, \ldots, a_i)$ be the first block. The sequence $S_n - S_i$ is also unbounded from above, and we continue this process. Apply Theorem 3.1 to the constructed sequence. It is clear that its block partition is in fact a block partition of the sequence $A$ satisfying the requirements. \hfill \Box

4. **More General Settings**

Assume now that our sequence is $A = (a_1, \ldots, a_n)$. Let $s$ be a function defined on the blocks that satisfies the conditions

(i) $s(\emptyset) = 0$ and $s(B) \geq 0$ for every block $B$,

(ii) $s(B_1) \leq s(B_2) \leq s(B_1) + 1$ if $B_1 \subset B_2$ and $B_2 \setminus B_1$ is a singleton.
Theorem 4.1. Assume $A = (a_1, \ldots, a_n)$ and $k$ is positive integer. Under the above conditions on $s$ there is a partition of $A$ into $k$ blocks $B_1, \ldots, B_k$ such that $|s(B_i) - s(B_j)| \leq 1$ for every $i, j \in [k]$.

The proof consists of checking that, under conditions (i) and (ii), the algorithm for Theorem 2.1 works without any change. □

For instance, assume every $a_i$ is a $d$-dimensional vector with non-negative coordinates in the unit ball of the $\ell_p$ norm, $p \geq 1$. When $s(B) = ||\sum_{a_i \in B} a_i||$, conditions (i) and (ii) are satisfied. (Simple examples show that this is not true for an arbitrary norm.) So there is a block partition $B_1, \ldots, B_k$ of $A$ such that the norms of the sums of the elements in the blocks differ by at most one. However, unlike in the 1-dimensional case, this does not mean that corresponding vectors are “almost” equal.

Now we relax condition (ii):

(iii) $|s(B_1) - s(B_2)| \leq 1$ if $B_1$ and $B_2$ differ by one element.

In this case we can prove the same bound as in (1.1).

Theorem 4.2. Assume $A = (a_1, \ldots, a_n)$ and $k$ is positive integer. If $s$ is non-negative and satisfies conditions (i) and (iii), then there is a partition of $A$ into $k$ blocks $B_1, \ldots, B_k$ such that $|s(B_i) - s(B_j)| \leq 2$ for every $i, j \in [k]$.

Before the proof some preparation is in place. We are to consider intervals $[x, y]$ where $0 \leq x \leq y \leq n$. The interval $[i-1, i)$ is identified with the element $a_i$ of the sequence $A$. The interval $[x, y)$ is a block if $x$ and $y$ are integers. Thus block $B = (a_i, \ldots, a_j)$ can and will be identified with the interval $[i-1, j)$; here $i - 1 \leq j$. Further, $[i, i)$ is the empty block positioned between $a_i$ and $a_{i+1}$.

Define next

$$T_n^{k-1} = \{ x = (x_1, \ldots, x_{k-1}) \in \mathbb{R}^{k-1} : 0 \leq x_1 \leq \cdots \leq x_{k-1} \leq n \},$$

and set $x_0 = 0, x_k = n$. Every $x \in T_n^{k-1}$ determines a unique partition of $[0, n)$ into $k$ intervals

$$[x_0, x_1), [x_1, x_2), \ldots, [x_{k-1}, x_k).$$

Let $P(x)$ denote this partition. Also, conversely, every partition of $[0, n)$ into $k$ intervals determines a unique $x \in T_n^{k-1}$ such that $P(x)$ is equal to this partition. Note that $P(x)$ is a block partition if and only if all coordinates of $x$ are integers. In this case $P(x) = (B_1, \ldots, B_k)$ and we define

$$S(x) = (s(B_1), \ldots, s(B_k)).$$

The size $s(B)$ of block $B = (a_i, \ldots, a_j)$ depends only on the interval $[i - 1, j)$ so we may (and do) define $s([i - 1, j)) = s(B)$. For simpler notation we write $s(i, j)$ instead of $s([i, j))$. Note that $s(i, j)$ is always
non-expanding and \( s[i, i] = 0 \). Condition (iii) says or rather implies that for all \( 0 \leq i \leq j \leq n \) and all \( 0 \leq i' \leq j' \leq n \)

\[ |s[i, j] - s[i', j']| \leq |i - i'| + |j - j'|. \]

In other words, the map \( s \) defined on pairs \( (i, j) \) (with \( 0 \leq i \leq j \leq n \)) is non-expanding in the \( \ell_1 \) norm.

We are going to extend \( s \) from blocks \( [i, j] \) to intervals \([x, y]\). Assume that \( i, j \in [n], i \leq j \) and \( x \in [i - 1, i], y \in [j - 1, j] \). The point \((x, y) \in \mathbb{R}^2\) is then either in the triangle with vertices \((i - 1, j - 1), (i, j - 1), (i, j)\) or in the triangle with vertices \((i - 1, j - 1), (i - 1, j), (i, j)\) or in both. Such triangles triangulate \( T_n^2 \) and so we can extend \( s \) on each triangle linearly. This is the usual simplicial extension of \( s \) onto \( T_n^2 \). We denote it invariably by \( s \) so we have now an \( s : T_n^2 \rightarrow \mathbb{R} \) map. It is very easy to check (and we omit the details) that the extended \( s \) is also non-expanding, that is, for all \( 0 \leq x \leq y \leq n \) and all \( 0 \leq x' \leq y' \leq n \)

\[ |s[x, y] - s[x', y']| \leq |x - x'| + |y - y'|. \]

The map \( S \) was defined on the lattice points of \( T_n^{k-1} \). We can extend it now to the whole \( T_n^{k-1} \): for \( x \in T_n^{k-1} \) let

\[ S(x) = (s[x_0, x_1], \ldots, s[x_{k-1}, x_k]) \in \mathbb{R}^k. \]

**Proof of Theorem 4.2.** Write \( R(a, b) \) for the halfline starting at \( a \) and going through \( b \) where \( a, b \in \mathbb{R}^k \) are distinct. Set \( e = (1, 1, \ldots, 1) \in \mathbb{R}^k \). We are going to show that there is an \( x \in T_n^{k-1} \) such that \( S(x) \) lies on the halfline \( R(0, e) \). This is trivial if \( S(x) \) coincides with the origin for some \( x \). So we assume that \( S(x) \neq 0 \) for any \( x \in T_n^{k-1} \).

Let \( e_1, \ldots, e_k \) be the standard basis of \( \mathbb{R}^k \) and write \( \triangle \) for the simplex with vertices \( e_1, \ldots, e_k \). We define a map \( g : T_n^{k-1} \rightarrow \triangle \) by setting \( g(x) = R(0, S(x)) \cap \triangle \). As all coordinates of \( S(x) \) are non-negative and \( S(x) \neq 0 \), \( g \) is a continuous map.

The simplex \( T_n^{k-1} \) has \( k \) facets, \( F_1, \ldots, F_k \), where \( F_i \) is given by the equation \( x_{i-1} = x_i \). The facet \( F_i \) is mapped by \( g \) to points whose \( i \)th coordinate is zero. This implies that a \((k - 1 - h)\)-dimensional face \( F_{i_1} \cap \cdots \cap F_{i_h} \) of \( T_n^{k-1} \) is mapped by \( g \) onto the \((k - 1 - h)\)-face of \( \triangle \), defined by \( z_{i_1} = \cdots = z_{i_h} = 0 \) where \( z_i \) is the \( i \)th coordinate of \( z \in \triangle \). In particular, \( g \) is a one-to-one correspondence between the vertices of \( T_n^{k-1} \) and the vertices of \( \triangle \). Let \( f : \triangle \rightarrow T_n^{k-1} \) be the linear (or simplicial) extension of \( g^{-1} \) from the vertices of \( \triangle \) to the whole simplex \( \triangle \). It follows that \( g \circ f : \triangle \rightarrow \triangle \) maps each face of \( \triangle \) onto itself.

**Lemma 4.3.** If \( h : \triangle \rightarrow \triangle \) is continuous and maps each face of \( \triangle \) onto itself, then \( h \) is surjective.

This result is known, see for instance Lemma 1 in [3] or Lemma 8.2 in [4] and also [5]. For the convenience of the reader we give another short proof at the end of this paper.
The lemma implies that $g$ is also surjective. So there is an $x^* \in T_n^{k-1}$ with $g(x^*) = (1/k, \ldots, 1/k)$. Then $S(x^*) = (t, \ldots, t)$ for some $t > 0$ or for $t = 0$ when $S(x^*) = 0$ for some $x^* \in \triangle$.

It is easy to finish the proof now. The point $x^*$ defines a partition $P(x^*)$ of $[0, n]$ into $k$ intervals $[x^*_{i-1}, x^*_i)$ and $s[x^*_{i-1}, x^*_i] = t$ for all $i \in [k]$. Round each $x^*_i$ to the nearest integer $y_i$, ties broken arbitrarily. So $|x^*_i - y_i| \leq 1/2$. Now $y = (y_1, \ldots, y_{k-1})$ defines a block partition $B_1, \ldots, B_k$ of $[0, n]$ (or $A$, if you wish). As $s$ is non-expanding,

$$|s(B_i) - s(B_j)| = |s[y_{i-1}, y_i] - s[y_{j-1}, y_j]| \leq |s[y_{i-1}, y_i] - s[x^*_{i-1}, x^*_i]| + |s[x^*_{i-1}, x^*_i] - s[x^*_{j-1}, x^*_j]| + |s[x^*_{j-1}, x^*_j] - s[y_{j-1}, y_j]| \leq 1 + |t - t| + 1 \leq 2,$$

for all $i, j \in [k]$, proving the theorem.

**Proof of Lemma 4.3** Given such an $h$, the map $h_\tau : \triangle \to \triangle$ where $\tau \in [0, 1]$ defined by $h_\tau(z) = (1 - \tau)z + \tau h(z)$ is a homotopy between $h$ and the identity. Note that $h_\tau(z) \in \partial \triangle$ if $z \in \partial \triangle$ for all $\tau \in [0, 1]$.

Take now two disjoint copies, $\triangle^+$ and $\triangle^-$, of $\triangle$ and identify their boundaries. This is an $S^d$, the $d$-dimensional unit sphere. Define a map $H : S^d \to S^d$ by setting $H(x)$ equal to $h(x)$ in $\triangle^+$ if $x \in \triangle^+$ and $h(x)$ in $\triangle^-$ if $x \in \triangle^-$. The map $H$ is well-defined and continuous since $x \in \partial \triangle^\pm$ is mapped to $H(x) \in \partial \triangle^\pm$. The homotopy $h_\tau$ extends to a homotopy $H_\tau$ between $H$ and the identity on $S^d$. Thus the degree of $H$ is one. This proves the lemma as $H|_{\triangle^\pm} : \triangle^+ \to \triangle^+$ is the same as $h : \triangle \to \triangle$ and $H_\tau(z) \in \triangle^+$ for all $\tau \in [0, 1]$ if $z \in \triangle^+$.  

**Acknowledgements.** We thank the anonymous referee for valuable comments. Research of the first author was partially supported by ERC Advanced Research Grant no 267165 (DISCONV), and by Hungarian National Research Grant K 83767.

**References**


**IMRE BÁRÁNY**

Rényi Institute of Mathematics
Hungarian Academy of Sciences
Victor S. Grinberg
5628 Hempstead Rd, Apt 102, Pittsburgh
PA 15217, USA
e-mail: victor.grinberg@yahoo.com