Regular graphs are antimagic

Kristóf Bérczi∗ Attila Bernáth† Máté Vizer‡

May 1, 2015

Abstract
In this note we prove - with a slight modification of an argument of Cranston et al. [2] - that k-regular graphs are antimagic for k ≥ 2.

1 Introduction
Throughout the note graphs are assumed to be simple. Given an undirected graph G = (V, E) and a subset of edges F ⊆ E, F(v) denotes the set of edges in F incident to node v ∈ V, and dF(v) := |F(v)| is the degree of v in F. A labeling f is an injective function f : E → {1, 2, ..., |E|}. Given a labeling f and a subset of edges F, let f(F) = ∑e ∈F f(e). A labeling is antimagic if f(E(u)) ̸= f(E(v)) for any pair of different nodes u, v ∈ V. A graph is said to be antimagic if it admits an antimagic labeling.

Hartsfield and Ringel conjectured [3] that all connected graphs on at least 3 nodes are antimagic. The conjecture has been verified for several classes of graphs (see e.g. [3]), but is widely open in general. In [2] Cranston et al. proved that every k-regular graph is antimagic if k ≥ 3 is odd. Note that 1-regular graphs are trivially not antimagic. We have observed that a slight modification of their argument also works for even regular graphs, hence we prove the following.

Theorem 1. For k ≥ 2, every k-regular graph is antimagic.

It is worth mentioning the following conjecture of Liang [5]. Let G = (S, T; E) be a bipartite graph. A path P = {uw, vw} of length 2 with u, w ∈ S is called an S-link.

Conjecture 2. Let G = (S, T; E) be a bipartite graph such that each node in S has degree at most 4 and each node in T has degree at most 3. Then G has a matching M and a family P of node-disjoint S-links such that every node v ∈ T of degree 3 is incident to an edge in M ∪ (∪P∈P P).

Liang showed that if the conjecture holds then it implies that every 4-regular graph is antimagic. The starting point of our investigations was proving Conjecture 2. As Theorem 1 provides a more general result, we leave the proof of Conjecture 2 for a forthcoming paper [1].

2 Proof of Theorem 1

A trail in a graph G = (V, E) is an alternating sequence of nodes and edges v0, e1, v1, ..., e_t, v_t such that e_i is an edge connecting v_{i−1} and v_i for i = 1, 2, ..., t, and the edges are all distinct (but there might be repetitions among the nodes). The trail is open if v_0 ̸= v_t, and closed otherwise. The length of a trail is the number of edges in it. A closed trail containing every edge of the graph is called an Eulerian trail. It is well known that a graph has an Eulerian trail if and only if it is connected and every node has even degree.

Lemma 3. Given a connected graph G = (V, E), let T = {v ∈ V : dE(v) is odd}. If T ̸= ∅, then E can be partitioned into |T|/2 open trails.

Proof. Note that |T| is even. Arrange the nodes of T into pairs in an arbitrary manner and add a new edge between the members of every pair. Take an Eulerian trail of the resulting graph and delete the new edges to get the |T|/2 open trails.

∗MTA-ELTE Egerváry Research Group, Department of Operations Research, Eötvös University, Budapest, Hungary. E-mail: berkri@cs.elte.hu.
†MTA-ELTE Egerváry Research Group, Department of Operations Research, Eötvös University, Budapest, Hungary. E-mail: bernath@cs.elte.hu.
‡MTA-Árpád Rényi Institute of Mathematics, P.O.B. 127, Budapest H-1364, Hungary. Email: vizermate@gmail.com
The main advantage of Lemma 3 is that the edge set of the graph can be partitioned into open trails such that at most one trail starts at every node of $V$. Indeed, there is a trail starting at $v$ if and only if $v$ has odd degree in $G$. This is how we see the Helpful Lemma of [2].

**Corollary 4 (Helpful Lemma of [2]).** Given a bipartite graph $G = (U, W; E)$ with no isolated nodes in $U$, $E$ can be partitioned into subsets $E^i', T_1, T_2, \ldots, T_l$ such that $d_{E'}(u) = 1$ for every $u \in U$, $T_i$ is an open trail for every $i = 1, 2, \ldots, l$, and the endpoints of $T_i$ and $T_j$ are different for every $i \neq j$.

**Proof.** Take an arbitrary $E' \subseteq E$ with the property $d_{E'}(u) = 1$ for every $u \in U$. A component of $G - E'$ containing more than one node is called nontrivial. If there exists a nontrivial component of $G - E'$ that only contains even degree nodes then let $uw_1 \in E - E'$ be an edge in this component with $u \in U$ and $w_1 \in W$, and let $uw_2 \in E'$. Replace $uw_2$ with $uw_1$ in $E'$. After this modification, the component of $G - E'$ that contains $u$ has an odd degree node, namely $w_1$. Iterate this step until every nontrivial component of $G - E'$ has some odd degree nodes. Let $E'' = E'$ and apply Lemma 3 to get the decomposition of $E - E''$ into open trails.

In what follows we prove that regular graphs are antimagic: for sake of completeness we include the odd regular case, too. We emphasize the differences from the proof appearing in [2].

**Proof of Theorem 7.** Note that it suffices to prove the theorem for connected regular graphs. Let $G = (V, E)$ be a connected $k$-regular graph and let $v^* \in V$ be an arbitrary node. Denote the set of nodes at distance exactly $i$ from $v^*$ by $V_i$ and let $q$ denote the largest distance from $v^*$. We denote the edge-set of $G[V_i]$ by $E_i$. Apply Corollary 4 to the induced bipartite graph $G[V_{i-1}, V_i]$ with $U = V_i$ to get $E_i^*$ and the trail decomposition of $G[V_{i-1}, V_i] - E_i^*$ for every $i = 1, \ldots, q$. The edge set of $G[V_{i-1}, V_i] - E_i^*$ is denoted by $E_i^*$.

Now we define the antimagic labeling $f$ of $G$ as follows. We reserve the $|E_i^*|$ smallest labels for labeling $E_i$, the next $|E_i^*| - 1$ smallest labels for labeling $E_i^*$, the next $|E_i^*| - 2$ smallest labels for labeling $E_i^*$, the next $|E_i^*| - 3$ smallest labels for labeling $E_i^*$, etc. There is an important difference here between our approach and that of [2] as we switched the order of labeling $E_i^*$ and $E_i$, and we don’t yet define the labels, we only reserve the intervals to label the edge sets. Next we prove a claim that tells us how to label the edges in $E_i^*$.

**Claim 5.** Assume that we have to label the edges of $E_i^*$ from interval $s, s + 1, \ldots, \ell$ (where $|E_i^*| = \ell - s + 1$), and that we are given a trail decomposition of $E_i^*$ into open trails. We can label $E_i^*$ so that successive labels (in a trail) incident to a node $v_i \in V_i$ have sum at most $s + \ell$, and successive labels (in a trail) incident to a node $v_{i-1} \in V_{i-1}$ have sum at least $s + \ell$.

**Proof.** Our proof of this claim is essentially the same as the proof in [2]: we merely restate it for self-containedness. Let $T$ be the trail decomposition of $E_i^*$ into open trails. Take an arbitrary trail $T = u_0, e_1, u_1, \ldots, e_t, u_t$ of length $t$ from $T$ and consider the following two cases (see Figure 4 for an illustration).

- **Case A:** If $u_0 \in V_{i-1} \subseteq V_{i-1}$ then label $e_1, \ldots, e_t$ by $s, \ell, s + 1, \ell - 1, \ldots$, in this order. In this case the sum of 2 successive labels is $s + \ell$ at a node in $V_i$, and it is $s + \ell + 1$ at a node in $V_{i-1}$.

- **Case B:** If $u_0 \in V_{i-1}$ then label $e_1, e_2, e_t$ by $\ell, s, \ell - 1, s + 1, \ldots$, in this order. In this case the sum of 2 successive labels is $s + \ell - 1$ at a node in $V_i$, and it is $s + \ell$ at a node in $V_{i-1}$.

We prove by induction on $|T|$. The proof is finished by the following cases.

1. If $T$ contains a trail of even length, then let $T$ be such a trail (and again $t$ denotes the length of $T$). If the endpoints of $T$ fall in $V_{i-1}$ then apply Case A. On the other hand, if the endpoints of $T$ fall in $V_i$ then apply Case B. In both cases we use $\frac{t}{2}$ labels from the lower end of the interval, and $\frac{t}{2}$ labels from the upper end, therefore we can label the edges of the trails in $T - T$ from the (remaining) interval $s + \frac{t}{2}, s + \frac{t}{2} + 1, \ldots, \ell - \frac{t}{2}$, so that the lower bound $s + \frac{t}{2} + \ell - \frac{t}{2} = s + \ell$ holds for the sum of two successive labels at every $v_{i-1} \in V_{i-1}$, and the same upper bound holds at each node $v_i \in V_i$.

2. Every trail in $T$ has odd length. If $T$ contains only one trail then label it using either of the two cases above and we are done. Otherwise let $T_1$ and $T_2$ be two trails from $T$, and let $t_i$ be the length of $T_i$ for both $i = 1, 2$. Label first the edges of $T_1$ using Case A (starting at the endpoint of $T_1$ that lies in $V_{i-1}$). Note that the remaining labels form the interval $s + \frac{t_1}{2} + 1, \ldots, \ell - \frac{t_1}{2}$. Next label the edges of $T_2$ using Case B (starting at the endpoint of $T_2$ that lies in $V_i$). Note that the sum of successive labels in the trail $T_2$ becomes $s + \frac{t_2}{2} + (\ell - \frac{t_2}{2}) - 1 = s + \ell$ at a node in $V_i$, and it is $s + \frac{t_1}{2} + (\ell - \frac{t_1}{2}) = s + \ell + 1$ at a node in $V_{i-1}$, which is fine for us. Finally, the remaining labels form the interval $s + \frac{t_1}{2} + 1 + \frac{t_2}{2}, \ldots, \ell - \frac{t_1}{2} - \frac{t_2}{2}$, therefore we can label the edges of the trails in $T - (T_1, T_2)$ from the remaining interval so that the lower bound $s + \frac{t_1}{2} + 1 + \ell - \frac{t_1}{2} - \frac{t_2}{2} = s + \ell$ holds for the sum of two successive labels at every node of $V_{i-1}$, and the same upper bound holds at every node of $V_i$. 

\[\square\]
Now we specify how the labels are determined to make sure \( f(E(u)) \neq f(E(v)) \) for every \( u \neq v \). We label the edges of every \( E_i \) arbitrarily from their dedicated intervals. Label the edges of every \( E_i' \) in the manner described by Claim 5. For any node \( v \in V_i \) with \( i > 0 \), let \( \sigma(v) \) denote the unique edge of \( E_i^\sigma \) incident to \( v \). Let \( p(v) = f(E(v)) - f(\sigma(v)) \) for every \( v \in V - v^* \). We label the edges in \( E_i^\sigma, E_{i-1}^\sigma, \ldots, E_1^\sigma \) as in [2]: if we already labeled \( E_i^\sigma, E_{i-1}^\sigma, \ldots, E_1^\sigma \), then \( p(v_i) \) is already determined for every \( v_i \in V_i \). So we order the nodes of \( V_i \) in an increasing order according to their \( p \)-value and assign the label to their \( \sigma \) edge in this order. This ensures that \( f(E(u)) \neq f(E(v)) \) for an arbitrary pair \( u, v \in V_i \).

We have fully described the labeling procedure. This labeling scheme ensures that \( f(E(v_i)) < f(E(v_j)) \) if \( v_i \in V_i, v_j \in V_j \) and \( i \geq j + 2 \) since \( G \) is regular and the edges in \( E(v_i) \) get larger labels than those in \( E(v_j) \). Similarly, \( f(E(v^*)) > f(E(v)) \) for every \( v \in V - v^* \) for the same reason. It is only left to show that \( f(E(v_i)) \neq f(E(v_{i-1})) \) for arbitrary \( v_i, v_{i-1} \in V_{i-1} \) and \( i \geq 2 \).

**Claim 6.** For arbitrary \( v_i \in V_i, v_{i-1} \in V_{i-1} \) and \( i \geq 2 \) we have

(i) \( p(v_i) \leq \frac{k-2}{2}(s + \ell) + \ell \) and \( p(v_{i-1}) \geq \frac{k-2}{2}(s + \ell) + s \), if \( k \) is even, and

(ii) \( p(v_i) \leq \frac{k-1}{2}(s + \ell) \) and \( p(v_{i-1}) \geq \frac{k-1}{2}(s + \ell) \), if \( k \) is odd.

**Proof.** Assume first that \( k \) is even. In this case \( p(v) \) is the sum of an odd number of labels. We pair up all but one of these labels using the trail decomposition of \( E_i^\sigma \) to get the bounds needed.

1. Take a node \( v_i \in V_i \). Note that \( f(e) < s \) for every \( e \in E(v_i) - E_i^\sigma \). Let \( t = d_{E_i^\sigma}(v_i) \).

   (a) If \( t \) is even then \( \sum_{e \in E_i^\sigma \cap E(v_i)} f(e) \leq \frac{t}{2}(s + \ell) \) by Claim 5 giving \( p(v_i) \leq \frac{t}{2}(s + \ell) + (k - 1 - t)s \leq \frac{k-2}{2}(s + \ell) + \ell \).

   (b) If \( t \) is odd then \( \sum_{e \in E_i^\sigma \cap E(v_i)} f(e) \leq \frac{t-1}{2}(s + \ell) + \ell \) by Claim 5 giving \( p(v_i) \leq \frac{t-1}{2}(s + \ell) + \ell + (k - 1 - t)s \leq \frac{k-2}{2}(s + \ell) + \ell \).

2. Now take a node \( v_{i-1} \in V_{i-1} \). Note that \( f(e) > \ell \) for every \( e \in E(v_{i-1}) - E_i^\sigma \). Let again \( t = d_{E_i^\sigma}(v_{i-1}) \).

   (a) If \( t \) is even then \( \sum_{e \in E_i^\sigma \cap E(v_{i-1})} f(e) \geq \frac{t}{2}(s + \ell) \) by Claim 5 giving \( p(v_{i-1}) \geq \frac{t}{2}(s + \ell) + (k - 1 - t)\ell \geq \frac{k-2}{2}(s + \ell) + s \).

   (b) If \( t \) is odd then \( \sum_{e \in E_i^\sigma \cap E(v_{i-1})} f(e) \geq \frac{t-1}{2}(s + \ell) + s \) by Claim 5 giving \( p(v_{i-1}) \geq \frac{t-1}{2}(s + \ell) + s + (k - 1 - t)\ell \geq \frac{k-2}{2}(s + \ell) + s \).

This concludes the proof of (i).

Although the proof of (ii) can be found in [2], we also present it here to make the paper self contained. The proof is very similar to the even case. So assume that \( k \) is odd. In this case \( p(v) \) is the sum of an even number of labels. We pair up these labels using the trail decomposition of \( E_i^\sigma \) to get the bounds needed.

1. Take a node \( v_i \in V_i \). Note that \( f(e) < s \) for every \( e \in E(v_i) - E_i^\sigma \). Let \( t = d_{E_i^\sigma}(v_i) \).

   (a) If \( t \) is even then \( \sum_{e \in E_i^\sigma \cap E(v_i)} f(e) \leq \frac{t}{2}(s + \ell) \) by Claim 5 giving \( p(v_i) \leq \frac{t}{2}(s + \ell) + (k - 1 - t)s \leq \frac{k-1}{2}(s + \ell) \).

Figure 1: An illustration for labeling trails.
(b) If \( t \) is odd then \( \sum_{e \in E_i \cap E(v_i)} f(e) \leq \frac{k-1}{2} (s + \ell) + (k - t)s \leq \frac{k-1}{2} (s + \ell) \).

2. Now take a node \( v_{i-1} \in V_{i-1} \). Note that \( f(e) > \ell \) for every \( e \in E(v_{i-1}) - E'_i \). Let again \( t = d_{E'_i}(v_{i-1}) \).

(a) If \( t \) is even then \( \sum_{e \in E_i \cap E(v_{i-1})} f(e) \geq \frac{k-1}{2} (s + \ell) \) by Claim 5, giving

\[
 p(v_{i-1}) \geq \frac{k-1}{2} (s + \ell) + \ell \geq \left( k - 1 \right) \frac{k-1}{2} (s + \ell).
\]

This concludes the proof of (ii), and we are done.

Remark 7. Observe that the proof of Theorem 1 does not really use the regularity of the graph, it merely relies on the fact that the degree of a node \( v_i \in V_i \) is not smaller than that of a node \( v_j \in V_j \) where \( i < j \). Hence the following result immediately follows.

Theorem 8. Assume that a connected graph \( G = (V, E) \) (\(|V| \geq 3\)) has a node \( v^* \in V \) of maximum degree such that \( d_{E_i}(v_i) \geq d_{E_j}(v_j) \) whenever \( v_i \in V_i, v_j \in V_j \) and \( i < j \), where \( V_\ell \) denotes the set of nodes at distance exactly \( \ell \) from \( v^* \). Then \( G \) is antimagic.

Acknowledgement

The first and the second authors were supported by the Hungarian Scientific Research Fund - OTKA, K109240. The third author would like to thank Zheijang Normal University, China - where he first heard about these problems - for their hospitality.

References


