(n, m)-Fold Covers of Spheres

Imre Bárány [*] Ruy Fabila-Monroy [†] Birgit Vogtenhuber [‡]

December 17, 2014

Abstract

A well known consequence of the Borsuk-Ulam theorem is that if the d-dimensional sphere S^d is covered with less than $d+2$ open sets, then there is a set containing a pair of antipodal points. In this paper we provide lower and upper bounds on the minimum number of open sets, not containing a pair of antipodal points, needed to cover the d-dimensional sphere n times, with the additional property that the northern hemisphere is covered $m > n$ times. We prove that if the open northern hemisphere is to be covered m times then at least $\lceil \frac{d-1}{2} \rceil + n + m$ and at most $d + n + m$ sets are needed. For the case of $n = 1$ and $d \geq 2$, this number is equal to $d + 2$ if $m \leq \lfloor \frac{d}{2} \rfloor + 1$ and equal to $\lfloor \frac{d-1}{2} \rfloor + 2 + m$ if $m > \lfloor \frac{d}{2} \rfloor + 1$. If the closed northern hemisphere is to be covered m times then $d + 2m - 1$ sets are needed, this number is also sufficient. We also present results on a related problem of independent interest. We prove that if S^d is covered n times with open sets, not containing a pair of antipodal points, then there exists a point that is covered at least $\lceil \frac{d}{2} \rceil + n$ times. Furthermore, we show that there are covers in which no point is covered more than $n + d$ times.

1 Introduction

The Lusternik-Schnirelmann [5] version of the Borsuk-Ulam theorem states that in any covering of the d-dimensional sphere S^d with at most $d+1$ open sets, there is a set containing a pair of antipodal points. A natural generalization of this result has been introduced by Stahl [9]; he considered *n-fold covers*, in which every point of S^d must be covered at least n times. He showed that in every n-fold cover of S^d with at most $d+2n-1$ open sets, there is a set containing a pair of antipodal points. Using a construction of Gale [2], it can easily be shown that this bound is tight. For every $n \geq 1$, Gale constructed a set of $d+2n$ points on the d-dimensional unit sphere, with the property that every open half-space that contains the origin contains at least n points of the set. Placing an open hemisphere with its pole at each of these points provides an n-fold cover of S^d with $d+2n$ open sets, in which no set contains a pair of antipodal points. We refer to this cover as the *Gale n-fold cover* of S^d. An n-fold cover of S^d is said to be *antipodal* if none of its sets contains a pair of antipodal points.

[*]Alfréd Rényi Mathematical Institute, Hungarian Academy of Sciences, Budapest, Hungary and Department of Mathematics, University College London. Partially supported by ERC Advanced Research Grant no 267165 (DISCONV), and by Hungarian National Research Grant K 83767. barany.imre@renyi.mta.hu

[†]Departamento de Matemáticas, Cinvestav, D.F. México, México. Partially supported by grant 153984 (CONACyT, Mexico). ruyfabila@math.cinvestav.edu.mx

[‡]Institute for Software Technology, University of Technology, Graz, Austria. bvogt@ist.tugraz.at

We consider a variation on this theme. Let $m > n \geq 1$. An (n,m)-*fold* cover of S^d is an n-fold cover of S^d, in which every point of the *open* northern hemisphere is covered m times. An $\overline{(n,m)}$-*fold* cover of S^d is an n-fold cover of S^d, in which every point of the *closed* northern hemisphere is covered m times. Let $f(d,n,m)$ be the minimum number of sets in an antipodal (n,m)-fold cover of S^d with open sets. In a similar way, let $\overline{f}(d,n,m)$ be the minimum number of sets in an antipodal $\overline{(n,m)}$-fold cover of S^d with open sets. Since the case of $d = 0$ is trivial, we will always assume that $d \geq 1$.

In this paper we show lower and upper bounds on $f(d,n,m)$ (Theorem 3.4) and provide the exact value of $\overline{f}(d,n,m)$ (Theorem 3.1). We also compute the exact value of $f(d,1,m)$ (Theorem 3.2 and Proposition 3.3). The search for a lower bound of $f(d,n,m)$ lead us to study the problem of finding a point covered many times in an antipodal n-fold cover of S^d with open sets. Let then $Q(d,n)$ be the maximum integer such that in every antipodal n-fold cover of S^d with open sets there exists a point that is covered $Q(d,n)$ times. This paper is organized as follows. In Section 2 we show upper and lower bounds on $Q(d,n)$ (Theorem 2.2). In Section 3 we give our results on $f(d,n,m)$ and $\overline{f}(d,n,m)$.

2 Bounds on $Q(d,n)$

The problem of determining $Q(d,1)$ has been studied before. Its exact value of $Q(d,1) = \left\lfloor \frac{d}{2} \right\rfloor + 2$ has been settled in a series of papers by Ščepin [6], Izydorek and Jaworowski [3], and Jaworowski [4]. An explicit cover yielding the upper bound for $Q(d,1)$ was given by Simonyi and Tardos [8]. They started by covering S^d with the projections from the origin of the closed facets of a regular $(d+1)$-simplex. Afterwards, they replaced the points of these sets that were covered more than $\left\lfloor \frac{d}{2} \right\rfloor + 1$ times with a new closed set. (Although this gives a cover with closed sets, sufficiently small open neighborhoods of these sets give the desired cover.) At first glance, it seems sensible to use a similar idea to upper bound $Q(d,n)$. However, our attempts of cutting out neighborhoods of often-covered points of an n-fold cover, and placing patches of n sets instead, always produced points that were covered an excessive amount of times. Finally, we decided to use Gale's n-fold cover of S^d.

Ky Fan's theorem [1] can be used to prove a lower bound of $\left\lceil \frac{d}{2} \right\rceil + 1$ for $Q(d,1)$. For proving a lower bound of $Q(d,n)$, we will use the following reformulation of Ky Fan's theorem that has been presented in [8].

Theorem 2.1. ***(Ky Fan's Theorem.)***
Let $\mathcal{F}$ be an antipodal cover of S^d. Assume that a linear order is given on $\mathcal{F}$. Then there exist $F_1 < F_2 < \cdots < F_{d+2}$ sets of $\mathcal{F}$ such that

$$F_1 \cap -F_2 \cap F_3 \cap -F_4 \cap \cdots \cap (-1)^{d+1} F_{d+2} \neq \emptyset.$$

□

We now give our bounds on $Q(d,n)$.

Theorem 2.2. $\left\lceil \frac{d}{2} \right\rceil + n \leq Q(d,n) \leq d + n.$

Proof. First we prove the lower bound. Let $\mathcal{F}$ be an antipodal n-fold cover of S^d with open sets. The intersections of all intersecting subsets of $\mathcal{F}$ consisting of n sets form an antipodal 1-fold cover $\mathcal{F}'$ of S^d with open sets. Explicitly, $\mathcal{F}' := \{\bigcap \mathcal{C} : \mathcal{C} \subset \mathcal{F}, |\mathcal{C}| = n \text{ and } \bigcap \mathcal{C} \neq \emptyset\}$. For an arbitrary linear order of $\mathcal{F}$, every set of $\mathcal{F}'$ is of the form $C = \bigcap_{i=1}^{n} F_i$ for some

$F_1 < F_2 < \cdots < F_n$ in $\mathcal{F}$. Hence, we may assign the tuple $v(C) := (F_1, F_2, \ldots, F_n)$ to C and define a linear order on $\mathcal{F}'$, by setting $C_1 < C_2$ if and only if $v(C_1) < v(C_2)$ in the lexicographical order of the tuples $v(C_1)$ and $v(C_2)$. By Ky Fan's theorem there exist sets $C_1 < C_2 < \cdots < C_{d+2}$ of $\mathcal{F}'$ such that $\bigcap_{i=1}^{d+2}(-1)^{i-1}C_i$ is not empty. Since $\mathcal{F}'$ is an antipodal cover, the first coordinates of the tuples associated to consecutive C_i's are different; by the additional assumption that $v(C_1) < \cdots < v(C_{d+2})$, all of these first coordinates are also pairwise different. Let $x \in \bigcap_{j=1}^{\lceil (d+2)/2 \rceil} C_{2j-1}$. This point is in all the first coordinates (sets) of the tuples $v(C_{2j-1})$, and in all the coordinates (sets) of the last tuple. As all of these sets are different, x is in at least $\lceil (d+2)/2 \rceil + n - 1 = \lceil \frac{d}{2} \rceil + n$ different sets of $\mathcal{F}$.

The upper bound is given by Gale's n-fold cover of S^d. In this cover a point $x \in S^d$ is not covered by precisely those hemispheres (sets) whose poles are separated from x by the hyperplane through the origin and orthogonal to $\vec{x}$. Since there are at least n of these sets and there are $d + 2n$ sets in this cover, x is covered at most $d + n$ times. □

We conjecture that the upper bound of Theorem 2.2 is tight.

Conjeture 2.3. *$Q(d, n) = d + n$ for $n \geq 2$.*

3 Bounds on $f(d, n, m)$ and $\overline{f}(d, n, m)$.

In this section we prove our results for $f(d, n, m)$ and $\overline{f}(d, n, m)$. We start by showing the exact values of $\overline{f}(d, n, m)$ and $f(d, 1, m)$.

Theorem 3.1. *$\overline{f}(d, n, m) = d + 2m - 1$ for $m > n$.*

Proof. Let $\mathcal{F}$ be an antipodal $\overline{(n, m)}$-fold cover of S^d with open sets. Note that the intersection of $\mathcal{F}$ with the equator is an antipodal m-fold cover of S^{d-1} with open sets. Therefore, as shown by Stahl [9], $|\mathcal{F}| \geq d + 2m - 1$, which proves the lower bound. For the upper bound, rotate Gale's m-fold cover of S^d so that one of the hemispheres (sets) in this cover coincides with the southern hemisphere. Remove this hemisphere to obtain an antipodal $\overline{(n, m)}$-fold cover of S^d with $d + 2m - 1$ open sets. □

Theorem 3.2. *For $d \geq 2$, $f(d, 1, m)$ is equal to:*

$$f(d, 1, m) = \begin{cases} d + 2 & \text{if } m \leq \lfloor \frac{d}{2} \rfloor + 1, \\ \lfloor \frac{d-1}{2} \rfloor + 2 + m & \text{if } m \geq \lfloor \frac{d}{2} \rfloor + 1. \end{cases}$$

Proof. First we prove the lower bound. Let $\mathcal{F}$ be an antipodal $(1, m)$-fold cover of S^d with open sets. Since $\mathcal{F}$ covers the equator once, there is a point in the equator of S^d covered at least $Q(d-1, 1) = \lfloor \frac{d-1}{2} \rfloor + 2$ times. Just below this point there is a point in the southern hemisphere covered by the same sets; its antipodal point in the northern hemisphere is covered by at least m other sets. Thus there are at least $\lfloor \frac{d-1}{2} \rfloor + 2 + m$ sets in $\mathcal{F}$. This lower bound is tight for $m \geq \lfloor \frac{d}{2} \rfloor + 1$. For $m < \lfloor \frac{d}{2} \rfloor + 1$, the better lower bound of $d + 2$ follows immediately from Lusternik-Schnirelmann theorem [5] and the fact that $\mathcal{F}$ is a 1-fold cover of S^d.

For the upper bound we carefully construct an antipodal $\left(1, \lfloor \frac{d}{2} \rfloor + 1\right)$-fold cover of S^d with $d + 2$ open sets. This cover proves the tight upper bound of $d + 2$ for $m \leq \lfloor \frac{d}{2} \rfloor + 1$.

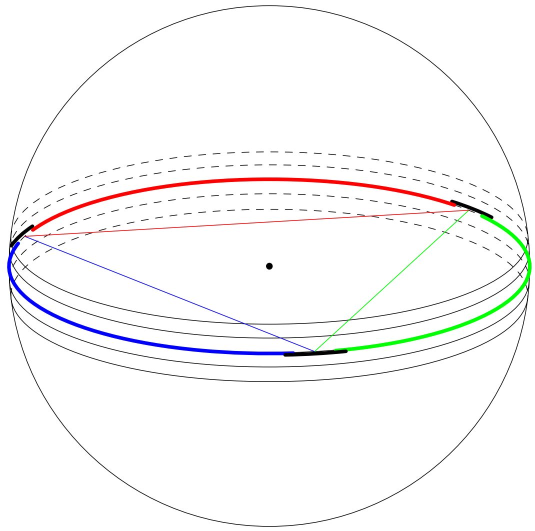

Figure 1: The 1-fold cover of the equator in the proof of Theorem 3.2.

For $m > \lfloor \frac{d}{2} \rfloor + 1$, we add $m - \lfloor \frac{d}{2} \rfloor - 1$ open northern hemispheres to this cover to produce an antipodal $(1, m)$-fold cover of S^d with $\lfloor \frac{d-1}{2} \rfloor + 2 + m$ open sets.

Assume that S^d is the unit sphere centered at the origin. We start by constructing a 1-fold cover of its equator S^{d-1} (the intersection of S^d with the hyperplane $x_{d+1} = 0$) in the following way. Fix a regular d-simplex τ centered at the origin and having its vertices on S^{d-1}. Project its closed facets from the origin to S^{d-1} and let $F'_1, \ldots, F'_{d+1}$, be these projections. This produces an antipodal cover of S^{d-1} with $d+1$ closed sets. Let D' be the set of points of S^{d-1} that are covered at least $\lceil \frac{d+2}{2} \rceil = \lceil \frac{d}{2} \rceil + 1$ times in this cover. Note that V' is closed, and since there are only $d+1$ sets F'_i, it does not contain a pair of antipodal points. Choose $\varepsilon_2 > \varepsilon_1 > 0$. Let D be the open ε_2-neighborhood of D', in S^{d-1}. Further, let F_i be the open ε_1-neighborhood of $F'_i \setminus D$, also in S^{d-1}. Choose ε_2 small enough such that none of $F_1, F_2, \ldots, F_{d+1}$ and D contain a pair of antipodal points. Choose ε_1 small enough with respect to ε_2, so that every point $x \in S^{d-1}$ has an open neighborhood that is covered by at most $\lceil \frac{d}{2} \rceil$ of the sets F_i. Then $\mathcal{F} := \{F_1, F_2, \ldots, F_{d+1}, D\}$ is an antipodal 1-cover of S^{d-1} with $d+2$ open sets. See Figure 1 for an illustration of the $d = 2$ case.

We now extend $\mathcal{F}$ to an antipodal $\left(1, \lfloor \frac{d}{2} \rfloor + 1\right)$-fold cover of S^d with open sets. Roughly speaking, we first extend all sets of $\mathcal{F}$ to parts of a "belt". Afterwards, we further extend the "facet" sets F_i of $\mathcal{F}$ to cover the northern hemisphere and the set D of $\mathcal{F}$ to cover the southern hemisphere. Let π be the orthogonal projection of $\mathbb{R}^{d+1}$ to the hyperplane $x_{d+1} = 0$. Let $0 < \delta'_1 < \delta'_2 < 1$. Let C''_i be the set of points $x \in \mathbb{R}^{d+1}$ with $\pi(x) \in F_i$, whose last coordinate satisfies $-\delta'_2 < x_{d+1} < \delta'_2$. Similarly, let C''_{d+2} be the set of points $x \in \mathbb{R}^{d+1}$ with $\pi(x) \in D$, whose last coordinate satisfies $-\delta'_2 < x_{d+1} < \delta'_1$. Note that while the facet sets F_i are extended symmetrically to the north and the south, the set D is extended further to the south than to the north.

Next, for each $1 \leq i \leq d+2$, let C'_i be the set of points $x \in S^d$ such that the infinite ray with apex in the origin and passing through x intersects C''_i. The sets C'_i the parts of "belts" on the sphere. Let $\delta_i = \sin \tan^{-1}(\delta'_i)$ be the corresponding heights of the belt parts,

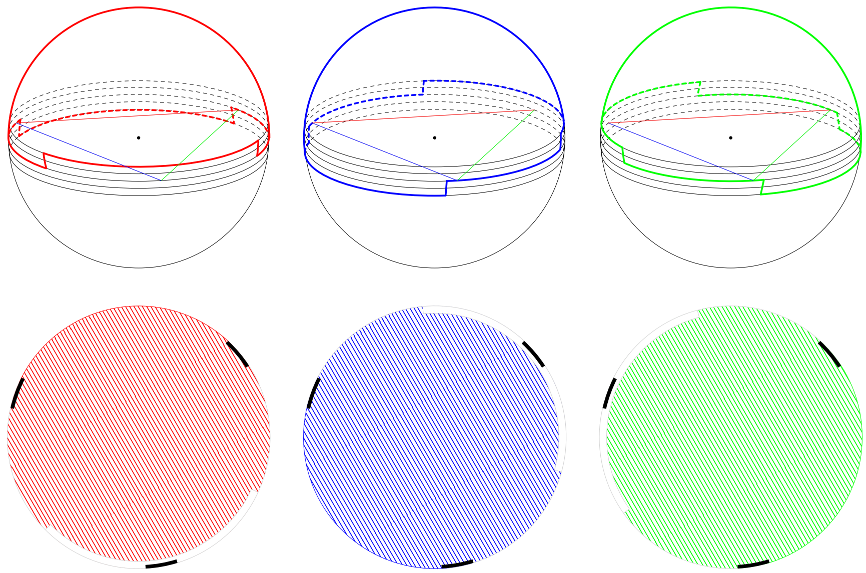

Figure 2: The covering of the northern hemisphere in the proof of Theorem 3.2

for $i = 1, 2$.

Finally, for $1 \leq i \leq d+1$, let C_i be the union of the open northern hemisphere with C'_i, minus the closure of $-C'_i$. Further, let C_{d+2} be the union of the southern hemisphere with C'_{d+2}, minus the closure of $-C'_{d+2}$. See Figures 2 and 3 for an illustration of the northern and southern hemispheres respectively (for the case $d = 2$ and $m = 2$). By construction, all these sets are open and do not contain a pair of antipodal points of S^d. What remains to show is that $\mathcal{C} := \{C_1, \ldots, C_{d+2}\}$ is in fact a $\left(1, \left\lfloor \frac{d}{2} \right\rfloor + 1\right)$-cover of S^d.

We distinguish a few cases with respect to the value of the last coordinate of the points of S^d. The set of points of S^d whose last coordinate is greater than δ_2 are covered $d+1$ times by $C_1, \ldots, C_{d+1}$. The set of points of S^d whose last coordinate is less than or equal to $-\delta_2$ are covered once by C_{d+2}.

To each point $x \in S^d$ whose last coordinate satisfies $-\delta_2 < x_{d+1} \leq \delta_2$, we assign the point x' in the equator such that $\pi(x)$ lies on the infinite ray from the origin to x'. Suppose that the last coordinate of x is greater than zero. By our choice of ε_1 and ε_2, there is an open neighborhood of x' in S^{d-1} that is covered by at most $\left\lceil \frac{d}{2} \right\rceil$ sets of $-F_1, \ldots, -F_{d+1}$. Assume without loss of generality that $-F_1, -F_2, \ldots, -F_{\left\lfloor \frac{d}{2} \right\rfloor + 1}$ do not cover this neighborhood, then at least $C_1, C_2, \ldots, C_{\left\lfloor \frac{d}{2} \right\rfloor + 1}$ cover x. Therefore $\mathcal{C}$ covers the open northern hemisphere $\left\lfloor \frac{d}{2} \right\rfloor + 1$ times. If the last coordinate of x is greater than $-\delta_2$ and at most zero, then x is covered by the extensions of the sets in $\mathcal{F}$ that cover x'. Thus, x is covered at least once and $\mathcal{C}$ is a $\left(1, \left\lfloor \frac{d}{2} \right\rfloor + 1\right)$-cover of S^d. □

Proposition 3.3. *$f(1, 1, m) = 2 + m$.*

Proof. For the upper bound, note that an antipodal $(1, m)$-cover of S^1 with $2 + m$ open sets can be obtained by adding $m - 1$ open northern hemispheres to an antipodal 1-cover of S^1

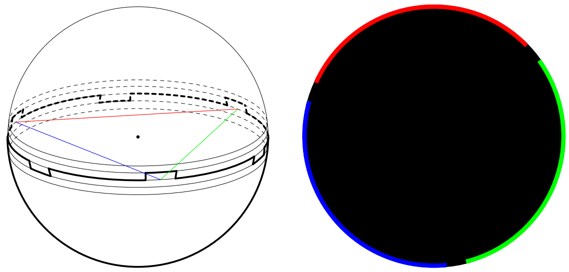

Figure 3: The covering of the southern hemisphere in the proof of Theorem 3.2

with three open sets. For the lower bound, suppose first that there exist two points x and y on the open upper hemisphere that are not covered by the same m sets. Then on the path from x to y on the northern hemisphere, there exists a point z that is covered by at least $m+1$ different sets. As its antipodal point $-z$ needs at least one set to be covered as well, this gives a total of at least $m+2$ sets. Hence, assume that all points on the open northern hemisphere are covered by the same m sets. But then none of the two points on the equator can be covered by any of these sets, again resulting in a total of at least $m+2$ sets. $\square$

In the remaining part of this section, we present lower and upper bounds for $f(d,n,m)$ for arbitrary values of n and m.

Theorem 3.4. $\lceil \frac{d-1}{2} \rceil + n + m \leq f(d,n,m) \leq d+n+m.$

Proof. Let $\mathcal{F}$ be an antipodal (n,m)-fold cover of S^d with open sets. Since $\mathcal{F}$ covers the equator once, by Theorem 2.2 there is a point in the equator of S^d covered at least $\lceil \frac{d-1}{2} \rceil + n$ times. Just below this point there is a point in the southern hemisphere covered by the same sets; its antipodal point in the northern hemisphere is covered by at least m other sets. Thus in total there are at least $\lceil \frac{d-1}{2} \rceil + n + m$ sets in $\mathcal{F}$. For the upper bound take Gale's n-fold cover of S^d with $d+2n$ sets. Add $m-n$ open northern hemispheres to obtain an antipodal (n,m)-fold cover of S^d with $d+n+m$ open sets. $\square$

In the case where $m-n < \lceil \frac{d}{2} \rceil$, the following proposition gives an improvement of the lower bound from Theorem 3.4. We omit the proof, which is a direct application of Stahl's result in combination with the fact that every (n,m)-fold cover is also an n-fold cover.

Proposition 3.5. $f(d,n,m) \geq d+2n.$

In the proof of the lower bound of $f(d,n,m)$ in Theorem 3.4 we used the lower bound of $Q(d,n)$ given in Theorem 2.2. Hence, any improvement on the lower bound of $Q(d,n)$ immediately improves the lower bound of $f(d,n,m)$. Assuming that Conjecture 2.3 holds, the proof of Theorem 3.4 would leave a gap of only one between the lower and upper bounds of $f(d,n,m)$.

Acknowledgments.

We are indebted to László Zádori [10] who asked whether $f(2,1,2)$ equals four or five, a question that started this piece of research.

References

[1] K. Fan. A generalization of Tucker's combinatorial lemma with topological applications. *Ann. of Math. (2)*, 56:431–437, 1952.

[2] D. Gale. Neighboring vertices on a convex polyhedron. In *Linear inequalities and related system*, Annals of Mathematics Studies, no. 38, pages 255–263. Princeton University Press, Princeton, N.J., 1956.

[3] M. Izydorek and J. Jaworowski. Antipodal coincidence for maps of spheres into complexes. *Proc. Amer. Math. Soc.*, 123(6):1947–1950, 1995.

[4] J. Jaworowski. Periodic coincidence for maps of spheres. *Kobe J. Math.*, 17(1):21–26, 2000.

[5] L. Lusternik and L. Schnirelmann. Méthodes topologiques dans les problèmes variationnels. I. Pt. Espaces à un nombre fini de dimensions. Translated from Russian by J. Kravtchenko. Paris: Hermann & Cie, 1934.

[6] E. V. Ščepin. A certain problem of L. A. Tumarkin. *Dokl. Akad. Nauk SSSR*, 217:42–43, 1974.

[7] G. Simonyi, G. Tardos, and S. T. Vrecíca. Local chromatic number and distinguishing the strength of topological obstructions. *Trans. Amer. Math. Soc.*, 361:889–908, 2009.

[8] G. Simonyi and G. Tardos. Local chromatic number, Ky Fan's theorem and circular colorings. *Combinatorica*, 26(5):587–626, 2006.

[9] S. Stahl. Reductions of n-fold covers. *Proc. Amer. Math. Soc.*, 72(2):422–424, 1978.

[10] L. Zádori, privite communication, April 2013.