BLOCK PARTITIONS OF SEQUENCES

IMRE BÁRÁNY, VICTOR S. GRINBERG

Abstract. Given a sequence $A = (a_1, \dots, a_n)$ of real numbers, a block B of the A is either a set $B = \{a_i, a_{i+1}, \dots, a_j\}$ where $i \le j$ or the empty set. The size b of a block B is the sum of its elements. We show that when each $a_i \in [0, 1]$ and k is a positive integer, there is a partition of A into k blocks $B_1, \dots, B_k$ with $|b_i - b_j| \le 1$ for every i, j. We extend this result in several directions.

1. Introduction

Assume $A = (a_1, \dots, a_n)$ is a sequence of real numbers $a_i \in [0, 1]$. A block B of the sequence is either the empty set or it is $\{a_i, a_{i+1}, \dots, a_j\}$ with $i \le j$. The size of the block B, to be denoted by b, is just the sum of the elements in B. Blocks $B_1, \dots, B_k$ form a partition of A if every element of A belongs to exactly one block. We always assume that if a_h is the last element of a non-empty block, then a_{h+1} is the first element of the next non-empty block.

It is easy to see that, for a given $k \in \mathbb{N}$, there is a k-partition of A into blocks $B_1, \dots, B_k$, of sizes $b_1, \dots, b_k$, such that

$$|b_i - b_j| \le 2 \text{ for all } i, j \in [k]. \tag{1.1}$$

Here and later, $[k]$ stands for the set $\{1, 2, \dots, k\}$. To see this define $S_j = \sum_1^j a_i$ and set $S = S_n$. The condition $a_i \in [0, 1]$ implies that for every $h \in [k-1]$ there is a subscript $m(h)$ such that $hS/k - 1/2 \le S_{m(h)} \le hS/k + 1/2$ and $m(h)$ is a non-decreasing function of h. The partial sums $S_0, S_{m(1)}, \dots, S_{m(k-1)}, S_n$ split A into k blocks $B_1, \dots, B_k$ that satisfy $S/k - 1 \le b_i \le S/k + 1$ for all i and consequently $|b_i - b_j| \le 2$ for all i, j.

In the first part of this paper we show the existence of a k-partition with $|b_i - b_j| \le 1$. Then we extend this result to infinite sequences. Finally we show that the bound in (1.1) holds under much weaker conditions. Related problems are treated in [1] and [2].

2010 *Mathematics Subject Classification.* Primary 05A18.

Key words and phrases. sequences of real numbers, blocks, partitions.

2. Finite sequences

Our starting result is the following theorem.

Theorem 2.1. *Given a sequence $A = (a_1, \dots, a_n)$ of real numbers $a_i \in [0,1]$ and a positive integer k, there is a partition of A into k blocks $B_1, \dots, B_k$ with* $\max b_i \le \min b_i + 1$.

Remarks. This result is best possible in the sense that, in general, $\max b_i - \min b_i$ cannot be made smaller than 1. There are many examples showing this, for instance $A = (1/2, 1, \dots, 1, 1/2)$ with $k-1$ ones in the middle, or when every $a_i = 1$ and k does not divide n. When $k > n$, the last example shows also that empty blocks have to be allowed. But no empty block can be present when $\sum_1^n a_i > k$, and actually even when $\sum_1^n a_i > k-1$.

Proof. Given a k-partition P of A with blocks $(B_1, \dots, B_k)$ let $M(P) = \max_{i \in [k]} b_i$ and $m(P) = \min_{i \in [k]} b_i$. We give an algorithm that finds the required partition. It starts with an arbitrary k-partition P. On each iteration, the current partition P is changed to another one, P^*, and the only difference is that either the last element of B_h is moved to B_{h+1} or the first element of B_h is moved to B_{h-1} for a unique $h \in [k]$.

Here comes the algorithm plus some comments.

(1) Fix $p \in [k]$ with $M(P) = b_p$. So B_p is a maximal block of P.
(2) If $M(P) \le m(P) + 1$, then stop.
(3) If $M(P) > m(P) + 1$, then let $m(P) = b_q$ for the $q \in [k]$ which is closest to p (ties broken arbitrarily). Thus B_q is a minimal block of P. Let B_h be the block next to B_q between B_p and B_q. (Note that B_h is a non-empty block: if it were, then $m(P) = 0$ and we should have chosen B_h instead of B_q.) So either $p < q$ and then $h = q-1$ and we define P^* by moving the last element from $B_h = B_{q-1}$ to B_q, or $q < p$, and then $h = q+1$ and P^* is obtained by moving the first element of $B_h = B_{q+1}$ to B_q. Set $P = P^*$. If $p = h$, then go to (1), else go to (2).

We prove next that this algorithm terminates with the required partition. Note first that in step (3) the size of every block in P^* is at most $M(P)$. Indeed, the size of B_h does not increase, and the size of B_q increases by some $a_i \le 1$ and since we have $m(P) + 1 < M(P)$, $m(P) + a_i < M(P)$ follows. This shows that $M(P^*) \le M(P)$, that is, the size of maximal block does not increase during the algorithm. Note also that no new block of size $M(P)$ is created in step (3).

Claim 2.2. *Step (3) is repeated at most kn times with B_p being the same block in P^* and in P.*

For the **proof**, let us define $f(P,p) = \sum_{i=1}^{k} |i-p||B_i|$ where, as usual, $|B_i|$ denotes the number of elements in B_i. It is clear that $f(P,p)$ takes positive integral values and is always less than kn. It is also evident that $f(P,p) < f(P^*,p)$, which proves the claim. □

Thus after at most kn iteration of (3), the algorithm decreases the size of B_p and so goes to (1). Consequently it decreases either the number of maximal blocks or $M(P)$. As there are only finitely many block partitions, the algorithm eventually terminates with (2). □

Proposition 2.3. *The above algorithm takes at most $O(kn^3)$ steps.*

The **proof** follows from three simple facts. Note that a block is just a set of consecutive elements of the sequence.

(a) No block can be fixed at step (1) more than once.
(b) No block can serve as maximal block for more than kn iterations of the loop "go to (2)" (and due to (a), in total, as well).
(c) There are no more than $O(n^2)$ blocks.

□

Theorem 2.1 can be strengthened by removing the condition $a_i \geq 0$:

Theorem 2.4. *Given a sequence $A = (a_1, \dots, a_n)$ of real numbers $a_i \leq 1$ for every $i \in [n]$ with $S = \sum_1^n a_i \geq 0$ and a positive integer k, there is a partition of A into k blocks $B_1, \dots, B_k$ with $\max b_i \leq \min b_i + 1$.*

The proof is based on a lemma that can suitably preprocess the sequence A.

Lemma 2.5. *Given a sequence $A = (a_1, \dots, a_n)$ of real numbers $a_i \leq 1$ for every $i \in [n]$ with $\sum_1^n a_i \geq 0$, there is a partition of A into blocks $(C_1, \dots, C_m)$ such that $c_i = \sum_{a_j \in C_i} a_j \in [0,1]$ for every $i \in [m]$.*

The **proof** is by induction on n and the case $n = 1$ is trivial. Assume the statement holds for sequences with fewer than n entries ($n \geq 2$). We show that the statement holds for $A = (a_1, \dots, a_n)$. If $S \leq 1$, then we can choose a single block $C_1 = A$. Otherwise $S > 1$ and we choose the smallest subscript $h \in [n]$ such that the size, c_1, of the block $C_1 = (a_1, \dots, a_h)$ is positive. It is clear that $c_1 \in (0,1]$. The sequence $A^* = (a_{h+1}, \dots, a_n)$ has fewer than n elements, every $a_i \leq 1$, and the sum of its elements is $S - c_1 > 0$. So induction gives a partition of A^* into blocks $(C_2, \dots, C_m)$ with all $c_i \in [0,1]$. They, together with C_1 form the required partition of A. □

Remark. There is a simple algorithm that produces the partition $(C_1, \dots, C_m)$. Namely, starting with $A = (a_1, \dots, a_n)$, check if there is an $i \in [n-1]$ with $a_i a_{i+1} \leq 0$. If there is no such i, then the partition with blocks $C_i = (a_i)$ satisfies the requirements. If there is such an i replace A by the sequence $(a_1, \dots, a_{i-1}, a_i + a_{i+1}, a_{i+2}, \dots, a_n)$ of length

$n-1$ and continue. The algorithm terminates either with the sequence (0) consisting a single zero, or with a sequence $(c_1, \dots, c_m)$ where each $c_i \in (0,1]$. Note that preprocessing takes $O(n)$ iterations with this algorithm.

The **proof** of Theorem 2.4 is quite easy now. Just apply the preprocessing lemma to A to obtain the partition into blocks $(C_1, \dots, C_m)$. The sequence $C = (c_1, \dots, c_m)$ satisfies the conditions $c_i \in [0,1]$ so Theorem 2.1 applies and gives the suitable partition of C which is, in fact, a suitable partition of A as well. □

Corollary 2.6. *Given a sequence $A = (a_1, ..., a_n)$ of real numbers with $a_i \in [-1,1]$ for all i and a positive integer k, there is a partition of A into k blocks $B_1, ..., B_k$ such that* $\max b_i - \min b_i \le 1$.

The **proof** follows immediately from Theorem 2.4 if $a_1 + ... + a_n \ge 0$. When $a_1 + ... + a_n < 0$, replace each a_i by $-a_i$, and apply the same theorem. The resulting block partition is a block partition of the original sequence which satisfies $\max b_i - \min b_i \le 1$. □

3. Infinite sequences

Assume now that $A = (a_1, a_2, \dots)$ is an infinite sequence of real numbers $a_n \in [0,1]$ and a is a non-negative real. To extend our main theorem, we wish to find a partition of A into blocks $(B_1, B_2, \dots)$ such that $\inf b_n \le a \le \sup b_n \le \inf b_n + 1$. This may not be possible if $\sum a_n$ is finite: for instance with $\sum a_n = 1000$ and $a = 400$ the size of the blocks must lie in [399,401]. No set of blocks of this type can partition A, clearly. The case is different when $\sum a_n = \infty$.

Theorem 3.1. *Given a sequence $A = (a_1, a_2, \dots)$ of real numbers $a_i \in [0,1]$ with $\sum a_n = \infty$ and a real number $a \ge 0$, there is a partition of A into blocks $B_1, B_2, \dots$ with* $\inf b_i \le a \le \sup b_i \le \inf b_i + 1$.

Proof. The case $a = 0$ is easy: just choose B_1 to be the empty block and $B_i = (a_{i-1})$, a singleton, for $i = 2, 3, \dots$. So assume $a > 0$. Recall that $S_n = \sum_1^n a_j$.

For every $k \in \mathbb{N}$ let $n(k)$ be the smallest subscript with

$$ka \le S_{n(k)} < ka + 1, \text{ so } S_{n(k)} = ka + \varepsilon(k), \tag{3.1}$$

where $\varepsilon(k) \in [0,1)$. We can apply Theorem 2.1 to the finite sequence $A_k = (a_1, \dots, a_{n(k)})$ giving a k-partition of A_k into blocks $(B_1^k, \dots, B_k^k)$ satisfying

$$\min_{i \in [k]} b_i^k \le a + \frac{\varepsilon(k)}{k} \le \max_{i \in [k]} b_i^k \le \min_{i \in [k]} b_i^k + 1,$$

where the middle inequality expresses the fact that the average is between the maximum and the minimum.

Assume now that, for some $k \in \mathbb{N}$, $\min_{i \in [k]} b_i^k \le a$. Then we can produce the required partition as $(B_1^k, \dots, B_k^k, B_{k+1} \dots)$ by defining

B_n for $n > k$ recursively as follows. If B_n has been constructed, then let B_{n+1} be the next block with $b_{n+1} \leq \min_{i\in[k]} b_i^k + 1$.

Assume now that $\min_{i\in[k]} b_i^k > a$ for every $k \in \mathbb{N}$. Then $b_i^k = a + \varepsilon_i^k$ for all $i \in [k]$ and $\sum_1^k \varepsilon_i^k = \varepsilon(k) < 1$. This implies that $\max_{i\in[k]} b_i^k = a + \varepsilon_i^k < a + 1$ (for some suitable $i \in [k]$).

It follows that there is an infinite subset I_1 of $\mathbb{N}$ such that B_1^k is the same block for all $k \in I_1$. Call this block B_1. Further, there is an infinite $I_2 \subset I_1$ such that B_2^k is the same block for all $k \in I_2$, call this block B_2, etc. We get a partition $(B_1, B_2, \dots)$. Here $\sup b_i \leq a + 1$ follows from the inequality at the end of the last paragraph.

We show finally that $\inf b_j = a$. Observe first that $b_j = a + \varepsilon_j^k$ for all $k \in I_j$ so we may write $b_j = a + \varepsilon_j$. Then, for $k \in I_j$,

$$\varepsilon_1 + \cdots + \varepsilon_j = \varepsilon_1^k + \cdots + \varepsilon_j^k \leq \varepsilon(k) < 1,$$

showing that $\lim \varepsilon_j = 0$. This proves that, indeed, $\inf b_j = a$. □

Remark. We could have chosen, instead of (3.1), $n(k)$ as the minimal subscript with

$$ka - 1 < S_{n(k)} \leq ka, \text{ so } S_{n(k)} = ka - \varepsilon(k),$$

starting only for $k > 1/a$, say. Essentially the same proof works with this choice.

Again one can get rid of the condition $a_n \geq 0$.

Theorem 3.2. *Let $A = (a_1, a_2, \dots)$ be a sequence of real numbers $a_i \leq 1$ satisfying the condition that for every $k \in \mathbb{N}$ there is $S_n > k$. Then for every real number $a \geq 0$, there is a partition of A into blocks $B_1, B_2, \dots$ with $\inf b_i \leq a \leq \sup b_i \leq \inf b_i + 1$.*

Proof. We prove the theorem by reducing it to Theorem 3.1. We construct a block partition of A so that the size of every block lies in $(0, 1]$. The construction is straightforward. Find the smallest i such that $S_i > 0$. Such an i exists because the sequence S_n is not bounded from above. Clearly, $S_i \leq 1$; let $(a_1, \dots, a_i)$ be the first block. The sequence $S_n - S_i$ is also unbounded from above, and we continue this process. Apply Theorem 3.1 to the constructed sequence. It is clear that its block partition is in fact a block partition of the sequence A satisfying the requirements. □

4. More general settings

Assume now that our sequence is $A = (a_1, \dots, a_n)$. Let s be a function defined on the blocks that satisfies the conditions

(i) $s(\emptyset) = 0$ and $s(B) \geq 0$ for every block B,
(ii) $s(B_1) \leq s(B_2) \leq s(B_1)+1$ if $B_1 \subset B_2$ and $B_2 \setminus B_1$ is a singleton.

Theorem 4.1. *Assume $A = (a_1, \dots, a_n)$ and k is positive integer. Under the above conditions on s there is a partition of A into k blocks $B_1, \dots, B_k$ such that $|s(B_i) - s(B_j| \le 1$ for every $i, j \in [k]$.*

The **proof** consists of checking that, under conditions (i) and (ii), the algorithm for Theorem 2.1 works without any change. □

For instance, assume every a_i is a d-dimensional vector with non-negative coordinates in the unit ball of the ℓ_p norm, $p \ge 1$. When $s(B) = ||\sum_{a_i \in B} a_i||$, conditions (i) and (ii) are satisfied. (Simple examples show that this in not true for an arbitrary norm.) So there is a block partition $B_1, \dots, B_k$ of A such that the norms of the sums of the elements in the blocks differ by at most one. However, unlike in the 1-dimensional case, this does not mean that corresponding vectors are "almost" equal.

Now we relax condition (ii):

(iii) $|s(B_1) - s(B_2)| \le 1$ if B_1 and B_2 differ by one element.

In this case we can prove the same bound as in (1.1).

Theorem 4.2. *Assume $A = (a_1, \dots, a_n)$ and k is positive integer. If s is non-negative and satisfies conditions (i) and (iii), then there is a partition of A into k blocks $B_1, \dots, B_k$ such that $|s(B_i) - s(B_j)| \le 2$ for every $i, j \in [k]$.*

Before the proof some preparation is in place. We are to consider intervals $[x, y)$ where $0 \le x \le y \le n$. The interval $[i-1, i)$ is identified with the element a_i of the sequence A. The interval $[x, y)$ is a block if x and y are integers. Thus block $B = (a_i, \dots, a_j)$ can and will be identified with the interval $[i-1, j)$; here $i - 1 \le j$. Further, $[i, i)$ is the empty block positioned between a_i and a_{i+1}.

Define next

$$T_n^{k-1} = \{x = (x_1, \dots, x_{k-1}) \in \mathbb{R}^{k-1} : 0 \le x_1 \le \cdots \le x_{k-1} \le n\},$$

and set $x_0 = 0, x_k = n$. Every $x \in T_n^{k-1}$ determines a unique partition of $[0, n)$ into k intervals

$$[x_0, x_1), [x_1, x_2), \dots, [x_{k-1}, x_k).$$

Let $P(x)$ denote this partition. Also, conversely, every partition of $[0, n)$ into k intervals determines a unique $x \in T_n^{k-1}$ such that $P(x)$ is equal to this partition. Note that $P(x)$ is a block partition if and only if all coordinates of x are integers. In this case $P(x) = (B_1, \dots, B_k)$ and we define

$$S(x) = (s(B_1), \dots, s(B_k)).$$

The size $s(B)$ of block $B = (a_i, \dots, a_j)$ depends only on the interval $[i-1, j)$ so we may (and do) define $s([i-1, j)) = s(B)$. For simpler notation we write $s[i, j)$ instead of $s([i, j))$. Note that $s[i, j)$ is always non-negative and $s[i,i)=0$. Condition (iii) says or rather implies that for all $0 \le i \le j \le n$ and all $0 \le i' \le j' \le n$

$$|s[i,j) - s[i',j')| \le |i-i'| + |j-j'|.$$

In other words, the map s defined on pairs (i,j) (with $0 \le i \le j \le n$) is non-expanding in the ℓ_1 norm.

We are going to extend s from blocks $[i,j)$ to intervals $[x,y)$. Assume that $i,j \in [n]$, $i \le j$ and $x \in [i-1,i)$, $y \in [j-1,j)$. The point $(x,y) \in \mathbb{R}^2$ is then either in the triangle with vertices $(i-1,j-1),(i,j-1),(i,j)$ or in the triangle with vertices $(i-1,j-1),(i-1,j),(i,j)$ or in both. Such triangles triangulate T_n^2 and so we can extend s on each triangle linearly. This is the usual simplicial extension of s onto T_n^2. We denote it invariably by s so we have now an $s : T_n^2 \to \mathbb{R}$ map. It is very easy to check (and we omit the details) that the extended s is also non-expanding, that is, for all $0 \le x \le y \le n$ and all $0 \le x' \le y' \le n$

$$|s[x,y) - s[x',y')| \le |x-x'| + |y-y'|.$$

The map S was defined on the lattice points of T_n^{k-1}. We can extend it now to the whole T_n^{k-1}: for $x \in T_n^{k-1}$ let

$$S(x) = (s[x_0,x_1), \dots, s[x_{k-1},x_k)) \in \mathbb{R}^k.$$

Proof of Theorem 4.2. Write $R(a,b)$ for the halfline starting at a and going through b where $a,b \in \mathbb{R}^k$ are distinct. Set $e=(1,1,\dots,1) \in \mathbb{R}^k$. We are going to show that there is an $x \in T_n^{k-1}$ such that $S(x)$ lies on the halfline $R(0,e)$. This is trivial if $S(x)$ coincides with the origin for some x. So we assume that $S(x) \ne 0$ for any $x \in T_n^{k-1}$.

Let $e_1,\dots,e_k$ be the standard basis of R^k and write $\triangle$ for the simplex with vertices $e_1,\dots,e_k$. We define a map $g : T_n^{k-1} \to \triangle$ by setting $g(x) = R(0,S(x)) \cap \triangle$. As all coordinates of $S(x)$ are non-negative and $S(x) \ne 0$, g is a continuous map.

The simplex T_n^{k-1} has k facets, $F_1,\dots,F_k$, where F_i is given by the equation $x_{i-1}=x_i$. The facet F_i is mapped by g to points whose ith coordinate is zero. This implies that a $(k-1-h)$-dimensional face $F_{i_1} \cap \dots \cap F_{i_h}$ of T_n^{k-1} is mapped by g onto the $(k-1-h)$-face of $\triangle$, defined by $z_{i_1} = \dots = z_{i_h} = 0$ where z_i is the ith coordinate of $z \in \triangle$. In particular, g is a one-to-one correspondence between the vertices of T_n^{k-1} and the vertices of $\triangle$. Let $f : \triangle \to T_n^{k-1}$ be the linear (or simplicial) extension of g^{-1} from the vertices of $\triangle$ to the whole simplex $\triangle$. It follows that $g \circ f : \triangle \to \triangle$ maps each face of $\triangle$ onto itself.

Lemma 4.3. *If $h : \triangle \to \triangle$ is continuous and maps each face of $\triangle$ onto itself, then h is surjective.*

This result is known, see for instance Lemma 1 in [3] or Lemma 8.2 in [4] and also [5]. For the convenience of the reader we give another short proof at the end of this paper.

The lemma implies that g is also surjective. So there is an $x^* \in T_n^{k-1}$ with $g(x^*) = (1/k, \dots, 1/k)$. Then $S(x^*) = (t, \dots, t)$ for some $t > 0$ or for $t = 0$ when $S(x^*) = 0$ for some $x^* \in \triangle$.

It is easy to finish the proof now. The point x^* defines a partition $P(x^*)$ of $[0, n)$ into k intervals $[x^*_{i-1}, x^*_i)$ and $s[x^*_{i-1}, x^*_i) = t$ for all $i \in [k]$. Round each x^*_i to the nearest integer y_i, ties broken arbitrarily. So $|x^*_i - y_i| \le 1/2$. Now $y = (y_1, \dots, y_{k-1})$ defines a block partition $B_1, \dots, B_k$ of $[0, n)$ (or A, if you wish). As s is non-expanding,

$$\begin{aligned}|s(B_i) \quad - \quad & s(B_j)| = |s[y_{i-1}, y_i) - s[y_{j-1}, y_j)| \\ \le \quad & |s[y_{i-1}, y_i) - s[x^*_{i-1}, x^*_i)| + |s[x^*_{i-1}, x^*_i) - s[x^*_{j-1}, x^*_j)| + \\ + \quad & |s[x^*_{j-1}, x^*_j) - s[y_{j-1}, y_j)| \le 1 + |t - t| + 1 \le 2,\end{aligned}$$

for all $i, j \in [k]$, proving the theorem. □

Proof of Lemma 4.3. Given such an h, the map $h_\tau : \triangle \to \triangle$ where $\tau \in [0, 1]$ defined by $h_\tau(z) = (1 - \tau)z + \tau h(z)$ is a homotopy between h and the identity. Note that $h_\tau(z) \in \partial\triangle$ if $z \in \partial\triangle$ for all $\tau \in [0, 1]$.

Take now two disjoint copies, $\triangle^+$ and $\triangle^-$, of $\triangle$ and identify their boundaries. This is an S^d, the d-dimensional unit sphere. Define a map $H : S^d \to S^d$ by setting $H(x)$ equal to $h(x) \in \triangle^+$ if $x \in \triangle^+$ and $h(x) \in \triangle^-$ if $x \in \triangle^-$. The map H is well-defined and continuous since $x \in \partial\triangle^\pm$ is mapped to $H(x) \in \partial\triangle^\pm$. The homotopy h_τ extends to a homotopy H_τ between H and the identity on S^d. Thus the degree of H is one. This proves the lemma as $H|_{\triangle^+} : \triangle^+ \to \triangle^+$ is the same as $h : \triangle \to \triangle$ and $H_\tau(z) \in \triangle^+$ for all $\tau \in [0, 1]$ if $z \in \triangle^+$. □

Acknowledgements. We thank the anonymous referee for valuable comments. Research of the first author was partially supported by ERC Advanced Research Grant no 267165 (DISCONV), and by Hungarian National Research Grant K 83767.

References

[1] I. Bárány, B. Doerr. Balanced partitions of vector sequences, *Lin. Alg. Appl.*, **414** (2006), 464–469.

[2] I. Bárány, V. S. Grinberg. On some combinatorial questions in finite dimensional spaces, *Lin. Alg. Appl.* **41** (1981), 1–9.

[3] R. N. Karasev. KKM-type theorems for products of simplices and cutting sets and measures by straight lines (2009) http://arxiv.org/pdf/0909.0604v1.pdf

[4] R. N. Karasev. Geometry of measures: partitions and convex bodies, (2013) http://www.rkarasev.ru/common/upload/mes_partition.pdf

[5] N. Strickland. Map from simplex to itself that preserves subsimplices, (2011) http://mathoverflow.net/questions/67318/

Imre Bárány

Rényi Institute of Mathematics
Hungarian Academy of Sciences

PO Box 127, 1364 Budapest, and
Department of Mathematics
University College London
Gower Street, London WC1E 6BT
England
e-mail: barany@renyi.hu

Victor S. Grinberg

5628 Hempstead Rd, Apt 102, Pittsburgh
PA 15217, USA
e-mail: victor_grinberg@yahoo.com