CHARACTER DEGREE SUMS OF FINITE GROUPS

ATTILA MARÓTI AND HUNG NGOC NGUYEN

Abstract. We present some results on character degree sums in connection with several important characteristics of finite groups such as p-solvability, solvability, supersolvability, and nilpotency. Some of them strengthen known results in the literature.

1. Introduction

For a finite group G, let $\mathrm{Irr}(G)$ denote the set of irreducible complex characters of G and $T(G)$ the sum of the degrees of these characters. That is,

$$T(G) := \sum_{\chi\in \mathrm{Irr}(G)} \chi(1).$$

Character degree sums of finite groups have been studied extensively by many authors. For example, Mann [18] has shown that $T(G)/|G|$ is bounded from below if and only if there exist normal subgroups $N \leq M$ of G so that $|N|$ and $|G/M|$ are bounded and M/N is abelian. Chapter 11 of the monograph *'Characters of Finite Groups'* [4] is devoted entirely to the study of character degree sums and consists of several up-to-date results. One of the highlighted theorems there is due to Berkovich and Nekrasov – it classifies all finite groups G with $T(G)/|G| > 1/p$ where p is the smallest prime divisor of $|G|$ such that a Sylow p-subgroup of G is not central. Note that all such groups are solvable even when $p = 2$.

Character degree sums provide a lot of information on the structure of finite groups. For instance, it has been proved recently by Isaacs, Loukaki, and Moretó [13] and Tong-Viet [22] that a finite group G must be solvable if $T(G) \leq 3k(G)$ or $T(G) > (4/15)|G|$ where $k(G)$ denotes the number of conjugacy classes of G. In [13], it was also proved that if $T(G) < (3/2)k(G)$ or $T(G) < (4/3)k(G)$, then G is respectively supersolvable or nilpotent. In [2], Barry, MacHale, and Ní Shé, by using the classification of Berkovich and Nekrasov mentioned above, proved that if

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$T(G) > (1/2)|G|$, then G is supersolvable (more precisely is nilpotent or has an abelian normal subgroup of index 2). In general, when the character degree sum of a finite group G is small in comparison with the class number of G or is large in comparison with the order of G, it is expected that G is close to abelian.

In this paper, we obtain more results in this direction. For convenience, we will call the quantity $T(G)/|G|$ the *character degree sum ratio* of G. Our first result provides a sufficient criterion for p-solvability of finite groups in terms of the character degree sum ratio.

Theorem 1. *Let G be a finite group and p a prime. If $k(G) \geq (3/p^2)|G|$ then G is p-solvable. Consequently, if $T(G) \geq (\sqrt{3}/p)|G|$, then G is p-solvable.*

Next, we extend the main result of [22] on the relation between character degree sums and solvability in finite groups. The following may be compared with [8, Theorem 11].

Theorem 2. *Let G be a finite group. If $T(G) > (1/4)|G|$ then G is solvable of Fitting height at most 4 or $G = \mathsf{A}_5 \times Z$ for some abelian group Z.*

Theorem 2 shows that groups isoclinic to A_5 have character degree sum ratio substantially larger than other non-solvable groups. Two groups are said to be *isoclinic* if there are isomorphisms between their inner automorphism groups and their derived subgroups such that the isomorphisms are compatible with the commutator map. Any group isoclinic to a simple group is isomorphic to a direct product of the simple group with an abelian group but this is not true for arbitrary groups. Since isoclinic groups have same proportions of degrees of irreducible complex representations, we observe that the character degree sum ratio is invariant under isoclinism, see Theorem 13. It is known that if $T(G) > (2/3)|G|$ then G is nilpotent (see [4, Chapter 11]), and that the bound here cannot be improved as $T(\mathsf{S}_3) = (2/3)|\mathsf{S}_3|$. Similar to Theorem 2, the following result shows that the character degree sum ratio of S_3 is substantially larger than that of other non-nilpotent groups not isoclinic to S_3.

Theorem 3. *Let G be a finite group. If $T(G) > (\sqrt{3/8})|G|$, then G is either abelian, isoclinic to a 2-group, to a 3-group, to S_3, or to D_{10}.*

As mentioned already, it was proved in [2] that if $T(G) > (1/2)|G|$ then G is supersolvable by using the long and complicated classification of finite groups G with $T(G)/|G| > 1/p$ where p is the smallest prime divisor of $|G|$ such that a Sylow p-subgroup of G is not central. We present here a short proof of this fact that is independent and indeed fundamentally different from the classification of Berkovich and Nekrasov.

Theorem 4. *Let G be a finite group. If $T(G) > (1/2)|G|$, then G is supersolvable.*

We note that the bound in this theorem cannot be improved since $T(\mathsf{A}_4) = (1/2)|G|$. It would be interesting if one can show that groups isoclinic to A_4 have significantly larger character degree sum ratio than other non-supersolvable groups.

To end this introduction, we remark that our proofs of Theorems 1 and 2, which are carried out respectively in Sections 3 and 4, rely on the classification of finite simple groups. On the other hand, proofs of Theorems 3 and 4 in Sections 5 and 6 are classification-free.

2. Preliminaries

There are two well-known bounds for $T(G)$, the sum of the complex irreducible character degrees of a finite group G. On one hand, by a formula of Frobenius and Schur (see [12, Corollary 4.6]), $T(G)$ can be bounded from below by the number $I(G)$ of elements of G of orders dividing 2. On the other hand, using the Cauchy-Schwarz inequality, we have $T(G) \leq \sqrt{k(G)|G|}$ where we recall that $k(G)$ is the number of complex irreducible characters of G. By introducing the notations $t(G) = T(G)/|G|$, $i(G) = I(G)/|G|$, and $d(G) = k(G)/|G|$, these inequalities can be stated as follows.

Lemma 5. *For a finite group G we have $i(G) \leq t(G) \leq \sqrt{d(G)}$.*

The invariant $d(G)$ is indeed the probability that a randomly chosen pair of elements of G commute. That is,

$$d(G) = \frac{1}{|G|^2}|\{(x, y) \in G \times G \mid xy = yx\}|.$$

This quantity is often referred to as the *commuting probability* (or *commutativity degree*) of G. The study of the commuting probability of finite groups dates back at least to work of Gustafson in the seventies. One of the earliest results is the following.

Lemma 6 (Gustafson [9])**.** *If G is a non-abelian finite group then $d(G) \leq 5/8$.*

In 1962, Nagao [19] showed that for a normal subgroup N of a finite group G we have $k(G) \leq k(N)k(G/N)$. This implies the following useful result.

Lemma 7. *For a normal subgroup N of a finite group G we have $d(G) \leq d(N)d(G/N)$.*

One of the deepest results on the commuting probability of a finite group is due to Guralnick and Robinson.

Lemma 8 (Guralnick and Robinson [8])**.** *Let $\mathbf{F}(G)$ be the Fitting subgroup of a finite group G. Then $d(G) \leq |G : \mathbf{F}(G)|^{-1/2}$.*

Two remarks are in order. First, the proof of Lemma 8 in [8] depends on the classification. As our proof of Theorem 2 uses Lemma 8 in both solvable and non-solvable cases, it depends on the classification as well. Second, Neumann [20] has shown that if $d(G)$ is bounded from below by some real positive number r then G

contains a normal subgroup H so that $|G : H|$ and H' are bounded by some function of r. Lemma 8 implies that this subgroup H can be taken to be nilpotent.

Finally, the following important result will also be used.

Lemma 9 (Gallagher [6]). *Let G be a finite group, N be a normal subgroup in G, χ be an irreducible character of N, and $I(\chi)$ be its inertia subgroup. Then the number of irreducible characters of G which lie over (G-conjugates of) χ is at most $k(I(\chi)/N)$.*

3. p-SOLVABILITY

This section is devoted to proving Theorem 1. We need two lemmas.

Lemma 10. *If G is a non-abelian finite simple group whose order is divisible by a prime p then $p^2/3 < |G|^{1/2}$ unless possibly if G is $\mathrm{PSL}_2(q)$, $\mathrm{PSL}_3(q)$, $\mathrm{PSU}_3(q)$, ${}^2\mathrm{B}_2(q)$, ${}^2\mathrm{G}_2(q)$, or ${}^3\mathrm{D}_4(q)$ for some prime power q.*

Proof. This is elementary computation using the list of orders of finite simple groups found in [15, pages 170-171], say. □

Lemma 11. *Let p be a prime divisor of the order of a non-abelian finite simple group G. Then we have the following:*

(1) *Let $G = \mathrm{PSL}_2(q)$. If $q \geq 4$ is even then $d(G) = 1/((q-1)q) < 3/p^2$. If q is odd then $d(G) = (q+5)/((q^2-1)q) < 3/p^2$.*
(2) *Let $G = \mathrm{PSL}_3(q)$. If $q \geq 4$ then*
$$d(G) \leq (q^2+3q)/((1/3)q^3(q^2-1)(q^3-1)) < 3/p^2.$$
Furthermore $d(\mathrm{PSL}_3(3)) = 1/468 < 3/169$.
(3) *Let $G = \mathrm{PSU}_3(q)$. Then $d(G) \leq (q^2+q+2)/((1/d)q^3(q^2-1)(q^3+1)) < 3/p^2$ where d is 3 if 3 divides $q+1$ and is 1 otherwise.*
(4) *Let $G = {}^2\mathrm{B}_2(q)$. Then $d(G) = (q+3)/(q^2(q^2+1)(q-1)) < 3/p^2$.*
(5) *Let $G = {}^2\mathrm{G}_2(q)$. Then $d(G) = (q+8)/(q^3(q^3+1)(q-1)) < 3/p^2$.*
(6) *Let $G = {}^3\mathrm{D}_4(q)$. Then*
$$d(G) = (q^4+q^3+q^2+q+6)/(q^{12}(q^8+q^4+1)(q^6-1)(q^2-1)) < 3/p^2.$$

Proof. Note that p can be chosen to be the largest prime divisor of $|G|$. (By Burnside's theorem p is at least 5.) From the formula for $|G|$, which can be found in [15, pages 170-171], we can derive a 'good' upper bound for p in terms of q. For such it is worthy to note that if 2^k+1 is a Fermat prime then k is a power of 2 and similarly if 3^k+1 is a prime then k is a power of 2. Upper bounds or the exact values for $k(G)$ can be found in [5, pages 3049-3050] and in [17].

We will prove (4) as a demonstration. Let $G = {}^2\mathrm{B}_2(q)$ with $q = 2^{2n+1}$ for some integer $n \geq 1$. Note that $|G| = q^2(q^2+1)(q-1)$. Now $q^2+1 = 2^{4n+2}+1$ cannot be a prime since $4n+2$ is not a power of 2. We deduce that $p \leq (q^2+1)/3$ if p is a prime divisor of $q^2(q^2+1)(q-1)$. Using the fact that $k(G) = q+3$, it is now straightforward to check that $d(G) < 3/p^2$. □

We are now in the position to prove Theorem 1.

Proof of Theorem 1. Let G be a finite group with $k(G) \geq (3/p^2)|G|$. Then we have $d(G) \geq 3/p^2$. By Lemma 7 we see that $d(G) \leq d(C)$ for any composition factor C of G. Therefore, to prove the first part of the theorem, it is sufficient to show $d(S) < 3/p^2$ for every non-abelian simple group S whose order is divisible by p. By Lemma 8, we have $d(S) = k(S)/|S| \leq |G|^{-1/2}$. This and Lemma 10 imply that $d(S) < 3/p^2$ unless S is isomorphic to one of the groups treated in Lemma 11. In all these exceptional cases we have $d(S) < 3/p^2$ by Lemma 11.

The second part of the theorem follows from the first part and Lemma 5. □

4. Solvability

In this section we will prove Theorem 2.

Proof of Theorem 2. Let G be a finite group with $t(G) > 1/4$. Then, by Lemma 5, we have that $d(G) > 1/16$. By using Lemma 8, we obtain that the index of the Fitting subgroup $\mathbf{F}(G)$ in G is less than 256. Hence if G is solvable then it has Fitting height at most 4 by Gap [7]. So we may assume that G is non-solvable.

Let S be the largest solvable normal subgroup of G. Clearly, S has index less than 256 in G by Lemma 8. Hence G/S is isomorphic to A_5, S_5, or $\mathrm{PSL}(2,7)$.

Suppose first that S is non-abelian. Then we have $d(S) \leq 5/8$ by Lemma 6. We also have $d(G/S) \leq 1/12$. So, by Lemma 7, we have

$$1/16 < d(G) \leq d(S)d(G/S) \leq (5/8)(1/12),$$

which is a contradiction. We conclude that S is abelian.

We may assume that G/S is isomorphic to A_5. For otherwise

$$1/16 < d(G) \leq d(G/S) \leq 7/120$$

which is impossible.

The factor group $G/S \cong \mathsf{A}_5$ acts naturally on $\mathrm{Irr}(S)$. Each orbit has size 1 or at least 5. Let r be the number of orbits of length 1. Since every subgroup of A_5 has at most 5 conjugacy classes, we have $k(G) \leq 5r + (|S| - r)$ by Clifford's theorem and Lemma 9. Thus we have

$$1/16 < d(G) \leq \frac{|S| + 4r}{60|S|}$$

which forces $r > (11/16)|S|$. Since more than half of the character group Irr(S) is fixed by G/S, we must have that G/S acts trivially on $\mathrm{Irr}(S)$. But then, by Brauer's permutation lemma, G/S must act trivially on S as well, which means that $Z = \mathbf{Z}(G) = S$.

Let H be the last term in the derived series of G. There exists a solvable normal subgroup T in H with $H/T \cong \mathsf{A}_5$. Since TZ/Z is a solvable normal subgroup in $G/Z \cong \mathsf{A}_5$ we must have $T \leq Z$. So H is perfect and a central extension of A_5. This means that H is either A_5 or $\mathrm{SL}(2,5)$. In the former case we have $G = \mathsf{A}_5 \times Z$, so assume that $H \cong \mathrm{SL}(2,5)$. We conclude that G is a central product of the normal subgroups H and Z.

Let the intersection of H and Z be $D = \langle a \rangle$. This is a central subgroup of order 2. Put $\overline{G} = H \times Z$. By [14, Lemma 5.2], the sum $s(\overline{G})$ of the degrees of those complex irreducible characters of $\overline{G}$ which have $\langle (a,a) \rangle$ in their kernel is at least $T(G)$. By the character table of $\mathrm{SL}(2,5)$ it is easy to see that

$$T(G) \leq s(\overline{G}) = T(\mathrm{SL}(2,5)) \cdot |Z|/2 = 15|Z|.$$

We conclude that

$$\frac{1}{4} < \frac{T(G)}{|G|} \leq \frac{15|Z|}{60|Z|} = \frac{1}{4},$$

which is a contradiction. This completes the proof. □

5. Nilpotency

The aim of this section is to prove Theorem 3. To do that, we need to recall some basic facts on *isoclinism*.

Two groups G and H are said to be *isoclinic* if there are isomorphisms $\varphi : G/\mathbf{Z}(G) \to H/\mathbf{Z}(H)$ and $\phi : G' \to H'$ such that

$$\begin{aligned} \text{if } &\varphi(g_1\mathbf{Z}(G)) = h_1\mathbf{Z}(H) \\ \text{and } &\varphi(g_2\mathbf{Z}(G)) = h_2\mathbf{Z}(H), \\ \text{then } &\phi([g_1, g_2]) = [h_1, h_2]. \end{aligned}$$

This concept was introduced by Hall in [10] as a structurally motivated classification for finite groups, especially for p-groups. It is well-known that several characteristics of finite groups such as nilpotency, supersolvability, or solvability are invariant under isoclinism, see [3]. We will see that the quantity $T(G)/|G|$ is also invariant under isoclinism.

Isoclinic groups have the same proportions of degrees of irreducible complex representations. We are aware that this result is known but we could not track down a formal reference. We refer the reader to the groupwiki webpage [23] for a proof.

Lemma 12. *Let G and H be isoclinic finite groups and let d be a positive integer. Suppose that G and H have respectively m and n irreducible characters of degree d. Then m is nonzero if and only if n is nonzero. In that case, we have*

$$\frac{m}{n} = \frac{|G|}{|H|}.$$

The next two results are crucial in the proof of Theorems 3.

Theorem 13. *Let G and H be isoclinic finite groups. Then $T(G)/|G| = T(H)/|H|$.*

Proof. By Lemma 12, we know that G and H have the same irreducible character degrees. So we assume that $d_1, d_2, ..., d_k$ are all character degrees of G and H. Let m_i and n_i denote the multiplicity of the degree d_i of G and H, respectively. We have, by Lemma 12,

$$T(G) = \sum_{i=1}^{k} m_i d_i = \sum_{i=1}^{k} n_i \frac{|G|}{|H|} d_i = \frac{|G|}{|H|} T(H),$$

and the theorem follows. □

From this proof it also follows that $d(G) = d(H)$ whenever G and H are isoclinic finite groups, a fact proved earlier by Lescot [16].

A *stem group* is defined to be a group whose center is contained inside its derived subgroup. It is known that every group is isoclinic to a stem group and if we restrict to finite groups, a stem group has the minimum order among all groups isoclinic to it, see [10] for more details.

Lemma 14. *For every finite group G, there is a finite group H isoclinic to G such that $|H| \leq |G|$ and $\mathbf{Z}(H) \subseteq H'$.*

Proof of Theorem 3. Let G be a finite group with $T(G) > \sqrt{3/8}|G|$. Then we have $d(G) > 3/8$ by Lemma 5. Using the table in [21, Page 246], we deduce that one of the following cases holds.

(1) Both $G/\mathbf{Z}(G)$ and G' are 2-groups;
(2) Both $G/\mathbf{Z}(G)$ and G' are 3-groups;
(3) $G/\mathbf{Z}(G) \cong \mathsf{S}_3$ and $|G'| = 3$;
(4) $G/\mathbf{Z}(G) \cong D_{10}$ and $|G'| = 5$.

By Lemma 14, we may assume that $\mathbf{Z}(G) \leq G'$. Thus G is a 2-group in case (1) and is a 3-group in case (2). In cases (3) and (4) the center of G cannot coincide with G' so $\mathbf{Z}(G) = 1$. This means that G must be isomorphic to S_3 and to D_{10} in the respective cases. The proof is now complete. □

6. Supersolvability

In this section we will prove Theorem 4. We first recall a well-known lemma, which can be found in [11, VI.8.6].

Lemma 15. *Let N be a normal subgroup of G that is contained in the Frattini subgroup of G. If G/N is supersolvable, then G is supersolvable.*

The next two lemmas are crucial in the proof of Theorem 4.

Lemma 16. *Let N be a normal subgroup of G that is contained in G'. If $T(G) > (1/2)|G|$, then $T(G/N) > (1/2)|G/N|$.*

Proof. We have

$$\begin{aligned} T(G) > \frac{1}{2}|G| &\Leftrightarrow \sum_{\chi\in \mathrm{Irr}(G)} \chi(1) > \frac{1}{2}\sum_{\chi\in \mathrm{Irr}(G)} \chi(1)^2 \\ &\Leftrightarrow \sum_{\chi\in \mathrm{Irr}(G)} (\chi(1)^2 - 2\chi(1)) < 0 \\ &\Leftrightarrow \sum_{\chi\in \mathrm{Irr}(G),\chi(1)\geq 3} (\chi(1)^2 - 2\chi(1)) < \sum_{\chi\in \mathrm{Irr}(G),\chi(1)=1} (2\chi(1) - \chi(1)^2) \\ &\Leftrightarrow \sum_{\chi\in \mathrm{Irr}(G),\chi(1)\geq 3} (\chi(1)^2 - 2\chi(1)) < |\{\chi \in \mathrm{Irr}(G), \chi(1) = 1\}| \\ &\Leftrightarrow \sum_{\chi\in \mathrm{Irr}(G),\chi(1)\geq 3} (\chi(1)^2 - 2\chi(1)) < |G : G'|. \end{aligned}$$

Since $N \subseteq G'$, we see that $[(G/N) : (G/N)'] = [(G/N) : (G'/N)] = [G : G']$. It follows, as every irreducible character of G/N can be considered as an irreducible character of G, that

$$\sum_{\chi\in \mathrm{Irr}(G/N),\chi(1)\geq 3} (\chi(1)^2 - 2\chi(1)) < |(G/N) : (G/N)'|,$$

which implies that $T(G/N) > (1/2)|G/N|$, as desired. □

The following lemma is a consequence of a result of Aschbacher and Guralnick [1].

Lemma 17. *Let p be a prime and let V be a finite dimensional vector space over the field of p elements. Let G be a group acting faithfully and irreducibly on V. Then $|G : G'| < |V|$.*

Proof. If N is a non-trivial normal subgroup of G then by Clifford's theorem V is a completely reducible N-module. Therefore, if N is furthermore a p-subgroup then the only irreducible N-submodule of V is the trivial module. So V must be trivial as an N-module. We conclude that $O_p(G)$, the maximal normal p-subgroup of G, is trivial. Now the lemma follows by [1, Theorem 3]. □

Remark. The proof of Aschbacher and Guralnick [1, Theorem 3] requires the finite simple group classification. However, when the group G is solvable, their proof actually does not depend on the classification. Our proof of Theorem 4 below uses Lemma 17 only in the case where G is solvable and therefore it neither does not depend on the classification. We thank the referees for bringing this discussion to our attention.

We are now ready to prove Theorem 4.

Proof of Theorem 4. Assume, to the contrary, that Theorem 4 is not true and let G be a minimal counterexample. In particular, we have $T(G) > (1/2)|G|$ and therefore G is solvable by [22, Theorem A]. If $G' = 1$ then G is abelian and we are done. So we can assume that G' is nontrivial. Let N be a minimal normal subgroup of G with $N \subseteq G'$, then N must be elementary abelian by the solvability of G. Also, using Lemma 16, we deduce that $T(G/N) > (1/2)|G/N|$, which implies that G/N is supersolvable by the minimality of G.

Again as G is not supersolvable, Lemma 15 implies that N is not contained in the Frattini subgroup of G. Hence, N is not contained in a maximal subgroup M of G so that $NM = G$. Since N is abelian, we see that $N \cap M \lhd G$. Now the minimality of N and the fact that N is not contained in M imply that $N \cap M = 1$. Equivalently,

$$G \cong N \rtimes M.$$

If $N \subseteq \mathbf{Z}(G)$ then N would be cyclic and hence G is supersolvable. Therefore, we assume that $N \nsubseteq \mathbf{Z}(G)$ or equivalently $[N, M] > 1$. It is clear that $[N, M]$ is normal in G. The minimality of N then implies that $[N, M] = N$. Therefore, no non-principal character of N is fixed under the conjugation action of M.

If there is a linear character α of N that is in an M-orbit of size 2, then the stabilizer $\mathrm{Stab}_M(\alpha)$ of α in M is a normal subgroup of M of index 2. The conjugation action of $M/\mathrm{Stab}_M(\alpha)$ on $\mathrm{Irr}(N)$ is irreducible and has no nontrivial fixed point. Therefore $|N|$ is a prime so that N is cyclic. This would imply that G is supersolvable since G/N is supersolvable, a contradiction.

From now on we can assume that every nontrivial orbit of the action of M on $\mathrm{Irr}(N)$ has size at least 3. It follows that every nontrivial orbit of the action of G on $\mathrm{Irr}(N)$ has size at least 3. Now consider the group $C = \mathbf{C}_M(N)$. This is normal in M and centralizes N, so it is normal in $G = MN$. Hence $K = N \times C$ is a normal subgroup of G. The subset

$$S = \mathrm{Irr}(K) \setminus \{1 \otimes \chi : \chi \in \mathrm{Irr}(C)\}$$

of $\mathrm{Irr}(K)$ is G-invariant and every G-orbit has size at least 3. By Clifford theory, each G-orbit of size d in S produces at least one irreducible character of G of degree divisible by d and different G-orbits in S produce different characters of G. Thus we have that

$$\sum_{d \geq 3} d \cdot n_d(G) \geq (|N| - 1) \cdot T(C),$$

where $n_d(G)$ denotes the number of irreducible complex characters of G of degree d. Since $d \leq d^2 - 2d$ for every $d \geq 3$, it follows that

$$\sum_{d \geq 3} (d^2 - 2d) \cdot n_d(G) \geq (|N| - 1) \cdot T(C).$$

Equivalently,

$$\sum_{\chi\in\mathrm{Irr}(G),\chi(1)\geq 3}(\chi(1)^2-2\chi(1))\geq(|N|-1)\cdot T(C).$$

Recall that N is elementary abelian. Therefore, we can consider N as a finite dimensional vector space over a field of p elements for some prime p. Also, since N is a minimal normal subgroup of G, the conjugation action of G on N is irreducible. In particular, the factor group M/C can be considered as a group of linear transformations acting faithfully and irreducibly on N. Since the group $M'C/C$ is normal in M/C and the quotient is abelian, we see by Lemma 17 that $|N|-1\geq |M|/|M'C|$.

We claim that $(|M|/|M'C|)\cdot T(C)\geq |M:M'|=|G:G'|$. This would be sufficient for our purposes since this would give

$$\sum_{\chi\in\mathrm{Irr}(G),\chi(1)\geq 3}(\chi(1)^2-2\chi(1))\geq |G:G'|,$$

and as we have already seen in the proof of Lemma 16, this inequality is equivalent to

$$T(G)\leq\frac{1}{2}|G|,$$

which violates the hypothesis.

To prove the claim it is sufficient to prove the inequality since the equality follows from $N\leq G'$. For that it is sufficient to verify

$$T(C)\geq |M'C|/|M'|=|M'C/M'|=|C:(M'\cap C)|.$$

But $M'\cap C\geq C'$ and so $T(C)\geq |C:C'|\geq |C:(M'\cap C)|$, as desired. □

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Alfréd Rényi Institute of Mathematics, Réaltanoda utca 13-15, H-1053, Budapest, Hungary
E-mail address: `maroti.attila@renyi.mta.hu`

Department of Mathematics, The University of Akron, Akron, Ohio 44325, USA
E-mail address: `hungnguyen@uakron.edu`