THE PACKING DENSITY OF THE n-DIMENSIONAL CROSS-POLYTOPE

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Abstract. The packing density of the regular cross-polytope in Euclidean n-space is unknown except in dimensions 2 and 4 where it is 1. The only non-trivial upper bound is due to Gravel, Elser, and Kallus [9] who proved that for $n = 3$ the packing density of the regular octahedron is at most $1 - 1.4\ldots \times 10^{-12}$. In this paper, we prove upper bounds for the packing density of the n-dimensional regular cross-polytope in the case that $n \geq 7$. We use a modification of Blichfeldt's method [2] due to G. Fejes Tóth and W. Kuperberg [7].

1. Introduction

Let $K \subset \mathbb{R}^n$ be a convex body (a compact convex set with interior points). A family $\mathcal{K} = \{K_1, K_2, \ldots\}$ of congruent copies of K is a *packing* in $\mathbb{R}^n$ if the elements of $\mathcal{K}$ are non-overlapping (their interiors are pairwise disjoint). The *density* of a packing is essentially the proportion of space covered by elements of the packing. The supremum of the densities of all packings with congruent copies of a convex body K is called the *packing density* of K and it is denoted by $\delta(K)$. For a more detailed introduction into the basic properties of density see, for example, [8] and [14].

One of the central problems of the theory of packing and covering is to determine the packing densities of particular convex bodies. The most important such body has always been the n-dimensional unit ball B^n. However, the exact value of $\delta(B^n)$ is known only in the cases $n = 2$ and 3. In particular, $\delta(B^2) = \pi/\sqrt{12}$, proved by Thue [16,17], and $\delta(B^3) = \pi/\sqrt{18}$, proved by Hales [10]. Recently, Cohn and Kumar [5] proved that in dimension 24 the density of no sphere packing can exceed the density of the Leech lattice by more than a multiplicative factor of $1+1.65\times 10^{-30}$. Thus, for any practical purpose, we may consider $\delta(B^{24}) = \pi^{12}/12!$.

The current best asymptotic upper bound for $\delta(B^n)$ is due to Kabatjanskiĭ and Levenšteĭn [11]:

$$\delta(B^n) \leq 2^{(-0.599+o(1))n}. \tag{1}$$

For small dimensions, the bounds proved by Cohn and Elkies [4] are better but asymptotically their upper bound is the same as (1).

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Other than the n-dimensional ball, the most interesting convex bodies are probably the n-dimensional regular polytopes that exist in every dimension: the simplex, the cube, and the cross-polytope. The n-cube is a tile, so its packing density is 1. However, very little is known about the packing densities of the regular simplex and the regular cross-polytope for $n \geq 3$. An exception is the case when $n = 4$, then the regular cross-polytope tiles $\mathbb{R}^4$ (cf. Section 22 in [8]), thus its packing density is 1. Very recently, Gravel, Elser and Kallus [9] proved upper bounds for the packing density of the regular tetrahedron ($1 - 2.6 \ldots \times 10^{-25}$) and the regular octahedron ($1 - 1.4 \ldots \times 10^{-12}$). These bounds are certainly not optimal. It seems unclear whether the method of Gravel, Elser and Kallus can be extended to higher dimensions.

We note that there has been much work done recently in order to construct efficient packings of regular tetrahedra, octahedra and other solids in $\mathbb{R}^3$. For an overview see Torquato and Jiao [19]. Dense packings of n-dimensional cross-polytopes were constructed by Rush [15]. Finally, we remark that, to the best of our knowledge, essentially nothing is known about the covering densities of the regular solids.

In the next section, we will prove an upper bound for the packing density of the regular n-dimensional cross-polytope using known upper bounds on $\delta(B^n)$ and the ratio of the volumes of the cross-polytope and its insphere. Subsequently, we significantly improve these upper bounds for small dimensions employing a modification of the method of Blichfeldt by G. Fejes Tóth and W. Kuperberg [7]. With these methods we establish non-trivial upper bounds on the packing density of the n-dimensional cross-polytope for $n \geq 7$, and we also show that the packing density of the cross-polytope approaches 0 exponentially fast as the dimension tends to infinity.

1.1. **Upper bound using the insphere.** If $K \subset \mathbb{R}^n$ is a convex body and $r(K)$ is the radius of the maximum size ball contained in K, then

$$\delta(K) \leq \frac{\text{Vol}\,(K)}{r(K)^n \text{Vol}\,(B^n)} \delta(B^n). \tag{2}$$

This may provide a non-trivial upper bound on the packing density of a convex body K whose insphere is sufficiently large in volume compared to K. Using this idea, Torquato and Jiao [18, 19] derived upper bounds for the packing densities of the regular dodecahedron and the regular icosahedron and for some of the Archimedean solids in $\mathbb{R}^3$, see Tables III and IV in [19]. However, it appears that they have not used the insphere volume ratio to investigate the packing densities of convex bodies in higher dimensions.

In $\mathbb{R}^4$, the cube, the cross-polytope and the 24-cell are tiles, so their packing densities are all equal to 1. The insphere volume ratio method gives an upper bound greater than 1 for the packing density of the regular simplex. However, one obtains non-trivial upper bounds for the packing densities of the 120-cell and the 600-cell using the $\delta(B^4) \leq 0.13126 \cdot \pi^2/2$ bound by Cohn and Elkies [4], see the numerical values in Table 1.

In dimensions higher than 4, there exist only three regular solids, the simplex, the cube and the cross-polytope. The n-dimensional cube is always a tile in $\mathbb{R}^n$,

n	P	Upper bound on $\delta(P)$
4	120-cell	0.74972
4	600-cell	0.69073

TABLE 1. Upper bounds on the packing densities of 4-dimensional regular solids obtained from their insphere volume ratios.

thus its packing density is 1. The insphere volume ratio method does not provide a non-trivial upper bound on the density of the regular simplex in any dimension. However, it gives an upper bound on the density of the n-dimensional cross-polytope, which approaches 0 exponentially fast as n tends to infinity.

Consider the regular cross-polytope

$$X^n := \text{conv}\,(\pm e_1, \ldots, \pm e_n),$$

where $e_i, i = 1, \ldots, n$, are the standard orthonormal basis vectors of $\mathbb{R}^n$. It is clear that

$$\text{Vol}\,(X^n) = \frac{2^n}{n!},$$

and the inradius of X^n is

$$r_n = r(X^n) = 1/\sqrt{n}.$$

We say that two non-negative sequences $f(n)$ and $g(n)$ are asymptotically equal if $\lim_{n\to\infty} f(n)/g(n) = 1$. The asymptotic equality of $f(n)$ and $g(n)$ will be denoted by $f(n) \sim g(n)$. We write $f(n) \ll g(n)$ if there exists a positive real number c such that $f(n) \leq c \cdot g(n)$ for all n.

Consider a packing of cross-polytopes in $\mathbb{R}^n$. Then (1), (2) and the Stirling formula yield that

$$\begin{aligned}
\delta(X^n) &\leq \frac{\text{Vol}\,(X^n)}{r_n^n \text{Vol}\,(B^n)}\delta(B^n) \\
&\leq \frac{\text{Vol}\,(X^n)}{r_n^n \text{Vol}\,(B^n)} 2^{-0.599n(1+o(1))} \\
&= \frac{2^n \sqrt{n}^n \Gamma(\frac{n}{2}+1)}{n! \sqrt{\pi}^n} 2^{-0.599n(1+o(1))} \\
&\sim \frac{1}{\sqrt{2}} \left(\frac{e}{\pi 2^{0.198}}\right)^{\frac{n}{2}} \qquad (3) \\
&\ll 0.86850^n. \qquad (4)
\end{aligned}$$

Thus, we conclude that

$$\delta(X^n) \to 0, \quad \text{as} \quad n \to \infty$$

exponentially fast. Note that the upper bound in (3) is asymptotic in nature and it says nothing about the packing density of X^n in specific dimensions. In order to obtain concrete bounds on $\delta(X^n)$, we must use specific upper bounds for $\delta(B^n)$. The Cohn-Elkies bounds on $\delta(B^n)$ for $5 \leq n \leq 36$ (Table 3 on page 711 in [4]), and the almost exact value of $\delta(B^{24})$ by Cohn and Kumar [5] yield by simple computations the upper bounds for $\delta(X^n)$ shown in Table 2.

We will improve on these bounds in Section 3.

n	Upper bound on $\delta(X^n)$	n	Upper bound on $\delta(X^n)$
24	0.98753	31	0.67265
25	0.95416	32	0.63268
26	0.90259	33	0.59472
27	0.85275	34	0.55877
28	0.80476	35	0.52476
29	0.75871	36	0.49264
30	0.71466		

TABLE 2. Upper bound on $\delta(X^n)$ for $24 \leq n \leq 36$ using the insphere volume ratio.

2. Blichfeldt's method and its extension

Let $K \subset \mathbb{R}^n$ be a convex body. A non-negative Lebesgue measurable function $f : [0, \infty) \to \mathbb{R}$ is a Blichfeldt gauge for K if it satisfies the following conditions.

i) $I_n(f) := \int_{\mathbb{R}^n} f(|x|)dx < \infty$.
ii) If $\{\varphi_i : i = 1, 2, \dots\}$ is a set of isometries of $\mathbb{R}^n$ such that the collection $\{\varphi_i K : i = 1, 2, \dots\}$ is a packing, then for any $x \in \mathbb{R}^n$ it holds that $\sum_{i=1}^{\infty} f(|\varphi_i^{-1}(x)|) \leq 1$.

For technical reasons, we assume the following extra condition on f.

iii) There exists an $r_0 > 0$ with $f(r) = 0$ for all $r > r_0$.

The idea of Blichfeldt [2] was that if f is a gauge for a convex body K, then

$$\delta(K) \leq \frac{\text{Vol}\,(K)}{I_n(f)}.$$

Blichfeldt applied this idea only to the unit ball in [2]. He used the gauge

$$f_0(r) = \begin{cases} 1 - \frac{r^2}{2} & \text{for } 0 \leq r \leq \sqrt{2}, \\ 0 & \text{for } r > \sqrt{2} \end{cases}$$

to show that $\delta(B^n) \leq (n+2)2^{-(n+2)/2}$ and noted that a slight improvement of this bound can be obtained by the gauge

$$f^*(r) = \begin{cases} f_0(r) & \text{for } r \geq 1, \\ 1 - f_0(2 - r) & \text{for } r \leq 1. \end{cases} \tag{5}$$

For $0 \leq \varrho \leq r(K)$ the inner parallel domain of K with radius ϱ is defined as

$$K_{-\varrho} := \{x \in K : \varrho B^n + x \subseteq K\}.$$

For $x \in \mathbb{R}^n$, let $d(x, K_{-\varrho})$ denote the Euclidean distance of x from $K_{-\varrho}$. It is proved in [7] that if f is a Blichfeldt gauge for B^n, then for any $0 < \rho \leq r(K)$,

$$g_\varrho(x) = f\left(\frac{d(x, K_{-\varrho})}{\varrho}\right)$$

is a Blichfeldt gauge for K. Thus, writing

$$G(\varrho) = \int_{\mathbb{R}^n} g_\varrho(x)dx,$$

we have

$$\delta(K) \leq \frac{\text{Vol}\,(K)}{G(\varrho)}. \tag{6}$$

This method gives an upper bound on $\delta(K)$ for each $0 < \varrho \leq r(K)$. Our objective is to find, or at least estimate, the best such upper bound.

Let $\kappa_n = \text{Vol}\,(B^n)$. For a convex body $K \subset \mathbb{R}^n$ and a non-negative real number λ, the radius λ parallel domain K_λ of K is the set of points in $\mathbb{R}^n$ whose distance from K is at most λ. The Steiner formula for the volume of K_λ can be written in the form

$$\text{Vol}\,(K + \lambda B^n) = \sum_{j=0}^{n} \lambda^{n-j} \kappa_{n-j} V_j(K),$$

where $V_j(K)$, $j = 0, \ldots, n$ are the intrinsic volumes of K introduced by McMullen [13]. Note that $V_n(K) = \text{Vol}\,(K)$ is the volume of K, and $2V_{n-1}(K) = S(K)$ is the surface volume of K.

Let $I_0(f) := f(0)$. Using an argument that is very similar to the proof of Steiner's formula, one obtains that

$$G(\varrho) = \sum_{j=0}^{n} \varrho^j I_j(f) V_{n-j}(K_{-\varrho}). \tag{7}$$

If (7) can be calculated or estimated explicitly, then (6) provides an upper bound for $\delta(K)$. We will see in the next section that it can give better results than the ones we can obtain from the insphere volume ratio.

The Blichfeldt technique may be used for estimating the packing density of a convex body for which the intrinsic volumes of its inner parallel domain can be calculated explicitly or at least estimated numerically. Two classes of such bodies were exhibited in [7]: cylinders and the radius 1 outer parallel domain of segments. In the next sections, we will apply this method to the cross-polytope X^n.

3. The case of the n-dimensional cross-polytope

Let $P \subset \mathbb{R}^n$ be a polytope such that all facets of P are tangent to its insphere. In this case we say that P is circumscribed around its insphere. It is not difficult to see that if P is such a polytope, then for all $0 \leq \varrho \leq r(P)$, the radius ϱ inner parallel domain $P_{-\varrho}$ of P is a polytope that is similar to P with similarity ratio $(r(P) - \varrho)/r(P)$. Since the jth intrinsic volume is homogeneous of degree j, it holds that

$$V_j(P_{-\varrho}) = \left(\frac{r(P) - \varrho}{r(P)}\right)^j V_j(P), \quad j = 0, \ldots, n.$$

Thus, the right hand side of (7) becomes a polynomial of degree n of ϱ in the case that P is circumscribed around its insphere, that is,

$$G(\varrho) = \sum_{j=0}^{n} \varrho^j I_j(f) \left(\frac{r(P) - \varrho}{r(P)}\right)^{n-j} V_{n-j}(P) \quad (0 < \varrho \leq (r(P))). \tag{8}$$

In particular, X^n is circumscribed about its insphere. Betke and Henk [1] determined the following formula for the jth intrinsic volume of X^n:

$$V_j(X^n) = 2^{j+1} \binom{n}{j+1} \cdot \frac{\sqrt{j+1}}{j!} \cdot \gamma(n, j), \tag{9}$$

where

$$\gamma(n,j)=\sqrt{\frac{j+1}{\pi}}\int_0^\infty e^{-(j+1)x^2}\left(\frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}dy\right)^{d-j-1}dx$$

is the outer angle at a j-dimensional face of X^n.

Although we only defined $G(\varrho)$ for $0<\varrho\le r_n$, it is, in fact, well-defined as a polynomial for all ϱ, and thus its derivatives of all orders exist at $\varrho=0$ and $\varrho=r_n$. Elementary calculus yields that

$$\begin{aligned}G'(r_n)&=-r_n^{n-2}I_{n-1}(f)V_1(X^n)+nr_n^{n-1}I_n(f)\\&=r_n^{n-2}(nr_nI_n(f)-I_{n-1}(f)V_1(X^n)).\end{aligned}\tag{10}$$

Böröczky and Henk [3] proved that for any fixed j,

$$\gamma(n,j)\sim\frac{1}{2}\frac{(j+1)!}{\sqrt{j+1}}\frac{(\pi\ln n)^{\frac{j}{2}}}{n^{j+1}},\ \text{as } n\to\infty.$$

In particular,

$$V_1(X^n)\sim\sqrt{\pi}\sqrt{\ln n},\ \text{as } n\to\infty.\tag{11}$$

It follows from property iii) of f and the Stirling formula that

$$\begin{aligned}\frac{I_n(f)}{I_{n-1}(f)}&=\frac{\omega_n}{\omega_{n-1}}\frac{\int_0^\infty f(r)r^{n-1}dr}{\int_0^\infty f(r)r^{n-2}dr}\\&\le\frac{\kappa_n}{\kappa_{n-1}}\frac{n}{n-1}\frac{\int_0^\infty f(r)r^{n-2}r_0dr}{\int_0^\infty f(r)r^{n-2}dr}\\&\sim r_0\sqrt{\frac{2\pi e}{n}}\ \text{as } n\to\infty,\end{aligned}$$

and thus

$$I_n(f)\ll r_0\sqrt{\frac{2\pi e}{n}}I_{n-1}(f).\tag{12}$$

Combining (10), (11) and (12), we obtain that

$$\begin{aligned}G'(r_n)&\ll r_n^{n-2}I_{n-1}(f)\left(r_0\sqrt{\frac{2\pi e}{n}}nr_n-\sqrt{\pi}\sqrt{\ln n}\right)\\&=\sqrt{\pi}r_n^{n-2}I_{n-1}(f)\left(r_0\sqrt{2e}-\sqrt{\ln n}\right)\\&<0\end{aligned}\tag{13}$$

for sufficiently large n.

Note that $G(r_n)=I_n(f)r_n^n$, and thus by (6),

$$\begin{aligned}\delta(X^n)&\le\frac{\text{Vol }(X^n)}{I_n(f)r_n^n}\\&=\frac{\text{Vol }(X^n)}{r_n^n\text{Vol }(B^n)}\cdot\frac{\text{Vol }(B^n)}{I_n(f)},\end{aligned}$$

which is exactly the upper bound on $\delta(X^n)$ that we obtain using the ratio of the volumes of K and its insphere multiplied by the upper bound on $\delta(B^n)$ from the Blichfeldt gauge f. Thus, (13) implies that for large n the Blichfeldt method gives a better upper bound on $\delta(X^n)$ than the insphere volume ratio combined with the upper bound $\delta(B^n)$ coming from the Blichfeldt gauge f.

4. Using the original Blichfeldt gauge function

In this section, we will use the Blichfeldt gauge f^* as defined in (5). Then $I_0(f^*) = 1$, and for $n \geq 1$,

$$I_n(f^*) = \frac{2\kappa_n}{n+2}(\sqrt{2})^n(1+b_n),$$

where

$$b_n = \frac{1}{(\sqrt{2})^n(n+1)} - (\sqrt{2}-1)^{n+1}\left(1+\frac{\sqrt{2}}{n+1}\right).$$

Routine calculations show that

$$\begin{aligned} G'(0) &= -\frac{n}{r_n}I_0(f^*)V_n(X^n) + I_1(f^*)V_{n-1}(X^n) \\ &= -n\sqrt{n}\frac{2^n}{n!} + 2\sqrt{n}\frac{2^n}{(n-1)!}\frac{1}{2} \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} G''(0) &= \frac{n(n-1)}{r_n^2}I_0(f^*)V_n(X^n) - 2\frac{n-1}{r_n}I_1(f^*)V_{n-1}(X^n) + 2I_2(f^*)V_{n-2}(X^n) \\ &= 2I_2(f^*)V_{n-2}(X^n) - \frac{n-1}{r_n}I_1(f^*)V_{n-1}(X^n) \\ &= \frac{n2^n}{(n-2)!}\left(\frac{\sqrt{n-1}}{2}\arccos\left(1-\frac{2}{n}\right)1.062097 - 1\right). \end{aligned}$$

It is easy to check that $G''(0) > 0$ when $n \geq 7$. Since Vol $(X^n)/G(0) = 1$, our method provides a non-trivial upper bound on $\delta(X^n)$ in the case that $n \geq 7$. Furthermore, since $G'(r_n) < 0$, the minimum of Vol $(X^n)/G(\varrho)$ is attained at an interior point of the interval $[0, r_n]$ and the method yields an upper bound on $\delta(X^n)$ that is better than the one obtained from the insphere volume ratio combined with the Blichfeldt upper bound on $\delta(X^n)$.

Although the quantities in (9) cannot be calculated explicitly, they can be approximated by numerical methods. By such numerical calculations, one obtains for $G(\varrho)$ a degree n polynomial in ϱ whose maximum may be approximated (again by numerical methods). The results of our calculations are described in Tables 3 and 4, and they are compared to the upper bounds in Table 2 in Figure 1.

We note that the numerical calculations suggest that the value of ϱ at which the maximum of $G(\varrho)$ is reached tends to $\frac{2}{3\sqrt{n}}$ as $n \to \infty$. Furthermore, it also appears from calculations that for sufficiently large n, the terms in which the exponent of ϱ is around $2n/3$ dominate the polynomial $G(\frac{2}{3\sqrt{n}})$.

Finally, we remark that we fitted and exponential function on the numerical results obtained from f^* and got the following approximate asymptotics

$$\delta(X^n) \ll 0.87434^n.$$

5. Concluding remarks

The upper bounds obtained from the Blichfeldt method depend on the gauge function used. In Section 4, we used f^* because $I_n(f^*)$ can be evaluated explicitly for all n. However, f^* is not the best possible such function, although in small dimensions it provides better upper bounds on $\delta(X^n)$ than other known Blichfeldt

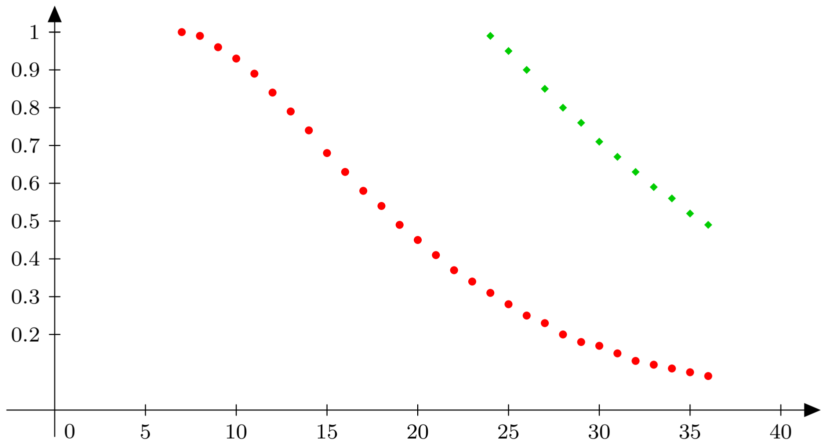

FIGURE 1. Comparison of upper bounds on $\delta(X^n)$ obtained from insphere volume ratio (diamonds) and the Blichfeldt method with f^* (dots) for $7 \le n \le 36$.

n	Upper bound	n	Upper bound
7	0.99805	22	0.37264
8	0.98606	23	0.33850
9	0.96188	24	0.30697
10	0.92730	25	0.27794
11	0.88500	26	0.25129
12	0.83754	27	0.22690
13	0.78705	28	0.20462
14	0.73524	29	0.18448
15	0.68339	30	0.16586
16	0.63247	31	0.14908
17	0.58317	32	0.13398
18	0.53596	33	0.12017
19	0.49116	34	0.10770
20	0.44896	35	0.09647
21	0.40944	36	0.08635

TABLE 3. Upper bounds on $\delta(X^n)$ using the Blichfeldt gauge f^* and calculated by Maple13.

gauges for B^n. For example, it is remarked in [7] that Levenšteĭn [12] introduced the following Blichfeldt gauge derived from spherical codes. Let $M(n,\varphi)$ denote the maximum number of points on S^{n-1} with the property that their pairwise angular distances are not smaller than φ. Then the following function

$$f_n(x) = \begin{cases} \frac{1}{M(n,\varphi)} & \text{for } 0 \le |x| < \sqrt{\frac{2}{1-\cos\varphi}} \\ 0 & \text{for} \sqrt{\frac{2}{1-\cos\varphi}} \le |x|. \end{cases}$$

is a Blichfeldt gauge for B^n if $\pi/3 \le \varphi \le \pi$. For a brief explanation why f_n is a Blichfeldt gauge, see page 726 in [7]. Kabatjanskĭ and Levenšteĭn (cf. Formula (52)

n	density	n	density
40	5.52108×10^{-2}	140	1.98099×10^{-7}
50	1.72421×10^{-2}	150	5.36214×10^{-8}
60	5.19017×10^{-3}	160	1.44520×10^{-8}
70	1.52250×10^{-3}	170	3.88033×10^{-9}
80	4.38143×10^{-4}	180	1.03837×10^{-9}
90	1.24242×10^{-4}	190	2.77031×10^{-10}
100	3.48295×10^{-5}	200	7.37113×10^{-11}
110	9.66572×10^{-6}	250	9.26781×10^{-14}
120	2.66200×10^{-6}	500	2.25312×10^{-28}
130	7.28254×10^{-7}	750	4.01494×10^{-43}
		1000	6.36493×10^{-58}

TABLE 4. Upper bounds on $\delta(X^n)$ using the Blichfeldt gauge f^* and calculated by Maple13.

in [11]) proved that

$$M(n,\varphi) \leq \frac{4\binom{k+n-2}{k}}{1-t^{\alpha,\alpha}_{1,k+1}} \quad \text{if } \cos\varphi \leq t^{\alpha,\alpha}_{1,k}, \tag{14}$$

where $t^{\alpha,\alpha}_{1,k}$ denotes the largest root of the Jacobi polynomial of degree k with parameters $\alpha = (n-3)/2$. For a definition of Jacobi polynomials see, for example, Formula (23) in [11]. Using (14), one can obtain Blichfeldt gauge functions for X^n in any dimension which yield concrete upper bounds on $\delta(X^n)$. However, these Blichfeldt gauges do not give better bounds on $\delta(X^n)$ than f^* up to (at least) dimension 300. On the other hand, in dimension 500 one obtains a better bound using (14) than with f^*. We note that the calculations with (14) become computationally very demanding for higher dimensions.

Kabatjanskiĭ and Levenšteĭn [11] derived from (14)

$$M(n,\varphi) \leq (\sin(\varphi/2))^{-n}2^{-(0.599+o(1))n}, \tag{15}$$

which holds for $\varphi \leq 63^0$ and is the best asymptotic upper bound on $M(n,\varphi)$. We note that if one uses the Blichfeldt gauge f_n that comes from (15), then Blichfeldt's theorem yields the Kabatjanskiĭ–Levenšteĭn upper bound (1) on $\delta(B^n)$. Together with (13), this indicates that the Blichfeldt method may provide a better asymptotic upper bound on $\delta(X^n)$ than the insphere volume ratio combined with (1).

We calculated upper bounds on $\delta(X^n)$ for $n \leq 1000$ using the Blichfeldt gauge f_n derived from (15) (omitting the unknown $o(n)$ term). The calculations suggest that the minimum of Vol $(X^n)/G(\varrho)$ is attained at a value ϱ which tends to roughly $0.767\ldots \times r_n$ as $n \to \infty$. We fitted an exponential function on the results based on which we conjecture that

$$\delta(X^n) \ll 0.82886^n.$$

In order to prove this we would need two things: an estimates on the $o(n)$ term in (15) and asymptotic formulae for the intrinsic volumes $V_j(X^n)$ for all $j = 0,\ldots,n$. To the best of our knowledge, no such asymptotic formulae are known at present.

References

[1] Ulrich Betke and Martin Henk, *Intrinsic volumes and lattice points of cross-polytopes*, Monatsh. Math. **115** (1993), no. 1-2, 27–33. MR1223242 (94g:52010)

[2] H. F. Blichfeldt, *The minimum value of quadratic forms, and the closest packing of spheres*, Math. Ann. **101** (1929), no. 1, 605–608. MR1512555

[3] Károly Böröczky Jr. and Martin Henk, *Random projections of regular polytopes*, Arch. Math. (Basel) **73** (1999), no. 6, 465–473. MR1725183 (2001b:52004)

[4] Henry Cohn and Noam Elkies, *New upper bounds on sphere packings. I*, Ann. of Math. (2) **157** (2003), no. 2, 689–714. MR1973059 (2004b:11096)

[5] Henry Cohn and Abhinav Kumar, *The densest lattice in twenty-four dimensions*, Electron. Res. Announc. Amer. Math. Soc. **10** (2004), 58–67. MR2075897 (2005e:11089)

[6] J. H. Conway and N. J. A. Sloane, *Sphere packings, lattices and groups*, 3rd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 290, Springer-Verlag, New York, 1999. With additional contributions by E. Bannai, R. E. Borcherds, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen and B. B. Venkov. MR1662447 (2000b:11077)

[7] G. Fejes Tóth and W. Kuperberg, *Blichfeldt's density bound revisited*, Math. Ann. **295** (1993), no. 4, 721–727. MR1214958 (94c:11055)

[8] L. Fejes Tóth, *Regular figures*, A Pergamon Press Book, The Macmillan Co., New York, 1964. MR0165423 (29 #2705)

[9] Simon Gravel, Veit Elser, and Yoav Kallus, *Upper bound on the packing density of regular tetrahedra and octahedra*, Discrete Comput. Geom. **46** (2011), no. 4, 799–818. MR2846180 (2012g:52035)

[10] Thomas C. Hales, *A proof of the Kepler conjecture*, Ann. of Math. (2) **162** (2005), no. 3, 1065–1185. MR2179728 (2006g:52029)

[11] G. A. Kabatjanskiĭ and V. I. Levenšteĭn, *Bounds for packings on the sphere and in space*, Problems of Information Transmission **14** (1978), no. 1, 1–17. MR0514023 (58 #24018)

[12] V. I. Levenshteĭn, *Bounds for packings of metric spaces and some of their applications*, Problemy Kibernet. **40** (1983), 43–110 (Russian). MR717357 (86c:52014)

[13] P. McMullen, *Non-linear angle-sum relations for polyhedral cones and polytopes*, Math. Proc. Cambridge Philos. Soc. **78** (1975), no. 2, 247–261. MR0394436 (52 #15238)

[14] C. A. Rogers, *Packing and covering*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 54, Cambridge University Press, New York, 1964. MR0172183 (30 #2405)

[15] J. A. Rush, *Constructive packings of cross polytopes*, Mathematika **38** (1991), no. 2, 376–380 (1992). MR1147836 (92k:11071)

[16] A. Thue, *Om Nogle Geometrisk Taltheoretiske Theoremer*, Forhdl. Skand. Naturforsk. **14** (1892), 352–353.

[17] A. Thue, *Über die dichteste Zusammenstellung von kongruenten Kreisen in einer Ebene*, Christiania Vid. Selsk. Skr. **1** (1910), 3–9.

[18] S. Torquato and Y. Jiao, *Dense packings of the Platonic and Archimedean solids*, Nature **460** (13 August 2009), 876–880.

[19] S. Torquato and Y. Jiao, *Dense packings of polyhedra: Platonic and Archimedean solids*, Phys. Rev. E (3) **80** (2009), no. 4, 041104, 21. MR2607418 (2010i:52027)

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