

Approximability of the upper chromatic number of hypergraphs*

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Abstract

A C -coloring of a hypergraph $\mathcal{H} = (X, \mathcal{E})$ is a vertex coloring $\varphi : X \rightarrow \mathbb{N}$ such that each edge $E \in \mathcal{E}$ has at least two vertices with a common color. The related parameter $\bar{\chi}(\mathcal{H})$, called the upper chromatic number of \mathcal{H} , is the maximum number of colors can be used in a C -coloring of \mathcal{H} . A hypertree is a hypergraph which has a host tree T such that each edge $E \in \mathcal{E}$ induces a connected subgraph in T . Notations n and m stand for the number of vertices and edges, respectively, in a generic input hypergraph.

We establish guaranteed polynomial-time approximation ratios for the difference $n - \bar{\chi}(\mathcal{H})$, which is $2 + 2 \ln(2m)$ on hypergraphs in general, and $1 + \ln m$ on hypertrees. The latter ratio is essentially tight as we show that $n - \bar{\chi}(\mathcal{H})$ cannot be approximated within $(1 - \epsilon) \ln m$ on hypertrees (unless $\text{NP} \subseteq \text{DTIME}(n^{\mathcal{O}(\log \log n)})$). Furthermore, $\bar{\chi}(\mathcal{H})$ does not have $\mathcal{O}(n^{1-\epsilon})$ -approximation and cannot be approximated within additive error $o(n)$ on the class of hypertrees (unless $\text{P} = \text{NP}$).

Keywords: approximation ratio, hypergraph, hypertree, C -coloring, upper chromatic number, multiple hitting set.

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1 Introduction

In this paper we study a hypergraph coloring invariant, termed upper chromatic number and denoted by $\overline{\chi}(\mathcal{H})$, which was first introduced by Berge (cf. [4]) in the early 1970's and later independently by several further authors [1, 17] from different motivations. The present work is the very first one concerning approximation algorithms on it.

We also consider the complementary problem of approximating the difference $n - \overline{\chi}$, the number of vertices minus the upper chromatic number. One of our main tools to prove a guaranteed upper bound on it is an approximation ratio established for the 2-transversal number of hypergraphs. As problems of this type are of interest in their own right, we also prove an approximation ratio in general for the minimum size of multiple transversals, i.e., sets of vertices intersecting each edge in a prescribed number of vertices at least. Earlier results allowed to select a vertex into the set several times; we prove bounds for the more restricted scenario where the set does not include any vertex more than once.

1.1 Notation and terminology

A *hypergraph* $\mathcal{H} = (X, \mathcal{E})$ is a set system, where X denotes the set of vertices and each edge $E_i \in \mathcal{E}$ is a nonempty subset of X . Here we also assume that for each edge E_i the inequality $|E_i| \geq 2$ holds, moreover we use the standard notations $|X| = n$ and $|\mathcal{E}| = m$. A hypergraph \mathcal{H} is said to be *r-uniform* if $|E_i| = r$ for each $E_i \in \mathcal{E}$.

We shall also consider hypergraphs with restricted structure, where some kind of host graphs are assumed. A hypergraph $\mathcal{H} = (X, \mathcal{E})$ admits a *host graph* $G = (X, E)$ if each edge $E_i \in \mathcal{E}$ induces a connected subgraph in G . The edges of the host graph G will be referred to as *lines*. Particularly, \mathcal{H} is called *hypertree* or *hyperstar* if it admits a host graph which is a tree or a star, respectively. Note that under our condition, which forbids edges of size 1, \mathcal{H} is a hyperstar if and only if there exists a fixed vertex $c^* \in X$ (termed the center of the hyperstar) contained in each edge of \mathcal{H} .

A *C-coloring* of \mathcal{H} is an assignment $\varphi : X \rightarrow \mathbb{N}$ such that each edge $E \in \mathcal{E}$ has at least two vertices of a common color (that is, with the same image). The *upper chromatic number* $\overline{\chi}(\mathcal{H})$ of \mathcal{H} is the maximum number of colors that can be used in a C-coloring of \mathcal{H} . We note that in the literature the

value $\overline{\chi}(\mathcal{H}) + 1$ is also called the ‘cochromatic number’ or ‘heterochromatic number’ of \mathcal{H} with the terminology of Berge [4, p. 151] and Arocha *et al.* [1], respectively. A C-coloring φ with $|\varphi(X)| = \overline{\chi}(\mathcal{H})$ colors will be referred to as an *optimal coloring* of \mathcal{H} . The *decrement* of $\mathcal{H} = (X, \mathcal{E})$, introduced in [2], is defined as $\text{dec}(\mathcal{H}) = n - \overline{\chi}(\mathcal{H})$. Similarly, the decrement of a C-coloring $\varphi : X \rightarrow \mathbb{N}$ is meant as $\text{dec}(\varphi) = |X| - |\varphi(X)|$. For results on C-coloring see the recent survey [8].

A *transversal* (also called hitting set or vertex cover) is a subset $T \subseteq X$ which meets each edge of $\mathcal{H} = (X, \mathcal{E})$, and the minimum cardinality of a transversal is the *transversal number* $\tau(\mathcal{H})$ of the hypergraph. An *independent set* (or stable set) is a vertex set $I \subseteq X$, which contains no edge of \mathcal{H} entirely. The maximum size of an independent set in \mathcal{H} is the *independence number* (or stability number) $\alpha(\mathcal{H})$. It is immediate from the definitions that the complement of a transversal is an independent set and vice versa, so the Gallai-type equality $\tau(\mathcal{H}) + \alpha(\mathcal{H}) = n$ holds for each hypergraph. Remark that selecting one vertex from each color class of a C-coloring yields an independent set, therefore $\overline{\chi}(\mathcal{H}) \leq \alpha(\mathcal{H})$ and, equivalently, $\text{dec}(\mathcal{H}) \geq \tau(\mathcal{H})$.

More generally, a *k-transversal* is a set $T \subseteq X$ such that $|E_i \cap T| \geq k$ for every $E_i \in \mathcal{E}$. A 2-transversal is sometimes called double transversal or strong transversal, and its minimum size is the *2-transversal number* $\tau_2(\mathcal{H})$ of the hypergraph.

For an optimization problem and a constant $c > 1$, an algorithm \mathcal{A} is called a *c-approximation algorithm* if, for every feasible instance \mathcal{I} of the problem,

- if the value has to be minimized, then \mathcal{A} delivers a solution of value at most $c \cdot \text{Opt}(\mathcal{I})$;
- if the value has to be maximized, then \mathcal{A} delivers a solution of value at least $\text{Opt}(\mathcal{I})/c$.

Throughout this paper, an approximation algorithm is always meant to be one with polynomial running time on every instance of the problem. We say that a value has guaranteed approximation ratio c if it has a c -approximation algorithm. In the other case, when no c -approximation algorithm exists, we say that the value cannot be approximated within ratio c . For a function $f(n, m)$, an $f(n, m)$ -approximation algorithm and the related notions can

be defined similarly. A polynomial-time approximation scheme, abbreviated as PTAS, means an algorithm for every fixed $\varepsilon > 0$ which is a $(1 + \varepsilon)$ -approximation and whose running time is a polynomial function of the input size (but any function of $1/\varepsilon$ may occur in the exponent).

For further terminology and facts we refer to [4, 6, 15] in the theory of graphs, hypergraphs, and algorithms, respectively. The notations $\ln x$ and $\log x$ stand for the natural logarithm and for the logarithm in base 2, respectively.

1.2 Approximability results on multiple transversals

The transversal number $\tau(\mathcal{H})$ of a hypergraph can be approximated within ratio $(1 + \ln m)$ by the classical greedy algorithm (see e.g. [15]). On the other hand, Feige [10] proved that $\tau(\mathcal{H})$ cannot be approximated within $(1 - \epsilon) \ln m$ for any constant $0 < \epsilon < 1$, unless $\text{NP} \subseteq \text{DTIME}(n^{\mathcal{O}(\log \log n)})$. As relates to the k -transversal number, in [15] a $(1 + \ln m)$ -approximation is stated under the less restricted setting which allows multiple selection of vertices in the k -transversal. In the context of coloring, however, we cannot allow repetitions of vertices. For this more restricted case, when the k -transversal consists of pairwise different vertices, we prove a guaranteed approximation ratio $(1 + \ln(km))$.

In fact we consider a more general problem, where the required minimum size of the intersection $E_i \cap T$ can be prescribed independently for each $E_i \in \mathcal{E}$.

Theorem 1 *Given a hypergraph $\mathcal{H} = (X, \mathcal{E})$ with m edges E_1, \dots, E_m and positive integers w_1, \dots, w_m associated with the edges, the minimum cardinality of a set $S \subset X$ satisfying $|S \cap E_i| \geq w_i$ for all $1 \leq i \leq m$ can be approximated within $\sum_{i=1}^m 1/i < 1 + \ln W$, where $W = \sum_{i=1}^m w_i$.*

This result, proved in the next section, implies a guaranteed approximation ratio $(1 + \ln 2m)$ for $\tau_2(\mathcal{H})$.

1.3 Approximability results on the upper chromatic number

The problem of determining the upper chromatic number is NP-hard, already on the class of 3-uniform hyperstars. On the other hand, the problems of

determining $\bar{\chi}(\mathcal{H})$ and finding a $\bar{\chi}(\mathcal{H})$ -coloring are fixed-parameter tractable in terms of maximum vertex degree on the class of hypertrees [9].

A notion closely related to our present subject was introduced by Voloshin [16, 17] in 1993. A *mixed hypergraph* is a triple $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ with two families of subsets called \mathcal{C} -edges and \mathcal{D} -edges. By definition, a coloring of a mixed hypergraph is an assignment $\varphi : X \rightarrow \mathbb{N}$ such that each \mathcal{C} -edge has two vertices of a common color and each \mathcal{D} -edge has two vertices of distinct colors. Then, the minimum and the maximum possible number of colors, that can occur in a coloring of \mathcal{H} , is termed the lower and the upper chromatic number of \mathcal{H} and denoted by $\chi(\mathcal{H})$ and $\bar{\chi}(\mathcal{H})$, respectively. For detailed results on mixed hypergraphs we refer to the monograph [18]. Clearly, the C-colorings of a hypergraph $\mathcal{H} = (X, \mathcal{E})$ are in one-to-one correspondence with the colorings of the mixed hypergraph $\mathcal{H}' = (X, \mathcal{E}, \emptyset)$, and also $\bar{\chi}(\mathcal{H}) = \bar{\chi}(\mathcal{H}')$ holds.

The following results are known on the approximation of the upper chromatic number of mixed hypergraphs:

- For mixed hypergraphs of maximum degree 2, the upper chromatic number has a linear-time $\frac{5}{3}$ -approximation and an $O(m^3 + n)$ -time $\frac{3}{2}$ -approximation. [13, Theorem 14 and Theorem 15]
- There is no PTAS for the upper chromatic number of mixed hypergraphs of maximum degree 2, unless $\text{P} = \text{NP}$. [13, Theorem 20]
- There is no $o(n)$ -approximation algorithm for the upper chromatic number of mixed hypergraphs, unless $\text{P} = \text{NP}$. [11, Corollary 5]

All these results assume the presence of \mathcal{D} -edges in the input mixed hypergraph. In this paper we investigate how hard it is to estimate $\bar{\chi}$ for C-colorings of hypergraphs.

On the positive side, we prove a guaranteed approximation ratio for the decrement of hypergraphs in general, furthermore we establish a better ratio on the class of hypertrees.

Theorem 2 *The value of $\text{dec}(\mathcal{H})$ is $(2 + 2 \ln(2m))$ -approximable on the class of all hypergraphs.*

Theorem 3 *The value of $\text{dec}(\mathcal{H})$ is $(1 + \ln m)$ -approximable on the class of all hypertrees.*

These theorems are essentially best possible concerning the ratio of approximation, moreover the upper chromatic number turns out to be inherently non-approximable already on hypertrees with rather restricted host trees, as shown by the next result.

Theorem 4

- (i) For every $\epsilon > 0$, $\text{dec}(\mathcal{H})$ cannot be approximated within $(1 - \epsilon) \ln m$ on the class of hyperstars, unless $\text{NP} \subseteq \text{DTIME}(n^{\mathcal{O}(\log \log n)})$.
- (ii) For every $\epsilon > 0$, $\bar{\chi}(\mathcal{H})$ cannot be approximated within $n^{1-\epsilon}$ on the class of 3-uniform hyperstars, unless $\text{P} = \text{NP}$.

As regards the *difference* between a solution determined by a polynomial-time algorithm and the optimum value, the situation is even worse.

Theorem 5 *Unless $\text{P} = \text{NP}$, neither of the following values can be approximated within additive error $o(n)$ for hypertrees of edge size at most 7:*

$$\bar{\chi}(\mathcal{H}), \text{dec}(\mathcal{H}), \alpha(\mathcal{H}) - \bar{\chi}(\mathcal{H}), \tau(\mathcal{H}) - \text{dec}(\mathcal{H}), \text{dec}(\mathcal{H}) - \tau_2(\mathcal{H})/2.$$

The relevance of the last quantity occurs in the context of Proposition 9 of Section 3.1.

We prove the positive results with guaranteed approximation ratio in Section 3, and the negative non-approximability results in Section 4.

1.4 Lemmas on connected colorings of hypertrees

Suppose that \mathcal{H} is a hypergraph over a host graph G , and φ is a C -coloring of \mathcal{H} . We say that φ is a *connected coloring* if each color class of φ induces a connected subgraph of G . We will use the following two lemmas concerning connected C -colorings of hypertrees, both established in [9]. A line uv of the host tree G is termed *monochromatic line* for a C -coloring φ if $\varphi(u) = \varphi(v)$.

Lemma 6 ([9, Proposition 2]) *If a hypertree admits a C -coloring with k colors, then it also has a connected C -coloring with k colors over any fixed host tree.*

Lemma 7 ([9, Proposition 3]) *If φ is a connected C -coloring of a hypertree \mathcal{H} over a fixed host tree G , then the decrement of φ equals the number of monochromatic lines in G .*

2 Multiple transversals

In this section, we describe a variation of the classical greedy algorithm, with the goal to produce a multiple transversal with pairwise different elements. Analyzing the greedy selection we will prove Theorem 1. We recall its statement.

Theorem 1. *Given a hypergraph $\mathcal{H} = (X, \mathcal{E})$ with m edges E_1, \dots, E_m and positive integers w_1, \dots, w_m associated with its edges, the minimum cardinality of a set $S \subset X$ satisfying $|S \cap E_i| \geq w_i$ for all $1 \leq i \leq m$ can be approximated within $\sum_{i=1}^m 1/i < 1 + \ln W$, where $W = \sum_{i=1}^m w_i$.*

Proof Denote by \mathcal{S} the collection of all feasible solutions, that are the sets $S \subset X$ such that $|S \cap E_i| \geq w_i$ holds for all $i = 1, \dots, m$. By definition, the optimum of the problem is the integer

$$M := \min_{S \in \mathcal{S}} |S|.$$

We will show that the greedy selection always yields an $S^* \in \mathcal{S}$ with

$$|S^*| \leq M \cdot (1 + 1/2 + \dots + 1/W).$$

To prove this, for any $Y \subset X$ and any $1 \leq i \leq m$ we define

$$w_{i,Y} := \max(0, w_i - |E_i \cap Y|)$$

which means the reduced number of elements to be picked further from E_i , once the set Y has already been selected. Moreover, to any vertex $x \in X \setminus Y$ we associate its usefulness

$$u_{x,Y} := |\{E_i \mid x \in E_i, w_{i,Y} > 0\}|.$$

The greedy algorithm then starts with $Y_0 = \emptyset$ and updates $Y_k := Y_{k-1} \cup \{x_k\}$ where $x_k \in X \setminus Y_{k-1}$ has maximum usefulness among all values $u_{x,Y_{k-1}}$ in the set $X \setminus Y_{k-1}$, as long as this maximum is positive. Reaching $u_{x,Y_t} = 0$ for all $x \in X \setminus Y_t$ (for some t), we set $S^* := Y_t$; we will prove that this S^* satisfies the requirements.

It is clear by the definition of $u_{x,Y}$ that S^* meets each E_i in at least w_i elements, i.e. $S^* \in \mathcal{S}$. We need to prove that S^* is sufficiently small. For this, consider the following auxiliary set of cardinality W :

$$Z := \{z(i, j) \mid 1 \leq i \leq m, 1 \leq j \leq w_i\}.$$

At the moment when Y_k is constructed by adjoining an element x_k to Y_{k-1} , we assign weight $1/u_{x_k, Y_{k-1}}$ to all elements $z(i, w_{i, Y_{k-1}})$ such that $x_k \in E_i$ and $w_{i, Y_{k-1}} > 0$. Note that $w_{i, Y_k} = w_{i, Y_{k-1}} - 1$ will hold after the selection of x_k . Moreover, total weight 1 is assigned in each step, hence the overall weight after finishing the algorithm is exactly $|S^*|$. We put the elements $z(i, j)$ in a sequence $Z^* = (z_1, z_2, \dots, z_W)$ such that the elements of Z occur in the order as they are weighted (i.e., those for x_1 first in any order, then the elements weighted for x_2 , and so on).

Just before the selection of x_k , the number of elements $z(i, j)$ to which a weight has been assigned is precisely $m_{k-1} := \sum_{\ell=1}^{k-1} u_{x_\ell, Y_{\ell-1}}$. We are going to prove that $u_{x_k, Y_{k-1}} \geq (W - m_{k-1})/M$. Assuming that this has already been shown, it follows that each z_q in Z^* has weight at most $M/(W + 1 - q)$ and consequently $|S^*| \leq M \cdot (1 + 1/2 + \dots + 1/W)$ as required.

Let now $S_0 \in \mathcal{S}$ be any fixed optimal solution. Consider the bipartite incidence graph B between the sets E_i and the elements of S_0 . That is, the first vertex class of B has m elements a_1, \dots, a_m representing the sets E_1, \dots, E_m while the second vertex class consists of the elements of S_0 ; we denote the latter vertices by b_1, \dots, b_M . There is an edge joining a_i with b_j if and only if $b_j \in E_i$.

Since $S_0 \in \mathcal{S}$, each a_i has degree at least w_i . Moreover, considering the moment just before x_k is selected, if we remove the vertices of $S_0 \cap Y_{k-1}$, in the remaining subgraph still each a_i has degree at least $w_{i, Y_{k-1}}$. We take a subgraph B' of this $B - Y_{k-1}$ (possibly B itself if $Y_{k-1} \cap S_0 = \emptyset$) such that each a_i has degree *exactly* $w_{i, Y_{k-1}}$. The number of edges in B' is then equal to $W - m_{k-1}$; hence, some b_j has degree at least $(W - m_{k-1})/M$. It follows that this b_j has usefulness at least $(W - m_{k-1})/M$ at the moment when x_k is selected; but x_k is chosen to have maximum usefulness, hence $u_{x_k, Y_{k-1}} \geq (W - m_{k-1})/M$. This completes the proof. \square

Corollary 8 *For each positive integer k , the k -transversal number τ_k has a $(1 + \ln(km))$ -approximation on the class of all hypergraphs.*

3 Guaranteed approximation ratios for the decrement

In this section we establish a connection between the parameters $\text{dec}(\mathcal{H})$ and $\tau_2(\mathcal{H})$, and then we prove our positive results stated in Theorems 2 and 3.

3.1 Decrement vs. 2-transversal number

First, we give an inequality valid for all hypergraphs without any structural restrictions and then, using this relation, we prove Theorem 2.

Proposition 9 *For every hypergraph \mathcal{H} we have $\tau_2(\mathcal{H})/2 \leq \text{dec}(\mathcal{H}) \leq \tau_2(\mathcal{H}) - 1$, and both bounds are tight. In particular, $\tau_2(\mathcal{H})$ is a 2-approximation for $\text{dec}(\mathcal{H})$.*

Proof *Lower bound:* If $\bar{\chi}(\mathcal{H}) \leq n/2$, then $\text{dec}(\mathcal{H}) \geq n/2 \geq \tau_2(\mathcal{H})/2$ automatically holds. If $\bar{\chi}(\mathcal{H}) > n/2$, then every $\bar{\chi}$ -coloring contains at least $2\bar{\chi}(\mathcal{H}) - n$ singleton color classes, therefore the total size of non-singleton classes is at most $n - (2\bar{\chi}(\mathcal{H}) - n) = 2(n - \bar{\chi}(\mathcal{H}))$. Since the union of the latter meets all edges at least twice, we obtain $2\text{dec}(\mathcal{H}) \geq \tau_2(\mathcal{H})$.

Upper bound: If S is a 2-transversal set of cardinality $\tau_2(\mathcal{H})$, we can assign the same color to the entire S and a new dedicated color to each $x \in X \setminus S$. This is a C-coloring with $n - |S| + 1$ colors and with decrement $\tau_2(\mathcal{H}) - 1$.

Tightness: The simplest example for equality in the upper bound is the hypergraph in which the vertex set is the only edge, i.e. $\mathcal{H} = (X, \{X\})$. Many more examples can be given. For instance, we can specify a proper subset $S \subset X$ with $|S| \geq 2$, and take all triples $E \subset X$ such that $|E \cap S| = 2$ and $|E \setminus S| = 1$. If $|S| \leq n - 2$, then S is the unique smallest 2-transversal set, and every C-coloring with more than two colors makes S monochromatic, hence the unique $\bar{\chi}$ -coloring uses $n - |S| + 1$ colors.

For the lower bound, we assume that $n = 3k + 1$. Let $X = \{1, 2, \dots, 3k + 1\}$ and

$$\begin{aligned} \mathcal{E} = & \{ \{3r + 1, 3r + 2, 3r + 3\} \mid 0 \leq r \leq k - 1 \} \\ & \cup \{ \{3r + 2, 3r + 3, 3r + 4\} \mid 0 \leq r \leq k - 1 \} \end{aligned}$$

Then $\tau_2(\mathcal{H}) = 2k$ because the k edges in the first line are mutually disjoint and hence need at least $2k$ vertices in any 2-transversal set, while the $2k$ -element set $\{3r + 2 \mid 0 \leq r \leq k - 1\} \cup \{3r + 3 \mid 0 \leq r \leq k - 1\}$ meets all edges twice. On the other hand, there exists a unique C-coloring with decrement k , obtained by making $\{3r + 2, 3r + 3\}$ a monochromatic pair for $r = 0, 1, \dots, k - 1$ and putting any other vertex in a singleton color class. This verifies equality in the lower bound. \square

Now, we are ready to prove Theorem 2. Let us recall its statement.

Theorem 2. *The value of $\text{dec}(\mathcal{H})$ is $(2 + 2 \ln(2m))$ -approximable on the class of all hypergraphs.*

Proof By Corollary 8, we have a $(1 + \ln(2m))$ -approximation algorithm \mathcal{A} for τ_2 . Hence, given a hypergraph $\mathcal{H} = (X, \mathcal{E})$, the algorithm \mathcal{A} outputs a 2-transversal T of size at most $(1 + \ln(2m))\tau_2(\mathcal{H})$. Then, assign color 1 to every $x \in T$, and color the $n - |T|$ vertices in $X \setminus T$ pairwise differently with colors $2, 3, \dots, n - |T| + 1$. As each edge $E_i \in \mathcal{E}$ contains at least two vertices of color 1, this results in a C-coloring φ with decrement satisfying

$$\text{dec}(\varphi) = |T| - 1 \leq (1 + \ln(2m))\tau_2(\mathcal{H}) - 1 < 2(1 + \ln(2m))\text{dec}(\mathcal{H}),$$

where the last inequality follows from Proposition 9. Therefore, algorithm \mathcal{A} together with the simple construction of coloring φ is a $(2 + 2 \ln 2m)$ -approximation for $\text{dec}(\mathcal{H})$. \square

3.2 Guaranteed approximation ratio on hypertrees

In this short subsection we prove Theorem 3. We recall its statement.

Theorem 3. *The value of $\text{dec}(\mathcal{H})$ is $(1 + \ln m)$ -approximable on the class of all hypertrees.*

Proof Given a hypertree $\mathcal{H} = (X, \mathcal{E})$ and $G = (X, L)$ which is a host tree of \mathcal{H} , construct the auxiliary hypergraph $\mathcal{H}^* = (L^*, \mathcal{E}^*)$ such that each vertex $l_i^* \in L^*$ represents a line l_i of the host tree, moreover each edge $E_i^* \in \mathcal{E}^*$ of the auxiliary hypergraph corresponds to the edge $E_i \in \mathcal{E}$ in the following way:

$$E_i^* = \{l_j^* \mid l_j \subseteq E_i\}.$$

Now, consider any connected C-coloring φ of \mathcal{H} . This coloring determines the set $S \subseteq L$ of monochromatic lines in the host tree, moreover the corresponding vertex set $S^* \subseteq L^*$ in \mathcal{H}^* . By Lemma 7, $\text{dec}(\varphi) = |S| = |S^*|$. As φ is a connected C-coloring, each edge of \mathcal{H} contains a monochromatic line and, consequently, S^* is a transversal of size $\text{dec}(\varphi)$ in \mathcal{H}^* . Similarly, in the opposite direction, if a transversal T^* of \mathcal{H}^* is given and the corresponding line-set is T in the host tree, then every edge E_i of \mathcal{H} contains two vertices,

say u and v , such that the line uv is contained in T . Then, the vertex coloring ϕ , whose color classes correspond to the components of (X, T) , is a connected C-coloring of \mathcal{H} , and in addition $\text{dec}(\phi) = |T| = |T^*|$ holds.

By Lemma 6, \mathcal{H} has a connected C-coloring φ with $\text{dec}(\varphi) = \text{dec}(\mathcal{H})$, therefore the correspondence above implies $\text{dec}(\mathcal{H}) = \tau(\mathcal{H}^*)$.

As \mathcal{H}^* can be constructed in polynomial time from the hypertree \mathcal{H} , and since a transversal T^* of size at most $(1 + \ln m)\tau(\mathcal{H}^*)$ can be obtained by greedy selection, a C-coloring ϕ of \mathcal{H} with

$$\text{dec}(\phi) = |T^*| \leq (1 + \ln m)\tau(\mathcal{H}^*) = (1 + \ln m)\text{dec}(\mathcal{H})$$

can also be constructed in polynomial time. This yields a guaranteed approximation ratio $(1 + \ln m)$ for the decrement on the class of hypertrees.

□

4 Approximation hardness

The bulk of this section is devoted to the proof of Theorem 5 on non-approximability for hypertrees. Then, we prove a lemma concerning parameters $\bar{\chi}(\mathcal{H})$ and $\text{dec}(\mathcal{H})$ of hyperstars. The section is closed with the proof of Theorem 4 and with some remarks.

4.1 Additive linear error

Our goal in this subsection is to prove Theorem 5. This needs the following construction, which was introduced in [7]. (We note that a similar construction was given already in [12].)

Construction of $\mathcal{H}(\Phi)$. Let $\Phi = C_1 \wedge \dots \wedge C_m$ be an instance of 3-SAT, with m clauses of size 3 over the set $\{x_1, \dots, x_n\}$ of n variables, such that the three literals in each clause C_j of Φ correspond to exactly three distinct variables. We construct the hypertree $\mathcal{H} = \mathcal{H}(\Phi)$ with the set

$$X = \{c^*\} \cup \{x'_i, t_i, f_i \mid 1 \leq i \leq n\}$$

of $3n + 1$ vertices, where the vertices x'_i, t_i, f_i correspond to variable x_i . First, we define the host tree $T = (X, E)$ with vertex set X and line-set

$$E = \{c^*x'_i, x'_it_i, x'_if_i \mid 1 \leq i \leq n\}.$$

Hypergraph \mathcal{H} will have 3-element “variable-edges” $H_i = \{x'_i, t_i, f_i\}$ for $i = 1, \dots, n$, and 7-element “clause-edges” F_j representing clause C_j for $j = 1, \dots, m$. All the latter contain c^* and six further vertices, two for each literal of C_j :

- If C_j contains the positive literal x_i , then F_j contains x'_i and t_i .
- If C_j contains the negative literal $\neg x_i$, then F_j contains x'_i and f_i .

Since H_1, \dots, H_n are disjoint edges, it is clear that $\text{dec}(\mathcal{H}) \geq n$ and $\overline{\chi}(\mathcal{H}) \leq 2n + 1$. We shall see later that equality holds if and only if Φ is satisfiable. In addition, since x'_1, \dots, x'_n is a transversal set of \mathcal{H} , the equalities $\tau(\mathcal{H}) = n$ and $\alpha(\mathcal{H}) = 2n + 1$ are valid for all Φ , no matter whether satisfiable or not. Also, $\tau_2(\mathcal{H}) = 2n$ for all Φ .

Optimal colorings of \mathcal{H} . By Lemma 6, we may restrict our attention to colorings where each color class is a subtree in T . This makes a coloring irrelevant if it 2-colors a variable-edge in such a way that $\{t_i, f_i\}$ is monochromatic but x'_i has a different color. Hence, at least one of the lines $x'_i t_i$ and $x'_i f_i$ is monochromatic (maybe both) for each i . Moreover, we may assume the following further simplification: there is no monochromatic line $c^* x'_i$. Indeed, if the entire H_i is monochromatic, then we would lose a color by making the line $c^* x'_i$ monochromatic. On the other hand, if say the monochromatic pair inside H_i is $x'_i t_i$, then every clause-edge F_j containing $c^* x'_i$ but avoiding t_i also contains the line $x'_i f_i$, therefore we get a coloring with the same number of colors if we assume that $x'_i f_i$ is monochromatic instead of $c^* x'_i$. Summarizing, we search an optimal coloring $\varphi : X \rightarrow \mathbb{N}$ with the following properties for all $i = 1, \dots, n$:

- $\varphi(c^*) \neq \varphi(x'_i)$
- $\varphi(x'_i) = \varphi(t_i)$ or $\varphi(x'_i) = \varphi(f_i)$

In the rest of the proof we assume that all vertex colorings occurring satisfy these conditions.

Truth assignments. Given a coloring φ , we interpret it in the following way for truth assignment and clause deletion:

- If H_i is monochromatic, delete all clauses from Φ which contain literal x_i or $\neg x_i$.

- Otherwise, assign truth value $x_i \mapsto \top$ if $\varphi(x'_i) = \varphi(t_i)$, and $x_i \mapsto \text{F}$ if $\varphi(x'_i) = \varphi(f_i)$.

It follows from the definition of $\mathcal{H}(\Phi)$ that this truth assignment satisfies the modified formula after deletion if and only if φ properly colors all edges of \mathcal{H} .

Also conversely, if Φ' is obtained from Φ by deleting all clauses which contain x_i or $\neg x_i$ for a specified index set $I \subseteq \{1, \dots, n\}$, then a truth assignment $a : \{x_i \mid i \in \{1, \dots, n\} \setminus I\} \rightarrow \{\top, \text{F}\}$ satisfies Φ' if and only if the following specifications for the monochromatic lines yield a proper coloring φ of \mathcal{H} :

- If $i \in I$, then $\varphi(x'_i) = \varphi(t_i) = \varphi(f_i)$.
- Otherwise, let $\varphi(x'_i) = \varphi(t_i)$ if $a(x_i) = \top$, and $\varphi(x'_i) = \varphi(f_i)$ if $a(x_i) = \text{F}$.

The observations above imply the following statement:

Lemma 10 *For any instance Φ of 3-SAT, the value of $\text{dec}(\mathcal{H}(\Phi))$ is equal to the minimum number of variables whose deletion from Φ makes the formula satisfiable. \square*

To complete our preparations for the proof of the theorem, let us quote an earlier result on formulas in which every positive and negative literal occurs in at most four clauses. The problem $\text{MAX 3SAT}(4, \bar{4})$ requires to maximize the number of satisfied clauses in such formulas. The following assertion states that this optimization problem is hard to approximate, even when the input is restricted to satisfiable formulas.

Lemma 11 *([3, Corollary 5]) Satisfiable $\text{MAX 3SAT}(4, \bar{4})$ has no PTAS, unless $\text{P} = \text{NP}$.*

Now we are ready to verify Theorem 5, which states:

Theorem 5. *Unless $\text{P} = \text{NP}$, neither of the following values can be approximated within additive error $o(n)$ for hypertrees of edge size at most γ :*

$$\bar{\chi}(\mathcal{H}), \text{dec}(\mathcal{H}), \alpha(\mathcal{H}) - \bar{\chi}(\mathcal{H}), \tau(\mathcal{H}) - \text{dec}(\mathcal{H}), \text{dec}(\mathcal{H}) - \tau_2(\mathcal{H})/2.$$

Proof We apply reduction from Satisfiable $\text{MAX 3SAT}(4, \bar{4})$. For each instance Φ of this problem, we construct the hypergraph $\mathcal{H} = \mathcal{H}(\Phi)$. Since

Φ is required to be satisfied, no variables have to be deleted from it to admit a satisfying truth assignment. This means precisely one monochromatic line inside each variable-edge. Hence, the above observations together with Lemma 6 imply that $\text{dec}(\mathcal{H}) = n$ and $\bar{\chi}(\mathcal{H}) = 2n + 1$.

On the other hand, Lemma 11 implies the existence of a constant $c > 0$ such that it is NP-hard to find a truth assignment that satisfies all but at most cm clauses in a satisfiable instance of MAX 3SAT(4, $\bar{4}$) with m clauses. Since each literal occurs in at most four clauses, this may require the cancelation of at least $cm/8 \geq c'n$ variables. Thus, for the coloring φ determined by a polynomial-time algorithm, $\text{dec}(\varphi) - \text{dec}(\mathcal{H}) = \Theta(n)$ may hold, and hence also $\bar{\chi}(\mathcal{H}) - |\varphi(X)| = \Theta(n)$. \square

4.2 No efficient approximation on hyperstars

Proposition 9 established a relation between $\text{dec}(\mathcal{H})$ and $\tau_2(\mathcal{H})$, valid for all hypergraphs. Here we show that for hyperstars there is a stronger correspondence between the parameters. After that, we prove Theorem 4 which states non-approximability results on hyperstars.

Given a *hyperstar* $\mathcal{H} = (X, \mathcal{E})$, let us denote by c^* the center of the host star. Hence, $c^* \in E$ holds for all $E \in \mathcal{E}$. We shall use the following notations:

$$E^- = E \setminus \{c^*\}, \quad \mathcal{E}^- = \{E^- \mid E \in \mathcal{E}\}, \quad \mathcal{H}^- = (X \setminus \{c^*\}, \mathcal{E}^-).$$

Proposition 12 *If \mathcal{H} is a hyperstar, then $\text{dec}(\mathcal{H}) = \tau(\mathcal{H}^-) = \tau_2(\mathcal{H}) - 1$ and $\bar{\chi}(\mathcal{H}) = \alpha(\mathcal{H}^-) + 1$.*

Proof If a 2-transversal set S does not contain c^* , then we can replace any $s \in S$ with c^* and obtain another 2-transversal set of the same cardinality. This implies $\tau(\mathcal{H}^-) = \tau_2(\mathcal{H}) - 1$.

Let us observe next that the equalities $\bar{\chi}(\mathcal{H}) = \alpha(\mathcal{H}^-) + 1$ and $\text{dec}(\mathcal{H}) = \tau(\mathcal{H}^-)$ are equivalent, due to the Gallai-type equality for $\alpha + \tau$ in \mathcal{H}^- .

Now, the particular case of Lemma 6 for hyperstars means that there exists a $\bar{\chi}$ -coloring of \mathcal{H} such that all color classes but that of c^* are singletons. Those singletons form an independent set in \mathcal{H}^- , because the color of c^* is repeated inside each E^- . Thus, we necessarily have $\bar{\chi}(\mathcal{H}) \leq \alpha(\mathcal{H}^-) + 1$.

Conversely, if S is a largest independent set in \mathcal{H}^- , i.e. $|S| = \alpha(\mathcal{H}^-) = |X| - 1 - \tau(\mathcal{H}^-)$ and $E^- \setminus S \neq \emptyset$ for all E^- , then making $X \setminus S$ a color class creates a monochromatic pair inside each $E \in \mathcal{E}$ because the color of c^* is

repeated in each E^- . Hence, assigning a new private color to each $x \in S$ we obtain that $\bar{\chi}(\mathcal{H}) \geq \alpha(\mathcal{H}^-) + 1$, consequently $\bar{\chi}(\mathcal{H}) = \alpha(\mathcal{H}^-) + 1$ and $\text{dec}(\mathcal{H}) = \tau(\mathcal{H}^-)$. \square

The following non-approximability results concerning $\bar{\chi}(\mathcal{H})$ and $\text{dec}(\mathcal{H})$ are valid already on the class of hyperstars. We recall the statement of Theorem 4.

Theorem 4.

- (i) For every $\epsilon > 0$, $\text{dec}(\mathcal{H})$ cannot be approximated within $(1 - \epsilon) \ln m$ on the class of hyperstars, unless $\text{NP} \subseteq \text{DTIME}(n^{\mathcal{O}(\log \log n)})$.
- (ii) For every $\epsilon > 0$, $\bar{\chi}(\mathcal{H})$ cannot be approximated within $n^{1-\epsilon}$ on the class of 3-uniform hyperstars, unless $\text{P} = \text{NP}$.

Proof By Proposition 12, the equalities $\bar{\chi}(\mathcal{H}) = \alpha(\mathcal{H}^-) + 1$ and $\text{dec}(\mathcal{H}) = \tau(\mathcal{H}^-)$ hold whenever \mathcal{H} is a hyperstar.

- (i) If \mathcal{H} is a generic hyperstar (with no restrictions on its edges), then \mathcal{H}^- is a generic hypergraph. Thus, approximating $\text{dec}(\mathcal{H})$ on hyperstars is equivalent to approximating $\tau(\mathcal{H}^-)$ on hypergraphs, which is known to be intractable within ratio $(1 - \epsilon)(\log m)$ unless $\text{NP} \subseteq \text{DTIME}(n^{\mathcal{O}(\log \log n)})$, by the result of Feige [10].
- (ii) If \mathcal{H} is a generic 3-uniform hyperstar, then \mathcal{H}^- is a generic graph. Thus, approximating $\bar{\chi}(\mathcal{H})$ on 3-uniform hyperstars is equivalent to approximating $\alpha(\mathcal{H}^-) + 1$ on graphs, which is known to be intractable within ratio $n^{1-\epsilon}$ unless $\text{P} = \text{NP}$, by the result of Zuckerman [19].

\square

In a similar way, we also obtain the following non-approximability result concerning τ_2 .

Corollary 13 *The value $\tau_2(\mathcal{H})$ does not have a polynomial-time $((1 - \epsilon) \ln m)$ -approximation on hyperstars, unless $\text{NP} \subseteq \text{DTIME}(n^{\mathcal{O}(\log \log n)})$.*

Proof By Proposition 12, the approximation of $\tau_2(\mathcal{H})$ on hyperstars \mathcal{H} is as hard as that of $\tau(\mathcal{H}^-)$ on general hypergraphs \mathcal{H}^- . \square

Remark 14 *In connection with Theorem 4 one may observe that, even if we restrict the problem instances to 3-uniform hypergraphs in which each vertex pair is contained in at most three edges, $\overline{\chi}(\mathcal{H})$ does not admit a PTAS. This follows from the fact that the determination of $\alpha(G)$ is MAX SNP-complete on graphs of maximum degree 3, by the theorem of Berman and Fujito [5].*

5 Concluding remarks

Our results on hyperstars show that $\text{dec}(\mathcal{H})$ admits a much better approximation than $\overline{\chi}(\mathcal{H})$ does. In a way this fact is in analogy with the following similar phenomenon in graph theory: The independence number $\alpha(G)$ is not approximable within $n^{1-\varepsilon}$, but $\tau(G) = n - \alpha(G)$ admits a polynomial-time 2-approximation because $\nu(G) \leq \tau(G) \leq 2\nu(G)$, and the matching number $\nu(G)$ can be determined in polynomial time. In this way, both comparisons $\text{dec}(\mathcal{H})$ with $\overline{\chi}(\mathcal{H})$ and $\tau(G)$ with $\alpha(G)$ demonstrate that there can occur substantial difference between the approximability of a graph invariant and its complement.

Perhaps hypertrees with not very large edges admit some fairly efficient algorithms:

Problem 15 *Determine the largest integer r such that there is a PTAS to approximate the value of $\overline{\chi}(\mathcal{H})$ for hypergraphs \mathcal{H} in which every edge has at most r vertices.*

Our results imply that $r \leq 6$ is necessary. From below, a very easy observation shows that for $r = 2$ there is a linear-time algorithm, because for graphs G , the value of $\overline{\chi}(G)$ is precisely the number of connected components.

For hypertrees with non-restricted edge size, the following open question seems to be the most important one:

Problem 16 *Is there a polynomial-time $o(n)$ -approximation for $\overline{\chi}$ on hypertrees?*

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