FULL GROUPS AND SOFICITY

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Abstract. First, we answer a question of Pestov, by proving that the full group of a sofic equivalence relation is a sofic group. Then, we give a short proof of the theorem of Grigorchuk and Medynets that the topological full group of a minimal Cantor homeomorphism is LEF. Finally, we show that for certain non-amenable groups all the generalized lamplighter groups are sofic.

1. Introduction

1.1. Sofic groups and LEF groups. The notion of sofic groups was introduced by Weiss [12] and Gromov [5] (in a somewhat different form). A group $\Gamma$ is sofic if for any finite set $F \subset \Gamma$ and $\epsilon > 0$ there exists a finite set $A$ and a mapping $\Theta : \Gamma \to \text{Map}(A)$ such that

- If $f, g, fg \in F$ then $d_H(\Theta(fg) - \Theta(f)\Theta(g)) \leq \epsilon$, where
  
  $$d_H(\alpha, \beta) = \frac{|\{x \in A \mid \alpha(x) \neq \beta(x)\}|}{|A|}.$$

- If $1 \neq f \in F$ then $d_H(\Theta(f), 1) > 1 - \epsilon$.
- $\Theta(1) = 1$.

All amenable and residually finite groups are sofic. It is an open question whether non-sofic groups exist. If we add the extra requirement that $\Theta(fg) = \Theta(f)\Theta(g)$, then we get the class of LEF-groups (locally embeddable into finite groups). This class of groups was introduced by Gordon and Vershik [11]. Clearly, all residually finite groups are LEF. However, simple, finitely presented groups are not LEF. Nevertheless, by a recent result of Juschenko and Monod [6] (and Theorem 2), there exist simple, finitely generated LEF-groups.

1.2. Sofic equivalence relations. Let $X = \{0, 1\}^\mathbb{N}$ be the standard Borel space with the natural product measure $\mu$. Let $\Phi : \mathbb{F}_\infty \curvearrowright X$ be a (not necessarily free) Borel action of the free group of countably infinite generators $\{\gamma_1, \gamma_1^{-1}, \gamma_2, \gamma_2^{-1}, \ldots\}$ preserving $\mu$. Note that $\mathbb{F}_\infty = \bigcup_{r=1}^{\infty} \mathbb{F}_r$, where $\mathbb{F}_r$ is the free group of rank $r$. Hence, we also have probability measure preserving (p.m.p) Borel actions $\Phi_r : \mathbb{F}_r \curvearrowright X$. We say that $x, y \in X$ are equivalent, $x \sim_\Phi y$ if there exists $w \in \mathbb{F}_\infty$, such that $w(x) = y$. Note that slightly abusing the notation we write $w(x)$ instead of $\Phi(w)(x)$. Thus, the action $\Phi$ represents a countable measured equivalence relation $E_\Phi$ on $X$. Similarly, each $\Phi_r$ represents a countable measured
equivalence relation $E_\Phi$, on $X$, and $E_\Phi = \cup_{i=1}^\infty E_{\Phi_i}$. Each equivalence relation $E_{\Phi_i}$ defines a graphing \([7\) $G_r$ on $X$:

- $V(G_r) = X$.
- $(x, y) \in E(G_r)$ if $\gamma_i x = y$ or $\gamma_i y = x$ for some $i$ (so, there may be loops in $G_r$).

Observe that each component of $G_r$ is a countable graph of bounded vertex degrees. We label each directed edge $(x, y)$ with all the generators mapping $x$ to $y$. Thus an edge, even a loop, may have multiple labels.

Now let us consider transitive actions of $F_r$ on countable sets. If $\alpha : F_r \curvearrowright Y$ is such an action then we have a bounded degree graph structure on $Y$ with multiple labels on the edges from the set \(\{\gamma_1, \gamma_1^{-1}, \ldots, \gamma_r, \gamma_r^{-1}\}\). Let $T_r$ be the set of graphs of all countable $F_r$-actions with a distinguished vertex (the root) such that all the vertices are labeled by the elements of \(\{0, 1\}^r\). Let $G \in T_r$. We define the the $k$-ball around the root $x$, $B_k(x)$ as the induced subgraph on vertices of $G$ in the form of $w(x)$, where $w \in F_r$ is a reduced word of length at most $k$. That is, $B_k(x)$ is the ball centered at $x$ of radius $k$ with respect to the shortest path metric of $G$. The ball $B_k(x)$ is a finite rooted graph with edge-colors from the set \(\{\gamma_1, \gamma_1^{-1}, \ldots, \gamma_r, \gamma_r^{-1}\}\) and vertex labels from the set \(\{0, 1\}^r\). We denote the set of all possible $k$-balls arising from $F_r$-actions by $U^k_r$. We can define a compact metric structure on the set $T_r$ the following way. Let $d_r(G, H) = \frac{1}{2^r}$ if $k$ is the maximal number such that the $k$-balls around the roots of $G$ resp. $H$ are isomorphic as rooted, labeled graphs.

Observe that if $\Theta : F_{\infty} \curvearrowright X$ is a p.m.p action then for each $r \geq 1$ and $x \in X$ one can associate an element $G(\Theta, x) \in T_r$. Namely, the orbit graph of $x$, where the vertex labels are given by the $X$-values, restricted on the first $r$ coordinates. Thus, we have a Borel map $\pi_{\Theta} : X \to T_r$. For $\kappa \in U^k_r$, let $\mu^{k}_{\Theta_{\kappa}}(\kappa) = (\pi_{\Theta})_{\kappa}(\mu)(L_{\kappa})$, where $L_{\kappa} \subset T_r$ is the set of elements $G$ such that the $k$-ball around the root of $G$ is isomorphic to $\kappa$. In other words, $\mu^{k}_{\Theta_{\kappa}}(\kappa)$ is the probability that the $k$-ball around a $\mu$-random element of $X$ is isomorphic to $\kappa$. Now let $\alpha : F_r \curvearrowright Y$ be an $F_r$-action on a finite set. Then for each element $y$ of $Y$, we can associate an element of $T_r$. Namely, $Y$ itself with root $y$. Hence, we can define a probability distribution $\mu^{k,r}_{\alpha_{\kappa}}$ on $U^k_r$. Following \([1\) we say that the action $\Theta : F_{\infty} \curvearrowright X$ is sofic if for all $r \geq 1$, there exists a sequence of finite $F_r$-actions $\{\alpha_{\kappa}\}_{n=1}^\infty$ such that for each $k \geq 1$ and $\kappa \in U^k_r$

$$\lim_{n \to \infty} \mu^{k,r}_{\alpha_{\kappa}}(\kappa) = \mu^{k}_{\Theta_{\kappa}}(\kappa).$$

In \([1\) the authors proved that

- Soficity is a property of the underlying equivalence relations. That is, if an action $\Theta_1$ is orbit equivalent to a sofic action $\Theta_2$, then $\Theta_2$ is sofic as well.
- Treeable equivalence relations are sofic.
- Actions associated to Bernoulli shifts of sofic groups are sofic.

1.3. **Full groups.** Let $E(X, \mu)$ be a countable, measured equivalence relation on a Borel set $X$ with invariant measure $\mu$. The Borel full group of $E$ is the group $[E]_B$ of all Borel bijections $T : X \to X$ such that for any $x \in X$, $T(x) \sim_E x$. We call two such bijections
$T_1, T_2$ equivalent if

$$\mu(\{x \in X \mid T_1(x) = T_2(x)\}) = 1.$$ 

The measurable full group $[E]$ is the group formed by the equivalence classes. Obviously, $[E] = [E]_B/N$, where $N$ is the normal subgroup of elements in $[E]_B$ fixing almost all points of $X$.

Now, let $T : C \to C$ be a homeomorphism of the Cantor set $C$. The topological full group $[[T]]$ is the group of homeomorphisms $S : C \to C$ such that $C$ can be partitioned into finitely many clopen sets $C = \bigcup_{i=1}^n A_i$ such that $S|_{A_i} = T^{n_i}$ for some integer $n_i$.

1.4. Results. Answering a question of Pestov\footnote{MR2566316-MathSciNet Review}, we prove the following theorem.

**Theorem 1.** The measurable full group of a sofic equivalence relation is sofic.

Then, we give a very short proof of a result of Grigorchuk and Medynets \cite{GrigorchukMedynets}.

**Theorem 2.** The topological full group of a minimal Cantor homeomorphism is LEF.

Let $X$ be a countably infinite set and $\Gamma$ be a countable group acting faithfully and transitively on $X$. Then $\Gamma$ can be represented by automorphisms on the Abelian group $\bigoplus_{x \in X} \{0, 1\}$. The groups $\bigoplus_{x \in X} \{0, 1\} \rtimes \Gamma$ are called the lamplighter group of the $\Gamma$-action. If the action is the natural translation action on $\Gamma$, then we get the classical lamplighter group of $\Gamma$. Paunescu \cite{Paunescu} proved that if $\Gamma$ is sofic, then the classical lamplighter group $\bigoplus_{\gamma \in \Gamma} \{0, 1\} \rtimes \Gamma$ is sofic. If $\Gamma$ is amenable, then all its generalized lamplighter groups are amenable hence sofic. Nevertheless, we show that there exist non-amenable groups for which all the generalized lamplighter groups are sofic.

**Theorem 3.** Let $\Gamma^k$ be the $k$-fold free product of the cyclic group of two elements. Then, for any transitive, faithful action of $\Gamma^k$ on a countable set the associated lamplighter group is LEF.

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2. Compressed sofic representations

Let $\Gamma$ be a countable sofic group with elements $\{\gamma_1, \gamma_2, \ldots\}$. A compressed sofic representation of $\Gamma$ is defined the following way. For any $i \geq 1$, we have a constant $\epsilon_i > 0$ and for any $n \geq 1$ we have mappings $\Theta_n : \Gamma \to \text{Map}(A_n)$ such that $|A_n| < \infty$ satisfying the following condition: For all $r > 0$ and $\epsilon > 0$ there exists $K_{r, \epsilon} > 0$ such that if $n > K_{r, \epsilon}$ then

- $d_H(\Theta_n(\gamma_i \gamma_j) \Theta_n(\gamma_i) \Theta_n(\gamma_j)) < \epsilon$ if $1 \leq i, j \leq r$.
- $d_H(\Theta_n(\gamma_i), \text{Id}) > \epsilon_i$ if $1 \leq i \leq r$.

Thus, in a compressed sofic representation we allow large amount of fixed points for each $\gamma \in \Gamma$.

**Lemma 2.1.** If $\Gamma$ has a compressed sofic representation then $\Gamma$ is sofic.
Proof. Let $\tilde{\Theta}_n^k : \Gamma \to \text{Map}(A_n^k)$ be defined by

$$\tilde{\Theta}_n^k(\gamma)(x_1, x_2, \ldots, x_k) = (\Theta_n(\gamma)(x_1), \Theta_n(\gamma)(x_2), \ldots).$$

Observe that if $\gamma, \delta \in \Gamma$, then

- $d_H(\tilde{\Theta}_n^k(\gamma\delta), \tilde{\Theta}_n^k(\gamma)\tilde{\Theta}_n^k(\delta)) \leq (1 - d_H(\Theta_n(\gamma\delta), \Theta_n(\gamma)\Theta_n(\delta)))^k$
- $d_H(\tilde{\Theta}_n^k(\gamma), Id) > (1 - d_H(\Theta_n(\gamma), Id))^k$

Hence, we can choose $\epsilon, n$ and $k$ appropriately to obtain for any $F \subseteq \Gamma$ and $\epsilon' > 0$ a map $\Theta$ as in the Introduction, proving the soficity of $\Gamma$.

3. The proof of Theorem[1]

Let $\Phi : \mathbb{F}_\infty \rhd \{0,1\}^N$ be a sofic action preserving the product measure $\mu$. Let $\Gamma \subseteq [E]$ be a finitely generated group, where $[E]$ is the equivalence relation defined by $\Phi$. So, we have an action $\Phi_\Gamma : \Gamma \rhd \{0,1\}^N$. Our goal is to construct a compressed sofic representation of $\Gamma$. Let $\{\gamma_n\}_{n=1}^\infty$ be an enumeration of the elements of $\Gamma$. Let $\epsilon_n = \mu(\text{Fix}(\Phi_\Gamma(\gamma_n))/2$. Since $\Gamma$ is in the full group, $\epsilon_n > 0$. Now, fix a subset $F \subseteq \Gamma$ and $\epsilon > 0$. We need to construct a map $\Theta : F \to \text{Map}(A)$ for some finite set $A$ such that if $\gamma_i, \gamma_j, \gamma_i\gamma_j \in F$ then

(1) $d_H(\Theta(\gamma_i\gamma_j)\Theta(\gamma_i)\Theta(\gamma_j)) < \epsilon$

(2) $d_H(\Theta(\gamma_i), 1) > \epsilon_i$

Let $\{s_1, s_1^{-1}, s_2, s_2^{-1}, \ldots, s_m, s_m^{-1}\}$ be a symmetric generating set for $\Gamma$. Observe that we have an action $\Sigma_\Gamma : \mathbb{F}_m \rhd \{0,1\}^N$ preserving $\mu$ such that $\Sigma_\Gamma(\delta) = \Phi_\Gamma(\tau(\delta))$, where $\tau : \mathbb{F}_m \to \Gamma$ is the natural quotient map. A dyadic $E$-map of depth $k$ is a Borel map $Q : X \to X$ is defined the following way. For each $\rho \in \{0,1\}^k$ we pick $w_Q(\rho) \in \mathbb{F}_k \subset \mathbb{F}_\infty$ and define $Q(x) = \Phi(w_Q(\rho))(x)$ if the first $k$-coordinates of $x$ is $\rho$. A dyadic approximation of $\Gamma$ is a sequence of families $\{Q_k(s_i)\}_{i=1}^m$, $\{Q_k(s_i^{-1})\}_{i=1}^m$, where for any $1 \leq i \leq m$

- $Q_k(s_i) : X \to X$, $Q_n(s_i^{-1}) : X \to X$ are dyadic $E$-maps of depth $k$.
- $\lim_{k \to \infty} \mu(\{x \in X \mid Q_k(s_i)(x) \neq \Sigma_\Gamma(s_i)(x)\}) = 0$
- $\lim_{k \to \infty} \mu(\{x \in X \mid Q_k(s_i^{-1})(x) \neq \Sigma_\Gamma(s_i)(x)\}) = 0$

We do not require $Q_k$ to be a bijection. Nevertheless, $Q_k$ can be extended to a homomorphism from $\mathbb{F}_m$ to $\text{Map}(X)$. Note that for simplicity we identified the generating set of $\mathbb{F}_m$ by the set $\{s_1, s_1^{-1}, s_2, s_2^{-1}, \ldots, s_m, s_m^{-1}\}$.

Since all the $\Sigma_\Gamma(s_i)$'s are Borel bijections such dyadic approximations clearly exist. The following lemma is an immediate consequence of the definition of the dyadic approximation.

Lemma 3.1. For any $\delta \in \mathbb{F}_m$

$$\lim_{k \to \infty} \mu(\text{Fix}(Q_k(\delta))) = \mu(\text{Fix}(\Sigma_\Gamma(\delta))).$$
Proposition 3.1. There exists a sequence of mappings \( \hat{\Theta}_k : F_m \to Map(B_k) \), where \( |B_k| < \infty \) such that for any \( \delta \in F_m \)

\[
\lim_{k \to \infty} \left( \mu(Fix(Q_k(\delta))) - \frac{|\text{Fix}(\hat{\Theta}_k(\delta))|}{|B_k|} \right) = 0.
\]

That is

\[
\lim_{k \to \infty} \frac{|\text{Fix}(\hat{\Theta}_k(\delta))|}{|B_k|} = \mu(Fix(\Sigma_{\Gamma}(\delta))).
\]

Proof. Let \( \Phi : F_k \curvearrowright \{0,1\}^N \) be the restriction of \( \Phi \). Since \( \Phi \) is sofic, there exists a sequence of mappings \( \{\nu_k^n : F_k \curvearrowright \operatorname{Perm}(C_{k,n})\}_{n=1}^\infty \), where \( C_{k,n} \) is a finite \( \{0,1\}^k \)-vertex labeled graph such that for any \( t \geq 1 \) and \( \kappa \in U_k^t \)

\[
\lim_{n \to \infty} \mu_{n_k}^{t,\kappa}(\kappa) = \mu_{\Phi_k}(\kappa).
\]

Recall that \( Q_k \) is not necessarily an action, only a homomorphism from \( F_m \) to \( Map(X) \). Hence, the local statistics of \( Q_k \) can not be described using the elements of \( U_k^t \) as in the case of honest \( F_m \)-actions. So, let \( W_k^t \) be the set of isomorphism classes of rooted \( t \)-balls of vertex degrees at most \( 2m \), where the vertices are labeled by elements of the set \( \{0,1\}^k \) and the edges (possibly loops) are labeled by subsets of \( \{s_1, s_1^{-1}, s_2, s_2^{-1}, \ldots, s_m, s_m^{-1}\} \). Note that \( U_k^t \subset W_k^t \). Let \( x, y \in X \) be points such that \( B_{\Phi_k}(x) \) and \( B_{\Phi_k}(y) \) represent the same element in \( U_k^{k^2} \). Here \( B_{\Phi_k}(x) \) denotes the \( k \)-ball with respect to the graphing associated to \( \Phi_k \). Then, by the definition of the dyadic approximations \( B_{\Phi_k}^{Q_k}(x) \) and \( B_{\Phi_k}^{Q_k}(y) \) represent the same elements in \( W_k^k \). Now we construct a sequence of maps \( \hat{\Theta}_k^n : F_m \curvearrowright Map(C_{k,n}) \) the following way.

\[
\hat{\Theta}_k^n(s_i)(x) = \nu_k^n(w_Q(s_i)(\rho(x)))(x),
\]

where \( \rho(x) \) is the \( \{0,1\}^k \)-label of \( x \). By the previous observation, for any \( \delta \in F_m \)

\[
\lim_{n \to \infty} \frac{|\text{Fix}(\hat{\Theta}_k^n(\delta))|}{|C_{k,n}|} = \mu(Fix(Q_k(\delta))).
\]

This finishes the proof of the proposition \( \square \)

Pick a section \( \sigma : \Gamma \to F_m \), that is a map such that \( \tau \sigma = Id \). Let \( \hat{\Theta}_k \) as in Proposition 3.1. Define \( \Theta_k : \Gamma \to Map(B_k) \) by

\[
\Theta_k(\gamma) = \hat{\Theta}_k(\sigma(\gamma)).
\]

Then \( \{\Theta_k\}_{k=1}^\infty \) is a compressed sofic representation of \( \Gamma \). \( \square \)

4. The proof of Theorem 2

Let \( T : C \to C \) be a minimal homeomorphism and \( \Gamma \subset [[T]] \) be a finitely generated subgroup of the topological full group of \( T \) with symmetric generating set \( S = \{a_1, a_2, \ldots, a_k\} \). It is enough to prove that \( \Gamma \) is LEF. Let \( x \in C \) and consider the \( T \)-orbit \( \{T^n(x)\}_{n=0}^\infty \). We define the map \( \phi : \Gamma \to \operatorname{Perm}(\mathbb{Z}) \) of \( \Gamma \) into the permutation group of the integers the
following way. Let \( \phi(\gamma)(n) = m \), if \( \gamma(T^m(x)) = T^n(x) \). Since \( T \) acts freely on \( C \), \( \phi \) is well-defined.

**Lemma 4.1.** \( \phi \) is an injective homomorphism.

**Proof.** If \( \phi(\gamma) = \text{Id} \), then \( \gamma \) fixes all the elements of the orbit of \( x \). Since all the orbits are dense, this implies that \( \gamma = 1 \). The fact that \( \phi \) is a homomorphism follows immediately, since \( \phi \) is the restriction of the \( \Gamma \)-action onto the orbit of \( x \).

Let \( a = \max |n| \), where for some \( p \in C \) and \( a_i \in S \), \( a_i(p) = T^m(p) \). We define a sequence

\[
l : \mathbb{Z} \to \{-a, -a + 1, \ldots, 0, 1, \ldots, a - 1, a\}^S
\]

the following way. Let \( l(n) = (t_{a_1}, t_{a_2}, \ldots, t_{a_k}) \), where \( a_i(T^n(x)) = T^{n+t_{a_i}}(x) \). The following lemma is well-known, we prove it for the sake of completeness.

**Lemma 4.2.** \( l \) is a repetitive sequence, that is, if we find a substring \( \sigma \) in \( l \), then there exists \( m \geq 1 \) such that for any interval of length \( m \) we can find \( \sigma \).

**Proof.** For a point \( p \in C \), we can define its \( n \)-pattern

\[
q_n(p) := \{-n, -n + 1, \ldots, 0, 1, \ldots, n - 1, n\} \to \{-a, -a + 1, \ldots, a - 1, a\}
\]

by \( q_n(p)(j) := (t_{a_1}, t_{a_2}, \ldots, t_{a_k}) \), where \( a_i(T^j(x)) = T^{j+t_{a_i}}(x) \). Observe that the set of points with a given \( n \) pattern is closed. Now, let us suppose that for a sequence \( \{k_r\}_{r=1}^\infty \subset \mathbb{Z} \) the intervals \( (k_r - r, k_r + r) \) do not contain \( \sigma \) as a substring. Then, if \( z \) is a limit point of \( \{T^{k_r}(x)\}_{r=1}^\infty \), no translates of \( z \) have \( \sigma \) as a part of their \( n \)-patterns. Therefore the orbit closure of \( z \) does not contain \( x \), in contradiction with the minimality of \( T \).

Now let \( r \geq 1 \) and consider the string \( \sigma_r = l_{\{-ar,-ar+1,\ldots,ar-1,ar\}} \), where \( a \) is the constant defined above. Note that if \( \gamma \in \Gamma \) is the product of at most \( r \) generators then \( |\phi(\gamma)(i) - i| \leq ar \). Pick \( n > 10a^r \) such that

- \( l_{\{-ar+n, -ar+1+n, \ldots, ar-1+n, ar+n\}} = \sigma_r \),
- for any \( \gamma \in \Gamma \) that is the product of at most \( r \) generators there is \( 0 < j < n \) such that \( \gamma(j) \neq j \).

Now we define \( \phi_r : W^r \to \text{Perm}(\mathbb{Z}_n) \), where \( W^r \) is the set of elements in \( \Gamma \) that are products of at most \( r \) generators by \( \phi_r(i) = \phi(i) (\text{mod} \ n) \). Clearly, \( \phi_r \) is injective and if \( x, y, xy \in W^r \) then \( \phi_r(x)\phi_r(y) = \phi_r(xy) \). This implies that \( \Gamma \) is LEF.

**5. The proof of Theorem 3**

Let \( \alpha : \Gamma^k \to X \) be a transitive and faithful action of the free product group. Consider the Schreier graph \( G_\alpha \) of the action with respect to the generators of the \( k \) cyclic groups \( \{a_1, a_2, \ldots, a_k\} \). Recall that \( V(G_\alpha) = X \) and \( (x, y) \in E(G) \) if \( y = a_ix \) for some \( i \geq 1 \). Hence \( G_\alpha \) is a connected graph of vertex degree bound \( k \).

**Proposition 5.1.** Let \( \alpha \) be as above. Then for any \( 1 \neq w \in \Gamma^k \), there exist infinitely many \( y \in X \) such that \( \alpha(w)(y) \neq y \).
Proof. We will need the following lemma.

**Lemma 5.1.** For any finite set $S \subseteq X$, there exists $g \in \Gamma^k$ such that $gS \cap S = \emptyset$.

**Proof.** We define a lazy random walk on $X$ the following way. For $y \in X$ the transition probability $p(x, y) = l/k$, where $l$ is the number of generators $a_i$ such that $a_ix = y$. It is well-known (see e.g. [9],[8]) that the probabilities $p_n(x, y)$ tend to zero for each pair $x, y \in X$. Now consider the standard random walk on the Cayley graph of $\Gamma^k$, the $k$-regular tree. Let $P_n(g)$ be the probability being at $g$ after taking $n$ steps starting from the identity. Then,

$$p_n(x, y) = \sum_{g \in \Gamma, gx = y} P_n(g).$$

By the previous observation, if $n$ is large enough, then

$$\sum P_n(g) < 1,$$

where the summation is taken for all $g \in \Gamma^k$ such that $gx \in S$, for some $x \in S$. Hence, there exists $g \in \Gamma^k$ such that $gS \cap S = \emptyset$.

Now let us suppose that $w \in \Gamma^k$ fixes all points of $X$ outside a finite set $S$. That is $\alpha(w)(S) = S$. Let $gS \cap S = \emptyset$. Then $gwg^{-1}$ fixes all the points of $X$ outside $gS$. Therefore the commutator $[w, gwg^{-1}]$ fixes all elements of $X$, in contradiction with the assumption that the action is faithful. \qed

Now fix a vertex $x \in X$ and consider the ball of radius $n$, $B_n(x)$ around $x$. We define an action $\alpha_n : \Gamma^k \curvearrowright B_n(x)$ the following way. Let $\partial B_n(x)$ be the boundary of the ball $B_n(x)$, that is, the set of all $y \in B_n(x)$ such that there exists $a_i$ for which $\alpha(a_i)y \notin B_n(x)$. If $y \notin \partial B_n(x)$, then let $\alpha_n(a_i)y = \alpha(a_i)y$. If $y \in \partial B_n(x)$ and $\alpha(a_i)y \notin B_n(x)$, then let $\alpha_n(a_i)(y) = y$. Finally, if $y \in \partial B_n(x)$ and $\alpha(a_i)y \in B_n(x)$, then let $\alpha_n(a_i)(y) = \alpha(a_i)(y)$. Now let $L_k^n = \{0, 1\}^{B_n(x)} \rtimes_\alpha \alpha_n(\Gamma^k)$ be the associated finite lamplighter group and $L^k = \otimes_{x \in X} \{0, 1\} \rtimes_\alpha \Gamma^k$. Our goal is to embed $L^k$ into $L_k^n$ locally. That is, for any finite set $F \subseteq L^k$ we construct an injective map $\Theta : F \to L_k^n$ such that $\Theta(fg) = \Theta(f)\Theta(g)$. Recall, that each element of $L^k$ can be uniquely written in the form $a \cdot w$, where $a \in \otimes_{x \in X} \{0, 1\}$ and $w \in \Gamma^k$. We regard the elements of the lamplighter group as permutations of the set $\otimes_{x \in X} \{0, 1\}$. If $\kappa \in \otimes_{x \in X} \{0, 1\}$ and $p \in X$ then

$$(a \cdot w)_{\kappa_p} = a(p) + \kappa(\alpha(w^{-1})(p)).$$

We will also use the product formula

$$(a_2 \cdot w_2)(a_1 \cdot w_1) = (a_2 + \alpha(w_2)(a_1), w_2 w_1),$$

where $\alpha(w_2)(a_1)(q) = a_1(\alpha(w_2^{-1})(q))$. For $l \geq 1$, let $H_l$ be the set of elements of $L^k$ in the form of $a \cdot w$, where $w$ is a word of length at most $l$ and the support of $a$ is contained in $B_l(x)$. For $n \geq l$ we define the map $\tau^n_l : H_l \to L_k^n$ by $\tau^n_l(a \cdot w) := a \cdot \alpha_n(w)$.

**Lemma 5.2.** If $n$ is large enough then $\tau^n_l$ is injective.
Proof. If \( n \) is large enough then \( B_n(x) \) contains a point \( y \) such that
- \( \alpha(w)(y) \neq y \)
- \( d(y, \partial B_n(x)) > l \)
- \( d(y, B_\kappa(x)) > l \),
where \( d \) is the shortest path distance on the Schreier graph \( G_\alpha \). Let \( \kappa \in \bigoplus_{x \in X} \{0, 1\} \) be the element which is 1 at \( y \) and zero otherwise. Then
\[
\tau^n_\iota(a \cdot w)(\kappa)_{|\alpha_n(w)(y)} = 1,
\]
hence \( \tau^n_\iota(a \cdot w) \) is not trivial. \( \square \)

The following lemma finishes the proof of Theorem \( \mathfrak{3} \).

**Lemma 5.3.** Suppose that \((a_1 \cdot w_1), (a_2 \cdot w_2)\) and \((a_2 \cdot w_2)(a_1 \cdot w_1) \in H_I \) and \( n \) is large enough. Then
\[
\tau^n_\iota((a_2 \cdot w_2))\tau^n_\iota((a_1 \cdot w_1)) = \tau^n_\iota((a_2 \cdot w_2)(a_1 \cdot w_1)).
\]

Proof. We need to prove that
\[
(a_2 \cdot \alpha_n(w_2))(a_1 \cdot \alpha_n(w_1)) = (a_2 + \alpha(w_2)(a_1)) \cdot \alpha_n(w_2w_1)
\]
holds in \( L_k^n \). Fix an element \( \kappa \in \{0, 1\}^{B_n(x)} \). Let \( n > 10l \) and \( d(p, \partial B_n(x)) > 5l \). Then
\[
(a_2 \cdot \alpha_n(w_2))(a_1 \cdot \alpha_n(w_1))(\kappa)_{|p} = (a_2 \cdot w_2)(a_1 \cdot w_1)(\kappa_{|p})
\]
and
\[
(a_2 + \alpha(w_2)(a_1) \cdot \alpha_n(w_2w_1))(\kappa)_{|p} = (a_2 + \alpha(w_2)(a_1) \cdot (w_2w_1)(\kappa_{|p})
\]
where \( \kappa \) is an extension of \( \kappa \) onto \( X \). On the other hand, if \( d(p, \partial B_n(x)) \leq 5l \), then
\[
(a_2 \cdot \alpha_n(w_2))(a_1 \cdot \alpha_n(w_1))(\kappa)_{|p} = \alpha_n(w_2)\alpha_n(w_1)(\kappa)_{|p} = \alpha_n(w_2w_1)(\kappa)_{|p}
\]
\( \square \)

**References**


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