A proof of the stability of extremal graphs, Simonovits' stability from Szemerédi's regularity

Zoltán Füredi [*]

Alfréd Rényi Institute of Mathematics,
13–15 Reáltanoda Street, 1053 Budapest, Hungary.
E-mail: `z-furedi@illinois.edu`

Abstract

The following sharpening of Turán's theorem is proved. Let $T_{n,p}$ denote the complete p–partite graph of order n having the maximum number of edges. If G is an n-vertex K_{p+1}-free graph with $e(T_{n,p}) - t$ edges then there exists an (at most) p-chromatic subgraph H_0 such that $e(H_0) \geq e(G) - t$.

Using this result we present a concise, contemporary proof (i.e., one using Szemerédi's regularity lemma) for the classical stability result of Simonovits [21].

1 The Turán problem

Given a graph G with vertex set $V(G)$ and vertex set $\mathcal{E}(G)$ its number of edges is denoted by $e(G)$. The neighborhood of a vertex $x \in V$ is denoted by $N(x)$, note that $x \notin N(x)$. For any $A \subset V$ the restricted neighborhood $N_G(x|A)$ stands for $N(x) \cap A$. Similarly, $\deg_G(x|A) := |N(x) \cap A|$. If the graph is well understood from the text we leave out subscripts. The *Turán graph* $T_{n,p}$ is the largest p-chromatic graph having n vertices, $n, p \geq 1$. Given a partition $(V_1, \ldots, V_p)$ of V the *complete multipartite* graph $K(V_1, \ldots, V_p)$ has vertex set V and all the edges joining distinct partite sets. $A \triangle B$ stands for the symmetric difference of the sets A and B. For further notations and notions undefined here see, e.g., the monograph of Bollobás [4].

Turán [23] proved that if an n vertex graph G has at least $e(T_{n,p})$ edges then it contains a complete subgraph K_{p+1}, except if $G = T_{n,p}$. Given a class of graphs $\mathcal{L}$, a graph G is called

$\mathcal{L}$-*free* if it does not contain any subgraph isomorphic to any member of $\mathcal{L}$. The *Turán number* $\mathrm{ex}(n, \mathcal{L})$ is defined as the largest size of an n-vertex, $\mathcal{L}$-free graph. Erdős and Simonovits [11] gave a general asymptotic for the Turán number as follows. Let $p + 1 := \min\{\chi(L) : L \in \mathcal{L}\}$. Then

$$\mathrm{ex}(n, \mathcal{L}) = \left(1 - \frac{1}{p}\right)\binom{n}{2} + o(n^2) \quad \text{as} \quad n \to \infty. \tag{1}$$

They also showed that if G is an extremal graph, i.e., $e(G) = \mathrm{ex}(n, \mathcal{L})$, then it can be obtained from $T_{n,p}$ by adding and deleting at most $o(n^2)$ edges. This result is usually called Erdős–Stone–Simonovits theorem, although it was proved first in [11], but indeed (1) easily follows from a result of Erdős and Stone [12].

The aim of this paper is to present a new proof for the following stronger version of (1), a structural stability theorem, originally proved by Erdős and Simonovits [11], Erdős [7, 8], and Simonovits [21]. For every $\varepsilon > 0$ and forbidden subgraph class $\mathcal{L}$ there is a $\delta > 0$, and n_0 such that if $n > n_0$ and G is an n-vertex, $\mathcal{L}$-free graph with

$$e(G) \geq \left(1 - \frac{1}{p}\right)\binom{n}{2} - \delta n^2,$$

then

$$|\mathcal{E}(G_n) \triangle \mathcal{E}(T_{n,p})| \leq \varepsilon n^2. \tag{2}$$

I.e., one can change (add and delete) at most εn^2 edges of G and obtain a complete p-partite graph. In other words, if an n-vertex $\mathcal{L}$-free graph G is almost extremal, $\min\{\chi(L) : L \in \mathcal{L}\} = p+1$, then the structure of G is close to a p-partite Turán graph. This result is usually called Simonovits' stability of the extremum.

Our main tool is a very simple proof for the case $\mathcal{L} = \{K_{p+1}\}$.

Stability results are usually more important than their extremal counterparts. That is why there are so many investigations concerning the *edit distance* of graphs. Let $G_1 = (V, \mathcal{E}_1)$ and $G_2 = (V, \mathcal{E}_2)$ be two (finite, undirected) graphs on the same vertex set. The *edit distance* from G_1 to G_2 is $\mathrm{ed}(G_1, G_2) := |\mathcal{E}_1 \triangle \mathcal{E}_2|$. Let $\mathcal{P}$ denote a class of graphs and G be a fixed graph. The edit distance from G to $\mathcal{P}$ is $\mathrm{ed}(G, \mathcal{P}) = \min\{\mathrm{ed}(G, F) : F \in \mathcal{P}, V(G) = V(F)\}$. This notion was explicitly introduced in [3], Alon and Stav [2] proved connections with Turán theory. For more recent results see Martin [18].

2   How to make a K_{p+1}-free graph p-chromatic

Ever since Erdős [5] observed that one can always delete at most $e/2$ edges from any graph G to make it bipartite there are many generalizations and applications of this (see, e.g., Alon [1] for a more precise form). Here we prove a version dealing with a narrower class of graphs. Recall that $e(T_{n,p}) := \max\{e(K(V_1, \ldots, V_p)) : \sum |V_i| = n\}$, the maximum size of a p-chromatic graph.

Theorem 1 *Suppose that $K_{p+1} \not\subset G$, $|V(G)| = n$, $t \geq 0$, and*

$$e(G) = e(T_{n,p}) - t.$$

Then there exists an (at most) p-chromatic subgraph H_0, $\mathcal{E}(H_0) \subset \mathcal{E}(G)$ such that

$$e(H_0) \geq e(G) - t.$$

Corollary 2 (Stability of $\mathrm{ex}(n, K_{p+1})$) *Suppose that G is K_{p+1}-free with $e(G) \geq e(T_{n,p}) - t$. Then there is a complete p-chromatic graph $K := K(V_1, \ldots, V_p)$ with $V(K) = V(G)$, such that*

$$|\mathcal{E}(G) \triangle \mathcal{E}(K)| \leq 3t.$$

Indeed, delete t edges of G to obtain the p-chromatic H_0. Since $e(H_0) \geq e(T_{n,p}) - 2t$ one can add at most $2t$ edges to make it a complete p-partite graph. (Here $V_i = \emptyset$ is allowed). □

There are other more exact stability results, e.g., Hanson and Toft [15] showed that for $t < n/(2p) - O(1)$ the graph G itself is p-chromatic, there is no need to delete any edge. Some results of E. Győri [14] implies a stronger form, namely that $e(H_0) \geq e(G) - O(t^2/n^2)$. Erdős, Győri, and Simonovits [10] considers only dense triangle-free graphs. The advantage of our Theorem 1 is that it contains no ε, δ, n_0, it is true for every n, p and t.

The inequality in Corollary 2 is simple because we estimate the edit distance of G from a not necessarily balanced p partite graph K. If we are interested in $\mathrm{ed}(G, T_{n,p})$ then we can use the following inequality. If $e(K((V_1, \ldots, V_p)) \geq e(T_{n,p}) - 2t$, then a simple calculation shows that the sizes of V_i's should be 'close' to n/p (more exactly we get $4t \geq \sum_i (|V_i| - (n/p))^2$) and hence

$$\mathrm{ed}(K, T_{n,p}) \leq n\sqrt{t/p} \tag{3}$$

Proof of Theorem 1. We find the large p-partite subgraph $H_0 \subset G$ by analyzing Erdős' degree majorization algorithm [6] what he used to prove Turán's theorem. Our input is the K_{p+1}-free graph G and the output is a partition $V_1, V_2, \ldots, V_p$ of $V(G)$ such that $\sum_i e(G|V_i) \leq t$.

Let $x_1 \in V(G)$ be a vertex of maximum degree and let $V_1 := V \setminus N(x_1)$, $V_1^+ := V \setminus V_1$. Note that $x_1 \in V_1$ and $\deg(x) \leq |V_1^+|$ for all $x \in V_1$. Hence

$$2e(G|V_1) + e(V_1, V_1^+) = \sum_{x \in V_1} \deg(x) \leq |V_1||V_1^+|.$$

In general, define $V_0^+ := V(G)$ and let x_i be a vertex of maximum degree of the graph $G|V_{i-1}^+$, let $V_i := V_{i-1}^+ \setminus N(x_i)$, $V_i^+ := V(G) \setminus (V_1 \cup \cdots \cup V_i)$. We have $x_i \in V_i$, $\deg(x_i, V_{i-1}^+) = |V_i^+|$ and

$$2e(G|V_i) + e(V_i, V_i^+) = \sum_{x \in V_i} \deg(x|V_{i-1}^+) \leq |V_i||V_i^+|. \tag{4}$$

The procedure stops in s steps when no more vertices left, i.e., if $V_1 \cup \cdots \cup V_s = V(G)$. Note that $s \leq p$ because $\{x_1, x_2, \ldots, x_s\}$ span a complete graph.

Add up the left hand sides of (4) for $1 \leq i \leq s$, we get $e(G) + (\sum_i e(G|V_i))$. The sum of the right hand sides is exactly $e(K(V_1, V_2, \ldots, V_s))$. We obtain

$$e(T_{n,p}) - t + \left(\sum_i e(G|V_i)\right) = e(G) + \left(\sum_i e(G|V_i)\right) \leq e(K(V_1, V_2, \ldots, V_p)) \leq e(T_{n,p})$$

implying $\sum_i e(G|V_i) \leq t$. □

3 Az application of the Removal Lemma

We only need a simple consequence of Szemerédi's Regularity Lemma. Recall that the graph H contains a homomorphic image of F if there is a mapping $\varphi : V(F) \to V(H)$ such that the image of each F-edge is an H-edge. There is a $\varphi : V(F) \to V(K_s)$ homomorphism if and only if $s \geq \chi(H)$. If there is no any $\varphi : V(F) \to V(H)$ homomorphism then H is called hom(F)-free.

Lemma 3 (A simple form of the Removal Lemma) *For every $\alpha > 0$ and graph F there is an n_1 such that if $n > n_1$ and G is an n-vertex, F-free graph then it contains a hom(F)-free subgraph H with $e(H) > e(G) - \alpha n^2$.*

This means that H does not contain any homomorphic image of F as a subgraph, especially if $\chi(F) = p + 1$ then H is K_{p+1}-free. The Removal Lemma can be attributed to Ruzsa and Szemerédi [20]. It appears in a more explicit form in [9] and [13]. For a survey of applications of Szemerédi's regularity lemma in graph theory see Komlós-Simonovits [16] or Komlós-Shokoufandeh-Simonovits-Szemerédi [17].

Proof of (2) using Lemma 3 and Corollary 2. Suppose that $F \in \mathcal{L}$, $\chi(F) = p + 1$ and $\alpha > 0$ an arbitrary real. Suppose that G is F-free with $n > n_1(F, \alpha)$ and $e(G) > e(T_{n,p}) - \alpha n^2$. We have to show that the edit distance of G to $T_{n,p}$ is small. First we claim that the edit distance of G to a complete p–partite graph $K(V_1, \ldots, V_p)$ is at most $7\alpha n^2$. Indeed, using the Removal Lemma we obtain a K_{p+1}-free subgraph H of G such that $e(H) > e(G) - \alpha n^2 > e(T_{n,p}) - 2\alpha n^2$. Apply Theorem 1 to H we get a p–partite H_0 with $e(H_0) > e(T_{n,p}) - 4\alpha n^2$. Then Corollary 2 yields a $K := K(V_1, \ldots, V_p)$ with $\mathrm{ed}(K, H) < 6\alpha n^2$, giving $\mathrm{ed}(K, G) \leq 7\alpha n^2$.

Since $e(K) \geq e(H_0) > e(T_{n,p}) - 4\alpha n^2$, we can use (3) with $t = 2\alpha n^2$ to get $\mathrm{ed}(K, T_{n,p}) \leq n^2\sqrt{2\alpha/p}$. This completes the proof that $\mathrm{ed}(G, T_{n,p}) \leq (7\alpha + \sqrt{2\alpha/p})n^2$. □

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