Optimal investment under behavioural criteria – a dual approach

Miklós Rásonyi[*]   José G. Rodríguez-Villarreal[†]

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Abstract

We consider a discrete-time, generically incomplete market model and a behavioural investor with power-like utility and distortion functions. The existence of optimal strategies in this setting has been shown in [2] under certain conditions on the parameters of these power functions.

In the present paper we prove the existence of optimal strategies under a different set of conditions on the parameters, identical to the ones in [5], which were shown to be necessary and sufficient in the Black-Scholes model. We also relax some assumptions of [2].

Although there exists no natural dual problem for optimisation under behavioural criteria (due to the lack of concavity), we will rely on techniques based on the usual duality between attainable contingent claims and equivalent martingale measures.

1 Introduction

This paper complements and improves results of [2] where the existence of an optimal strategy for an investor with behavioural criteria was proved under certain parameter restrictions (Assumption 2.3b below). Here we show the same result under different restrictions on the parameters (Assumption 2.3a) which are identical to the ones in [5] but they are neither stronger nor weaker than Assumption 2.3b. Assumption 2.3a is necessary and sufficient in certain continuous-time models (this is shown in [5] except a borderline case whose proof is yet unpublished). Furthermore, we manage to reprove the main result of [2] under somewhat weaker assumptions.

The key new ideas are imported from [5] and rely on the construction of an equivalent martingale measure for the price process whose density has nice integrability properties (see Lemma 3.1 below). It is this martingale measure that permits us to prove the tightness of an optimising sequence of strategies (Lemma 3.13 below), that's why we call our approach a dual one.

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[*]MTA Alfréd Rényi Institute of Mathematics, Budapest and University of Edinburgh, e-mail: rasonyi@renyi.mta.hu

[†]University of Edinburgh, e-mail: J.G.Rodriguez-Villarreal@sms.ed.ac.uk

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2 Model description

Fix an integer $T > 0$ acting as time horizon in the sequel and a filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t=0,\ldots,T}, \mathbb{P}\right)$. We consider a financial market evolving in discrete time consisting of d risky assets whose discounted prices are given by an $\mathbb{R}^d$-valued adapted stochastic process, $S = (S_t)_{t=0,\ldots,T}$ where $S_t = \left(S_t^1, \ldots, S_t^d\right)$.

In addition, we are assuming the financial market to be liquid and frictionless, that is, all costs and constraints associated with transactions are non-existent, investors are allowed to short-sell stocks and to borrow money, and it is always possible to buy or sell an unlimited number of shares of any asset.

We denote by Ξ_t^d the set of $d-$dimensional $\mathcal{F}_t-$measurable random variables. Let $\mathscr{W}$ be the set of $\mathbb{R}$-valued (or $\mathbb{R}^d$-valued) random variables Y such that $\mathbf{E}_{\mathbb{P}} |Y|^p < \infty$ for all $p > 0$.

Trading stategies are characterised by an initial capital z and a d-dimensional process $\{\theta_t : 1 \leqslant t \leqslant T\}$ representing the holdings in the respective assets. We assume θ to be predictable, i.e. $\theta_t \in \Xi_{t-1}^d$ for all t. The class of all such strategies is denoted by Φ.

We define $X_t^z(\theta) := z + \sum_{k=1}^t \theta_k \cdot \Delta S_k$, the value process of a portfolio with initial investment z and trading strategy θ, where $\Delta S_k := S_k - S_{k-1}$ and $\cdot$ denotes scalar product. For $x \in \mathbb{R}$ the notations x_+, x_- stand for positive and negative parts, respectively.

Assumption 2.1. *For all $t \geqslant 1$, $\Delta S_t \in \mathscr{W}$. Furthermore, for $0 \leqslant t \leqslant T - 1$, there exist $\mathcal{F}_t$-measurable $\kappa_t, \beta_t > 0$ satisfying $\frac{1}{\kappa_t}, \frac{1}{\beta_t} \in \mathscr{W}$ such that*

$$\operatorname*{ess\,inf}_{\xi \in \Xi_t^d} \mathbb{P}\left(\xi \cdot \Delta S_{t+1} \leqslant -\kappa_t |\xi| \,\middle|\, \mathcal{F}_t\right) \geqslant \beta_t \text{ a.s.} \tag{1}$$

We may and will assume $\kappa_t, \beta_t \leqslant 1$ in the sequel. As pointed out in [2], (1) is a strengthened form of the absence of arbitrage condition. We denote by $\mathcal{M}^e(S)$ the set of equivalent martingale measures for S. Recall that, under the standard no arbitrage hypothesis, $\mathcal{M}^e(S) \neq \emptyset$, see e.g. [3]. Assumption 2.1 will allow us to construct a particular $\mathbb{Q} \in \mathcal{M}^e(S)$ with favourable properties, see Lemma 3.1 below.

Now we turn to the description of an economic agent. Her attitude towards gains and loses will be described in terms of functions u_+ and u_-. In addition, she will be assumed to distort the "real world" distributions (probabilities) by means of functions w_+ and w_-. She will further have a "benchmark" or reference point B which is used when evaluating portfolio payoffs at the terminal time T.

Assumption 2.2. *We assume that $u_\pm : \mathbb{R}^+ \to \mathbb{R}^+$ and $w_\pm : [0,1] \to [0,1]$ are measurable functions such that $u_\pm(0) = 0$, $w_\pm(0) = 0$, $w_\pm(1) = 1$, and*

$$u_+(x) \leqslant k_+ (x^\alpha + 1), \tag{2}$$

$$k_-\left(x^\beta - 1\right) \leqslant u_-(x), \tag{3}$$

$$w_+(p) \leqslant g_+ p^\gamma, \tag{4}$$

$$w_-(p) \geqslant g_- p^\delta, \tag{5}$$

with $\alpha, \beta, \gamma, \delta > 0$, $k_\pm, g_\pm > 0$ fixed constants.

Assumption 2.3. *This concerns the parameters involved in Assumption 2.2. For convenience, we shall consider two separate cases.*
Assumption 2.3a. The parameters α, β, γ and δ satisfy

$$\alpha < \beta \text{ and } \frac{\alpha}{\gamma} < 1 < \frac{\beta}{\delta}. \tag{6}$$

Assumption 2.3b. The parameters α, β, γ and δ are such that

$$\delta \leq 1, \ \alpha < \beta, \ \text{and } \frac{\alpha}{\gamma} < \beta. \tag{7}$$

Assumption 2.4. *Similarly to the previous assumption, we consider two cases.*
Assumption 2.4a. The reference point $B \in \Xi_T^1$ belongs to $L^{1+r}(\mathbb{P})$ for some $r > 0$.

Assumption 2.4b. For the reference point $B \in \Xi_T^1$ there is a trading strategy $\phi \in \Phi$ and initial capital $b \in \mathbb{R}$ satisfying

$$X_T^b(\phi) = b + \sum_{t=1}^{T} \phi_t \cdot \Delta S_t \leqslant B. \tag{8}$$

Clearly, in Assumption 2.3b above, one of the two conditions $\alpha < \beta$ and $\alpha/\gamma < \beta$ subsumes the other, depending on whether $\gamma > 1$ or $\gamma \leq 1$. An economic interpretation can easily be given to Assumption 2.4b. It means that the losses occurring in the behavioural investor's benchmark are comparable to the value of some self-financing portfolio.

Given a real-valued random variable X representing the outcome of an investment, a behavioural agent measures her satisfaction distorting the expected utility of profits as well as the expected "dissatisfaction" of losses. Consider the nonlinear functionals $V_+(X)$ and $V_-(X)$ defined below. Let

$$V_+(X) := \int_0^\infty w_+ \left(\mathbb{P}\left(u_+(X_+) > y \right) \right) dy. \tag{9}$$

Notice that V_+ incorporates the utility of the investor on gains and w_+ produces a non-linear alteration of the given probability distribution. If $w_+(x) = x$ then we return to the expected utility framework since in this case $V_+(X) = \mathbf{E}u_+\left((X - B)_+\right)$. Similarly, let

$$V_-(X) := \int_0^\infty w_- \left(\mathbb{P}\left(u_-(X_-) > y \right) \right) dy, \tag{10}$$

and, finally, the objective or performance functional we aim to optimise is defined by

$$V(X) := V_+(X) - V_-(X), \tag{11}$$

provided that at least one of the summands is finite.

According to the cummulative prospective theory (CPT) developed in [4] and [7], behavioural investors assess their satisfaction from a given portfolio at terminal time T by means of the functional defined in (11) and the benchmark B. So we define the functionals V_+, V_- below by

$$V_+(z, \theta_1, \ldots, \theta_T) := V_+\left(X_T^z(\theta)\right) = \int_0^\infty w_+ \left(\mathbb{P}\left(u_+\left(\left(X_T^z(\theta) - B\right)_+ \right) > y \right) \right) dy, \tag{12}$$

$$V_- (z, \theta_1, \ldots, \theta_T) := V_- (X_T^z (\theta)) = \int_0^\infty w_- \left(\mathbb{P} \left(u_- \left((X_T^z (\theta) - B)_- \right) > y \right) \right) dy. \quad (13)$$

We say that a trading strategy $\theta \in \Phi$ is admissible for initial capital z if $V_- (X_T^z (\theta)) < \infty$. We denote the set of such trading strategies by $\mathcal{A}(z)$ and define, for $\theta \in \mathcal{A}(z)$,

$$V (z, \theta_1, \ldots, \theta_T) := V (X_T^z (\theta)) = V_+ (z, \theta_1, \ldots, \theta_T) - V_- (z, \theta_1, \ldots, \theta_T).$$

The optimal portfolio problem for a behavioural investor consists in finding $\theta^\star = (\theta_1^\star, \ldots, \theta_T^\star) \in \mathcal{A}(z)$ such that

$$\sup_{\theta \in \mathcal{A}(z)} V (z, \theta_1, \ldots, \theta_T) = V (z, \theta_1^\star, \ldots, \theta_T^\star). \quad (14)$$

3 Main results

As it is well known, most discrete-time market models are incomplete, i.e. $\mathcal{M}^e(S)$ is not a singleton, hence the problem of how to choose a suitable equivalent martingale measure $\mathbb{Q}$ arises.

Lemma 3.1. *Under Assumption 2.1 there exists $\mathbb{Q} \in \mathcal{M}^e(S)$ such that for $\rho := d\mathbb{Q}/d\mathbb{P}$ we have both $\rho, 1/\rho \in \mathscr{W}$.*

Proof. We rely on [6], which provides a utility maximisation framework where the existence of a martingale measure with desirable properties can be guaranteed. Define the continuously differentiable, concave function

$$U (x) = \begin{cases} x - \frac{1}{2} & \text{if } x \geqslant 0 \\ -\frac{1}{2} (x - 1)^2 & \text{if } x < 0. \end{cases} \quad (15)$$

The hypotheses of Proposition 7.1 in [6] hold by Assumption 2.1 and by (15), hence there is $\mathbb{Q} \in \mathcal{M}^e(S)$ such that

$$\rho = \frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{U'(X_T^0(\phi^*))}{EU'(X_T^0(\phi^*))}$$

for some $\phi^* \in \Phi$. Inspecting the proof of Proposition 7.1 in [6] one can easily check that $\phi_t^* \in \mathscr{W}$ for all t. Hence $\rho \in \mathscr{W}$ and ρ is bounded away from 0, a fortiori, $1/\rho \in \mathscr{W}$. $\square$

We fix the probability $\mathbb{Q}$ just constructed for later use. It will be key in establishing moment estimates which underlie our main results. Note also that, under Assumption 2.4a, $B \in L^{1+\epsilon}(\mathbb{Q})$ for all $0 < \epsilon < r$ by Hölder's inequality and $\rho \in \mathscr{W}$.

We first address the well-posedness of the optimal portfolio problem for a behavioural investor. We say that the optimal investment problem (14) is *well-posed* if the supremum in (14) is finite. If the supremum is infinite then the problem is called *ill-posed*.

We know from section 3 of [2] that $\alpha/\gamma \leqslant \beta/\delta$ and $\alpha < \beta$ are necessary for well-posedness. It is an open problem whether they are sufficient as well. We show below, however, that either (6) or (7) are sufficient.

Theorem 3.2. *Under Assumptions 2.1, 2.2, 2.3a and 2.4a, the optimisation problem (14) is well-posed. In other words,*

$$\sup_{\theta \in \mathcal{A}(z)} V (z, \theta_1, \ldots, \theta_T) < \infty. \quad (16)$$

We shall use the auxiliary results given below which were shown in [5] (see Lemmas 3.12, 3.13 and 3.14 there). We include their statements for the sake of completeness.

Lemma 3.3. *If a, b and s are positive numbers satisfying $\frac{b}{sa} > 1$ then there exists a constant D such that*

$$\mathbf{E}_{\mathbb{P}}\left(X^s\right) \leqslant 1 + D\left(\int_0^\infty \mathbb{P}\left(X^b > y\right)^a dy\right)^{\frac{1}{a}}. \tag{17}$$

for all non-negative random variables X. $\square$

Lemma 3.4. *Let $d\mathbb{Q}/d\mathbb{P}, d\mathbb{P}/d\mathbb{Q} \in \mathscr{W}$, $\alpha < \beta$ and $\frac{\alpha}{\gamma} < 1 < \frac{\beta}{\delta}$. Fix $m \in \mathbb{R}$. Then there is some $\eta > 0$ satisfying $\eta < \beta$, $\alpha < \eta$ and $\delta < \eta$, and there exist constants $L_1 = L_1(m)$ and $L_2 = L_2(m)$ such that*

$$\int_0^\infty \mathbb{P}\left((X_+)^\alpha > y\right)^\gamma dy \leqslant L_1 + L_2 \int_0^\infty \mathbb{P}\left((X_-)^\eta > y\right)^\delta dy, \tag{18}$$

for all random variables X with $\mathbf{E}_{\mathbb{Q}}[X] = m$. $\square$

Lemma 3.5. *Let a, b and s be strictly positive real numbers such that $s < a < b$ and $s \leqslant 1$. Then there exist $0 < \zeta < 1$ and constants R_1, R_2 such that*

$$\int_0^\infty \mathbb{P}\left(X^a > y\right)^s dy \leqslant R_1 + R_2 \left[\int_0^\infty \mathbb{P}\left(X^b > y\right)^s dy\right]^\zeta, \tag{19}$$

for all non-negative random variables X. $\square$

Remark 3.6. *Note that in the paper [5] it was assumed that $u_\pm, w_\pm$ are power functions (and not only comparable to power functions as in Assumption 2.2 above). Furthermore, $\alpha, \beta, \gamma, \delta \leq 1$ were stipulated, in line with the literature. One can check in [5] that the proof of Lemma 3.4 above goes through without this restriction.*

These Lemmas allow us to prove Theorem 3.2.

Proof of Theorem 3.2. We imitate the proof of Theorem 3.15 in [5]. By contradiction, let us suppose that the optimisation problem is ill-posed. Then for a sequence $\phi(n) \in \mathcal{A}(z)$, $n \in \mathbb{N}$ we have $V_+([X_T^z(\phi(n)) - B]_+) \to +\infty$ as $n \to +\infty$. Note that, for any non-negative X,

$$V_+(X) \leqslant \int_0^\infty g_+ \mathbb{P}(X^\alpha > (y/k_+) - 1)^\gamma dy \leq \int_0^\infty g_+ k_+ \mathbb{P}(X^\alpha > t)^\gamma dt + g_+ k_+.$$

Thus it follows from Lemma 3.4 (with the choice $m := z - \mathbf{E}_{\mathbb{Q}}[B]$) that

$$\lim_{n\to+\infty} \int_0^{+\infty} \mathbb{P}\left([X_T^z(\phi(n)) - B]_-^\eta > y\right)^\delta dy = +\infty$$

for some η satisfying $\eta < \beta$, $\alpha < \eta$ and $\delta < \eta$. Notice that

$$V_-(X) \geqslant \int_0^\infty g_- \mathbb{P}(k_- X^\beta - k_- > y)^\delta dy \geq \int_1^\infty g_- k_- \mathbb{P}(X^\beta > t)^\delta dt. \tag{20}$$

Consequently, we can apply Lemma 3.5 to conclude that also

$$\lim_{n\to+\infty} V_-([X_T^z(\phi(n)) - B]_-) = +\infty.$$

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Therefore, using Lemmas 3.4 and 3.5 again (and recalling that $0 < \zeta < 1$),

$$
\begin{aligned}
V(X_T^z(\phi(n)) - B) &\leq g_+ k_+ (L_1 + 1) + g_+ k_+ L_2 \int_0^{+\infty} \mathbb{P}(([X_T^z(\phi(n)) - B]_-)^\eta > y)^\delta \, dy \\
- \quad V_-([X_T^z(\phi(n)) - B]_-) &\leq g_+ k_+ (L_1 + 1 + L_2 R_1) \\
+ \quad g_+ k_+ L_2 R_2 &\left[\frac{V_-([X_T^z(\phi(n)) - B]_-)}{g_- k_-} + 1 \right]^\zeta - V_-([X_T^z(\phi(n)) - B]_-) \xrightarrow[n \to +\infty]{} -\infty,
\end{aligned}
$$

which is absurd. Hence, as claimed, the problem is well-posed. $\qquad\square$

We present a result about well-posedness under the alternative conditions Assumptions 2.3b and 2.4b as well. It is worth pointing out that while the conclusions of Theorems 3.2 and 3.7 are identical, the methods for proving them are significantly different.

Theorem 3.7. *Under Assumptions 2.1, 2.2, 2.3b and 2.4b the problem is well-posed, i.e.*

$$
\sup_{\theta \in \mathcal{A}(z)} V(z, \theta_1, \ldots, \theta_T) < \infty. \tag{21}
$$

Proof. Notice that $\delta \leq 1$ and (5) imply the fourth inequality in Assumption 4.1 of [2]. Hence our result follows from Theorem 4.4 in [2]. Note that in [2] $\alpha, \beta, \gamma \leq 1$ were also assumed. As already indicated in Remark 4.2 of [2], the proofs go through without this restriction. $\qquad\square$

From now on, the existence of optimal strategies will be our main concern. We will need to assume that the filtration is rich enough in the sense of Assumption 3.8 below. This Assumption means that investors randomize their strategies or, from a mathematical point of view, that we enlarge the underlying probability space. We will comment on this in section 4 as well.

Assumption 3.8. *Define $\mathcal{G}_0 = \{\emptyset, \Omega\}$, and $\mathcal{G}_t = \sigma(Z_1, \ldots, Z_t)$ for $1 \leq t \leq T$, where the Z_i, $i = 1, \ldots, T$ are $\mathbb{R}^N$-valued independent random variables. S_0 is constant, ΔS_t is $\mathcal{G}_t$-adapted and B is $\mathcal{G}_T$-measurable.*

Furthermore, $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{F}_0$, $t \geq 0$, where $\mathcal{F}_0 = \sigma(\varepsilon)$ with ε uniformly distributed on $[0, 1]$ and independent of $(Z_1, \ldots, Z_T)$.

Remark 3.9. *The above Assumption clearly implies that $\Delta S_t = f^{(t)}(Z_1, \ldots, Z_t)$ for some Borel functions $f^{(t)}$, for all t, and $B = g_B(Z_1, \ldots, Z_T)$ for some Borel function g_B. We may and will suppose without loss of generality that each of the Z_i is bounded.*

In [5] the existence of optimal strategies was shown under Assumption 2.3a (and $B \in L^1(\mathbb{Q})$ for some reference probability $\mathbb{Q} \in \mathcal{M}^e(S)$) in a (narrow) class of continuous-time models. In [2] existence was shown under Assumptions 2.3b, 2.4b and 3.8 in discrete-time models assuming also the continuity of $f^{(t)}, g_B$. In the present paper we shall prove existence of an optimiser in discrete-time models under Assumption 3.8 and either Assumption 2.3a or Assumption 2.3b, and we do not need continuity of $f^{(t)}, g_B$. We first present some preparatory results.

Proposition 3.10. *Let Assumptions 2.1, 2.2, 2.3a and 2.4a hold and take $\mathbb{Q} \sim \mathbb{P}$ as constructed in Lemma 3.1. Further, suppose that a sequence of trading strategies $\{\theta^n\} \subset \mathcal{A}(z)$ satisfies*

$$
\sup_n V_-(z, \theta_1^n, \ldots, \theta_T^n) < \infty. \tag{22}
$$

Then there exists $\pi > 1$ such that

$$\sup_n \mathbf{E}_{\mathbb{Q}} \left(X_T^z \left(\theta^n \right) \right)_-^{\pi} < \infty, \tag{23}$$

and

$$\sup_n \mathbf{E}_{\mathbb{Q}} \left(X_T^z \left(\theta^n \right) \right)_+ < \infty. \tag{24}$$

It follows also that

$$\sup_n \mathbf{E}_{\mathbb{Q}} \left[\sup_{t \leqslant T} \left(X_t^z \left(\theta^n \right) \right)_-^{\pi} \right] < \infty, \tag{25}$$

$$\sup_{n,t} \mathbf{E}_{\mathbb{Q}} \left[\left| X_t^z \left(\theta^n \right) \right| \right] < \infty. \tag{26}$$

Proof. This is a direct application of Lemma 3.5. Indeed, choose $1 < s < \frac{\beta}{\delta}$ and λ such that $1 < \lambda < s < \frac{\beta}{\delta}$. Applying Hölder's inequality,

$$\mathbf{E}_{\mathbb{Q}} \left[\left(X_T^z \left(\theta^n \right) - B \right)_-^{\frac{s}{\lambda}} \right] = \mathbf{E}_{\mathbb{Q}} \left[\rho^{1/\lambda} \frac{1}{\rho^{1/\lambda}} \left(X_T^z \left(\theta^n \right) - B \right)_-^{\frac{s}{\lambda}} \right] \leqslant$$

$$\leqslant C \mathbf{E}_{\mathbb{P}} \left[\left(X_T^z \left(\theta^n \right) - B \right)_-^s \right]^{1/\lambda},$$

where $C = \mathbf{E}_{\mathbb{Q}} \left[\rho^{q/\lambda} \right]^{1/q} < \infty$ and q is the conjugate number of λ. Lemma 3.3 yields that, for all n,

$$C \mathbf{E}_{\mathbb{P}} \left[\left(X_T^z \left(\theta^n \right) - B \right)_-^s \right]^{1/\lambda} \leqslant C \left(1 + D \left(\int_0^\infty \mathbb{P} \left(\left(X_T^z \left(\theta^n \right) - B \right)_-^\beta > y \right)^\delta dy \right)^{1/\delta} \right)^{1/\lambda} \tag{27}$$

for some $D < \infty$. Hence (22) and (20) imply (23), setting $\pi := \min\{\frac{s}{\lambda}, 1 + \frac{r}{2}\}$ (note that, as we have pointed out after Lemma 3.1, $\mathbf{E}_{\mathbb{Q}} |B|^{1+(r/2)} < \infty$). Moreover, Hölder's inequality gives

$$\sup_n \mathbf{E}_{\mathbb{Q}} \left(X_T^z \left(\theta^n \right) \right)_- < \infty.$$

It follows from Theorem 2 in [3] that $\{X_t^z \left(\theta^n \right)\}_{t \leqslant T}$ is a martingale under $\mathbb{Q}$, thus

$$\mathbf{E}_{\mathbb{Q}} \left| X_t^z \left(\theta^n \right) \right| \leqslant \mathbf{E}_{\mathbb{Q}} \left| X_T^z \left(\theta^n \right) \right|,$$

for all n, t. From $\mathbf{E}_{\mathbb{Q}} \left[X_t^z \left(\theta^n \right) \right] = z$ and (23) we have

$$\sup_n \mathbf{E}_{\mathbb{Q}} \left(X_T^z \left(\theta^n \right) \right)_+ \leqslant |z| + \sup_n \mathbf{E}_{\mathbb{Q}} \left(X_T^z \left(\theta^n \right) \right)_- < \infty$$

Hence $\sup_n \mathbf{E}_{\mathbb{Q}} \left| X_T \left(\theta^n \right) \right| < \infty$ and this implies (24) as well as (26). In order to prove (25), Doob's inequality is applied, noting that $f(x) = x_-$ is convex and hence the process $\{\left(X_t^z \left(\theta^n \right) \right)_-\}_{t \leqslant T}$ is a positive submartingale. $\square$

Notice that we could show Proposition 3.10 only in a discrete time and finite horizon setting since it relies on Theorem 2 of [3] which fails in more general (e.g. continuous-time) settings.

Remark 3.11. *In [5] admissible strategies θ were required to satisfy* both $V_- (X_T^z (\theta)) < \infty$ *and the martingale property for $X_t^z (\theta)$ (under some fixed $\mathbb{Q} \in \mathcal{M}^e(S)$). The proof above shows that, in the present discrete-time setting,* $V_- (X_T^z (\theta)) < \infty$ *implies the martingale property for $X_t^z (\theta)$ under $\mathbb{Q}$. So the domain of optimisation in the present paper is the same as the one in [5].*

Remark 3.12. *Let $\theta = (\theta_1, \theta_2, \ldots, \theta_T) \in \mathcal{A}(z)$ be as in Proposition 3.10. Clearly,*

$$(\theta_T \cdot \Delta S_T)_+ \leqslant (X_T^z(\theta))_+ + \left(X_{T-1}^z(\theta)\right)_-. \tag{28}$$

Thus

$$\mathbf{E}_{\mathbb{Q}} (\theta_T \cdot \Delta S_T)_+ \leqslant \mathbf{E}_{\mathbb{Q}} (X_T^z(\theta))_+ + \mathbf{E}_{\mathbb{Q}} \left(X_{T-1}^z(\theta)\right)_-$$

implies

$$\sup_n \mathbf{E}_{\mathbb{Q}} \left[(\theta_T^n \cdot \Delta S_T)_+\right] < \infty. \tag{29}$$

We will now proceed to proving that trading strategies satisfying (22) have some uniformly bounded moments.

Lemma 3.13. *Let $\{\theta^n\} \subset \mathcal{A}(z)$ be a sequence of trading strategies. Let Assumptions 2.1, 2.2, 2.3a and 2.4a hold and assume that* (22) *holds. Then*

$$\sup_n \mathbf{E}_{\mathbb{Q}} |\theta_t^n|^{1/2} < \infty \text{ for } t = 1, 2, \ldots, T. \tag{30}$$

Proof. Brackets $\langle \cdot, \cdot \rangle$ are used to denote scalar product in this proof. A uniform bound for $\mathbf{E}_{\mathbb{Q}} \left[(\theta_T^n \cdot \Delta S_T)_+\right]$ can obtained as in Remark 3.12. Using the same idea for $t \leqslant T$,

$$\mathbf{E}_{\mathbb{Q}} \langle \theta_t^n, \Delta S_t \rangle_+ \leqslant \mathbf{E}_{\mathbb{Q}} (X_t^z(\theta^n))_+ + \mathbf{E}_{\mathbb{Q}} \left(X_{t-1}^z(\theta^n)\right)_- \leqslant$$
$$\leqslant \mathbf{E}_{\mathbb{Q}} |X_t^z(\theta^n)| + \mathbf{E}_{\mathbb{Q}} \left|X_{t-1}^z(\theta^n)\right|, \tag{31}$$

and the right-hand side is bounded uniformly in n by Proposition 3.10.

Denote $\rho_t := \mathbf{E}_{\mathbb{P}}[\rho | \mathcal{F}_t]$ for $0 \leq t \leq T$. From Assumption 2.1,

$$\mathbf{E}_{\mathbb{Q}} \langle \theta_T^n, \Delta S_T \rangle_+ \geqslant \mathbf{E}_{\mathbb{Q}} \left[|\theta_T^n| \left\langle \frac{\theta_T^n}{|\theta_T^n|}, \Delta S_T \right\rangle \mathbf{1}_{\left\{ \left\langle \frac{\theta_T^n}{|\theta_T^n|}, \Delta S_T \right\rangle \geqslant \kappa_{T-1} \right\}} \right] \geqslant \tag{32}$$

$$\geqslant \mathbf{E}_{\mathbb{Q}} \left[|\theta_T^n| \kappa_{T-1} \mathbb{Q} \left(\left\langle \frac{\theta_T^n}{|\theta_T^n|}, \Delta S_T \right\rangle \geqslant \kappa_{T-1} \middle| \mathcal{F}_{T-1} \right) \right]. \tag{33}$$

Using the property

$$\mathbf{E}_{\mathbb{Q}} [\eta | \mathcal{G}] = \mathbf{E}_{\mathbb{P}} \left[\eta \frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{G} \right] \middle/ \mathbf{E}_{\mathbb{P}} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{G} \right],$$

of conditional expectations which holds for any sigma-algebra $\mathcal{G}$ and for any positive random variable η, we get

$$\mathbf{E}_{\mathbb{Q}} \langle \theta_T^n, \Delta S_T \rangle_+ \geqslant \mathbf{E}_{\mathbb{Q}} \left[\frac{|\theta_T^n|}{\rho_{T-1}} \kappa_{T-1} \mathbf{E}_{\mathbb{P}} \left[\mathbf{1}_{\left\{ \left\langle \frac{\theta_T^n}{|\theta_T^n|}, \Delta S_T \right\rangle \geqslant \kappa_{T-1} \right\}} \rho_T \middle| \mathcal{F}_{T-1} \right] \right]. \tag{34}$$

Denote $A_T = \left\{ \left\langle \frac{\theta_T^n}{|\theta_T^n|}, \Delta S_T \right\rangle \geqslant \kappa_{T-1} \right\}$ and apply the (conditional) Cauchy inequality to the right-hand side:

$$\mathbf{E}_{\mathbb{P}} \left[\mathbf{1}_{A_T} \rho_T \middle| \mathcal{F}_{T-1} \right] \geqslant \mathbb{P}^2 \left(A_T \middle| \mathcal{F}_{T-1} \right) \middle/ \mathbf{E}_{\mathbb{P}} \left[\frac{1}{\rho_T} \middle| \mathcal{F}_{T-1} \right].$$

From Assumption 2.1 and Cauchy's inequality, the right-hand side of (34) can be minorised by

$$\mathbf{E}_{\mathbb{Q}} \left[\frac{|\theta_T^n|}{\rho_{T-1}} \kappa_{T-1} \beta_{T-1}^2 \Big/ \mathbf{E}_{\mathbb{P}} \left[\rho_T^{-1} \big| \mathcal{F}_{T-1} \right] \right] \geqslant$$

$$\mathbf{E}_{\mathbb{Q}} \left[|\theta_T^n|^{1/2} \right]^2 \Big/ \mathbf{E}_{\mathbb{Q}} \left[\rho_{T-1} \kappa_{T-1}^{-1} \beta_{T-1}^{-2} \mathbf{E}_{\mathbb{P}} \left[\rho_T^{-1} \big| \mathcal{F}_{T-1} \right] \right], \qquad (35)$$

thus

$$\mathbf{E}_{\mathbb{Q}} \langle \theta_T, \Delta S_T \rangle_+ \mathbf{E}_{\mathbb{Q}} \left[\rho_{T-1} \kappa_{T-1}^{-1} \beta_{T-1}^{-2} \mathbf{E}_{\mathbb{P}} \left[\rho_T^{-1} \big| \mathcal{F}_{T-1} \right] \right] \geqslant \mathbf{E}_{\mathbb{Q}} \left[|\theta_T|^{1/2} \right]^2.$$

The same procedure applies to θ_t, $t = 1, \ldots, T$ by (31). Thus, for all t,

$$\sup_n \mathbf{E}_{\mathbb{Q}} \left[|\theta_t^n|^{1/2} \right] \leqslant \sup_n \left[\mathbf{E}_{\mathbb{Q}} \langle \theta_t^n, \Delta S_t \rangle_+ \mathbf{E}_{\mathbb{Q}} \left[\rho_{t-1} \kappa_{t-1}^{-1} \beta_{t-1}^{-2} \mathbf{E}_{\mathbb{P}} \left[\rho_t^{-1} \big| \mathcal{F}_{t-1} \right] \right] \right]^{1/2} < \infty,$$

by Assumption 2.1 and (31). $\qquad \square$

Remark 3.14. *Applying Hölder's inequality, the estimates above can be carried out with no significant alteration for any $0 < \xi < 1$, i.e.*

$$\sup_n \mathbf{E}_{\mathbb{Q}} |\theta_t^n|^\xi < \infty \text{ for } t = 1, 2, \ldots, T \qquad (36)$$

can be shown. For simplicity we did this only for $\xi = \frac{1}{2}$.

From the last Lemma the next one follows trivially.

Lemma 3.15. *Under the Assumptions 2.1, 2.2, 2.3a and 2.4a, let $\{\theta^n\}_{n \geqslant 1} \subset \mathcal{A}(z)$ a sequence of admissible trading strategies such that $\sup_n V_-(z, \theta_1^n, \ldots \theta_T^n) < \infty$. Then $\{\theta^n\}_{n \geqslant 1}$ is a tight sequence of $\mathbb{R}^{dT}$-valued random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.* $\qquad \square$

Proposition 3.16. *Let $\{\theta^n\}_{n \geqslant 1}$ be a sequence of trading strategies whose set of laws is tight. Let μ_n be the law of $X_T^z(\theta^n) - B$ for all n. Under Assumption 3.8, there exists a law $\mu^\star$ and a trading strategy $\theta^\star$ such that $\mu^\star = Law(X_T^z(\theta^\star) - B)$ and $\mu^\star$ is an accumulation point of the sequence $\{\mu_n\}_{n \geq 1}$ in the weak (narrow) topology.*

Proof. Lemma 9.4 of [2] provides independent random variables $\varepsilon', \tilde{\varepsilon}$, uniform on $[0, 1]$, which are both functions of ε. Now following the proof of Theorem 6.8 of [2] verbatim (with $\phi_1 = \ldots = \phi_T = 0$) we obtain an $\mathcal{F}_t$-predictable process $\theta_t^\star$ such that, by Prokhorov's theorem, the law of

$$Y_k := (\varepsilon', \theta_1^{n_k}, \ldots, \theta_T^{n_k}, Z_1, \ldots, Z_T)$$

tends to that of

$$Y := (\varepsilon', \theta_1^\star, \ldots, \theta_T^\star, Z_1, \ldots, Z_T)$$

for a subsequence n_k, as $k \to \infty$. Skorokhod's theorem provides random variables

$$\bar{Y}_k = (\bar{\varepsilon}'(n_k), \bar{\theta}_1^{n_k}, \ldots, \bar{\theta}_T^{n_k}, \bar{Z}_1^{n_k}, \ldots, \bar{Z}_T^{n_k})$$

and

$$\bar{Y} = (\bar{\varepsilon}', \bar{\theta}_1, \ldots, \bar{\theta}_T, \bar{Z}_1, \ldots, \bar{Z}_T)$$

on some probability space such that $Law(Y_k) = Law(\bar{Y}_k)$ for all k, $Law(Y) = Law(\bar{Y})$ and $\bar{Y}_k$ tends to $\bar{Y}$ a.s.

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By assumption, we also have that $\Delta S_i = f^{(i)}(Z_1, \ldots, Z_i)$ and $B = g_B(Z_1, \ldots, Z_T)$. Denoting $\Delta \bar{S}_i = f^{(i)}(\bar{Z}_1, \ldots, \bar{Z}_i)$ and $\bar{B} := g_B(\bar{Z}_1, \ldots, \bar{Z}_T)$ we have that

$$Law\left((Z_i)_{i \leqslant T}, (\theta_i^\star)_{i \leqslant T}, (\Delta S_i)_{i \leqslant T}, B\right) = Law\left((\bar{Z}_i)_{i \leqslant T}, (\bar{\theta}_i)_{i \leqslant T}, (\Delta \bar{S}_i,)_{i \leqslant T}, \bar{B}\right).$$

Therefore

$$Law\left(\sum_{i=1}^T \theta_i^\star \cdot \Delta S_i - B\right) = Law\left(\sum_{i=1}^T \bar{\theta}_i \cdot \Delta \bar{S}_i - \bar{B}\right). \tag{37}$$

Denote $\bar{B}^k := g_B(\bar{Z}_1^{n_k}, \ldots, \bar{Z}_T^{n_k})$ and $\Delta \bar{S}_i^k := f^{(i)}(\bar{Z}_1^{n_k}, \ldots, \bar{Z}_i^{n_k})$. By Théorème 1 in [1], $\Delta \bar{S}_i^k \to \Delta \bar{S}_i$ for all i in probability and also $\bar{B}^k \to \bar{B}$ in probability, $k \to \infty$.

It follows that

$$\sum_{i=1}^T \bar{\theta}_i^k \cdot \Delta \bar{S}_i^k - \bar{B}^k \to \sum_{i=1}^T \bar{\theta}_i \cdot \Delta \bar{S}_i - \bar{B}, \tag{38}$$

in probability, hence in law. In other words, we have

$$\sum_{i=1}^T \theta_i^k \cdot \Delta S_i - B \to \sum_{i=1}^T \theta_i^\star \cdot \Delta S_i - B \tag{39}$$

in law. This finishes the proof. □

Our first main result on the existence of optimal strategies now follows easily from Proposition 3.16 above.

Theorem 3.17. *Let Assumptions 3.8, 2.1, 2.2, 2.3a and 2.4a be in force and let $u_\pm$, $w_\pm$ be continuous. Then the supremum in (14) is attained by an optimal strategy $\theta^\star$.*

Proof. Let us take a maximising sequence of admissible strategies $\{\theta^j\}_{j \geqslant 1}$, i.e.

$$V\left(z, \theta_1^j, \ldots, \theta_T^j\right) \to \sup_{\theta \in \mathcal{A}(z)} V\left(z, \theta_1, \ldots, \theta_T\right), \ j \to \infty.$$

The proof of Theorem 3.2 shows that we necessarily have $\sup_j V_-\left(z, \theta_1^j, \ldots, \theta_T^j\right) < \infty$, showing (22).

Due to Lemma 3.15, we can conclude that the sequence $\{\theta^j\}_{j \geqslant 1}$ is tight. Proposition 3.16 then shows that there is a strategy $\theta^\star \in \Phi$, $\theta^\star = (\theta_1^\star, \theta_2^\star, \ldots, \theta_T^\star) \in \mathbb{R}^{dT}$ such that $X_T^z(\theta^j) - B \to X_T^z(\theta^\star) - B$ in law (along a subsequence which we assume to be the original sequence). Now our aim is to prove

$$\limsup_j V\left(z, \theta_1^j, \theta_2^j, \ldots, \theta_T^j\right) \leqslant V\left(z, \theta_1^\star, \theta_2^\star, \ldots, \theta_T^\star\right). \tag{40}$$

By the continuous mapping theorem we have

$$\left(X_T^z(\theta^j) - B\right)_\pm \to (X_T^z(\theta^\star) - B)_\pm$$

and

$$u_\pm\left(\left(X_T^z(\theta^j) - B\right)_\pm\right) \to u_\pm\left((X_T^z(\theta^\star) - B)_\pm\right) \tag{41}$$

in law. Let D be the set of discontinuity points of the limiting distributions in (41). Then

$$\mathbb{P}\left(u_\pm\left(\left(X_T^z(\theta^j) - B\right)_\pm\right) \geqslant y\right) \to \mathbb{P}\left(u_\pm\left((X_T^z(\theta^\star) - B)_\pm\right) \geqslant y\right) \ \text{for all } y \in \mathbb{R}^+ \backslash D,$$

in particular, for Lebesgue-a.e. y. By Assumption 2.2,

$$w_+ \left(\mathbb{P} \left(u_+ \left(\left(X_T^z \left(\theta^j \right) - B \right)_+ \right) \geqslant y \right) \right) \leqslant g_+ \left[\mathbb{P} \left(u_+ \left(\left(X_T^z \left(\theta^j \right) - B \right)_+ \right) \geqslant y \right) \right]^\gamma. \quad (42)$$

Take $1/\gamma < \lambda < 1/\alpha$. Applying Markov's inequality and Assumption 2.2 again,

$$g_+ \left[\mathbb{P} \left(u_+^\lambda \left(\left(X_T^z \left(\theta^j \right) - B \right)_+ \right) \geqslant y^\lambda \right) \right]^\gamma \leqslant \frac{c'}{y^{\lambda\gamma}} \left\{ \mathbf{E}_{\mathbb{P}} \left(1 + \left(X_T^z \left(\theta^j \right) - B \right)_+^\alpha \right)^\lambda \right\}^\gamma, \quad (43)$$

for some $c' > 0$, hence

$$c' \left[\mathbb{P} \left(u_+^{\lambda\gamma} \left(\left(X_T^z \left(\theta^j \right) - B \right)_+ \right) \geqslant y^\lambda \right) \right]^\gamma \leqslant \frac{c''}{y^{\lambda\gamma}} \mathbf{E}_{\mathbb{P}}^\gamma \left(1 + \left(X_T^z \left(\theta^j \right) - B \right)_+^{\alpha\lambda} \right).$$

with some $c'' > 0$. Furthermore,

$$\frac{c''}{y^{\lambda\gamma}} \mathbf{E}_{\mathbb{P}}^\gamma \left(1 + \left(X_T^z \left(\theta^j \right) - B \right)_+^{\alpha\lambda} \right) \leqslant c''' \left(1 + \left[\mathbf{E}_{\mathbb{P}} \left(X_T^z \left(\theta^j \right) \right)_+^{\alpha\lambda} \right]^\gamma + \left[\mathbf{E}_{\mathbb{P}} B_-^{\alpha\lambda} \right]^\gamma \right). \quad (44)$$

The last term is finite by Assumption 2.4a. Hölder's inequality applied with $p = \frac{1}{\alpha\lambda}$ gives

$$\mathbf{E}_{\mathbb{P}} \left(X_T^z \left(\theta^j \right) \right)_+^{\alpha\lambda} = \mathbf{E}_{\mathbb{Q}} \left[\frac{1}{\rho} \cdot \left(X_T^z \left(\theta^j \right) \right)_+^{\alpha\lambda} \right] \leqslant C_1 \left[\mathbf{E}_{\mathbb{Q}} \left(X_T^z \left(\theta^j \right) \right)_+ \right]^{\alpha\lambda}$$

with $C_1 = \mathbf{E}_{\mathbb{Q}} \left[\rho^{1/(\alpha\lambda - 1)} \right]^{1 - \alpha\lambda}$. Thus

$$w_+ \left(\mathbb{P} \left(u_+ \left(\left(X_T^z \left(\theta^j \right) - B \right)_+ \right) \geqslant y \right) \right) \leqslant \frac{1}{y^{\lambda\gamma}} \left\{ D_1 + D_2 \left[\mathbf{E}_{\mathbb{Q}} \left(X_T^z \left(\theta^j \right) \right)_+ \right]^{\alpha\lambda\gamma} \right\},$$

with suitable constants D_1, D_2. The condition $\sup_j V_- \left(z, \theta_1^j, \ldots, \theta_T^j \right) < \infty$ implies that $\sup_j \mathbf{E}_{\mathbb{Q}} \left(X_T^z \left(\theta^j \right) \right)_+ < \infty$ (see Proposition 3.10). This in turn gives that the sequence of positive functions $w_+ \left(\mathbb{P} \left(u_+ \left(\left(X_T^z \left(\theta^j \right) - B \right)_+ \right) \geqslant y \right) \right)$ can be dominated by $\frac{K}{y^{\lambda\gamma}}$ for some $K > 0$.

This estimate allows to apply Lebesgue's theorem since

$$w_+ \left(\mathbb{P} \left(u_+ \left(\left(X^z \left(\theta^j \right) - B \right)_+ \right) \geqslant y \right) \right) \leqslant \mathbf{I}_{[0,1]} \left(y \right) + \frac{K}{y^{\lambda\gamma}} \cdot \mathbf{I}_{(1,\infty)} \left(y \right), \quad (45)$$

which yields

$$\limsup_j V_+ \left(z, \theta_1^j, \ldots, \theta_T^j \right) \leqslant \int_0^\infty \limsup_j w_+ \left(\mathbb{P} \left(u_+ \left(\left(X^z \left(\theta^j \right) - B \right)_+ \right) \geqslant y \right) \right) dy,$$

and the latter equals $V_+ \left(z, \theta_1^\star, \ldots, \theta_T^\star \right)$.

On the other hand, by Fatou's lemma applied to $V_- \left(z, \theta_1^j, \ldots, \theta_T^j \right)$ we have

$$V_- \left(z, \theta_1^\star, \ldots, \theta_T^\star \right) \leqslant \liminf_j V_- \left(z, \theta_1^j, \ldots, \theta_T^j \right) \leqslant \sup_j V_- \left(x_0, \theta_1^j, \ldots, \theta_T^j \right) < \infty,$$

so $\theta^\star \in \mathcal{A}(z)$ and

$$\limsup_j V^+ \left(z, \theta_1^j, \ldots, \theta_T^j \right) - \liminf_j V^- \left(z, \theta_1^j, \ldots, \theta_T^j \right) \leqslant$$
$$V^+ \left(z, \theta_1^\star, \ldots, \theta_T^\star \right) - V^- \left(z, \theta_1^\star, \ldots, \theta_T^\star \right),$$

thus

$$\limsup_j \left\{ V^+\left(z, \theta_1^j, \ldots, \theta_T^j\right) - V^-\left(z, \theta_1^j, \ldots, \theta_T^j\right) \right\} \leqslant$$
$$V^+\left(z, \theta_1^\star, \ldots, \theta_T^\star\right) - V^-\left(z, \theta_1^\star, \ldots, \theta_T^\star\right),$$

which yields (40). Hence $\theta^\star$ is optimal and the supremum is attainable. $\qquad\square$

Now we turn to the case of Assumptions 2.3b and 2.4b.

Theorem 3.18. *Let Assumptions 3.8, 2.1, 2.2, 2.3b and 2.4b be in force and let $u_\pm$, $w_\pm$ be continuous. Then the supremum is attained in (14) by some $\theta^\star \in \mathcal{A}(z)$.*

Proof. We can follow the proof of Theorem 6.8 in [2] verbatim up to the point of constructing

$$Y := (\varepsilon', \theta_1^\star, \ldots, \theta_T^\star, Z_1, \ldots, Z_T).$$

Then the argument of Proposition 3.16 shows that

$$\sum_{i=1}^T \theta_i^k \cdot \Delta S_i - B \to \sum_{i=1}^T \theta_i^\star \cdot \Delta S_i - B \qquad (46)$$

in law. From this point on the Fatou-lemma argument of Theorem 6.8 in [2] applies verbatim and optimality of $\theta^\star$ can be established. $\qquad\square$

4 A sufficient condition

Assumption 3.8 may look restrictive at first sight. Hence we provide a simple sufficient condition for its validity.

Proposition 4.1. *Let $\mathcal{G}_t := \sigma(\tilde{Z}_1, \ldots, \tilde{Z}_t)$ for $t = 1, \ldots, T$ and $\mathcal{G}_0 = \{\emptyset, \Omega\}$ where the $\tilde{Z}_i$, $i = 1, \ldots, T$ are N-dimensional random variables with a Lebesgue-a.e. positive joint density on $\mathbb{R}^{TN}$. Then there are independent $\mathbb{R}^N$-valued random variables Z_i, $i = 1, \ldots, T$ such that $\mathcal{G}_t = \sigma(Z_1, \ldots, Z_t)$ for $t = 1, \ldots, T$.*

We first recall a foklore-type result (see Lemma 9.6 of [2]).

Lemma 4.2. *Let X be a real-valued random variable with atomless law. Let $F(x) := P(X \leq x)$ denote its cumulative distribution function. Then $F(X)$ has uniform law on $[0, 1]$.* $\qquad\square$

The following results are parallel to Lemma 9.8 and Corollary 9.9 of [2]. In the sequel, when we write "measurable bijection" we mean that both the function and its inverse are measurable.

Lemma 4.3. *Let (Y, W) be an $\mathbb{R} \times \mathbb{R}^k$-valued random variable with Lebesgue almost everywhere positive density $f(x^1, \ldots, x^{k+1})$. Then there is a measurable bijection H from $\mathbb{R}^{k+1}$ into $[0, 1] \times \mathbb{R}^k$ such that $H^i(x^1, \ldots, x^{k+1}) = x^i$ for $i = 2, \ldots, k+1$ and $Z := H^1(Y, W)$ is uniform on $[0, 1]$, independent of W.*

Proof. The conditional distribution function of Y knowing $W = (x^2, \ldots, x^{k+1})$,

$$F(x^1, \ldots, x^{k+1}) := \frac{\int_{-\infty}^{x^1} f(z, x^2, \ldots, x^{k+1}) dz}{\int_{-\infty}^{\infty} f(z, x^2, \ldots, x^{k+1}) dz},$$

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is clearly measurable (in all its variables). By a.e. positivity of f, F is also strictly increasing in x^1 hence the function

$$H : (x^1, \ldots, x^{k+1}) \to (F(x^1, \ldots, x^{k+1}), x^2, \ldots, x^{k+1})$$

is a measurable bijection. By Lemma 4.2 the conditional law $P(H^1(Y, W) \in \cdot | W = (x^2, \ldots, x^{k+1}))$ is uniform on $[0, 1]$ for Lebesgue-almost all $(x^2, \ldots, x^{k+1})$, which shows that $H^1(Y, W)$ is independent of W with uniform law on $[0, 1]$. $\qquad\square$

Corollary 4.4. *Let $(\tilde{W}_1, \ldots, \tilde{W}_k)$ be an $\mathbb{R}^k$-valued random variable with a.e. positive density (w.r.t. the k-dimensional Lebesgue measure). Then there are independent random variables $W_1, \ldots, W_k$ and measurable bijections $g_l(k) : \mathbb{R}^l \to \mathbb{R}^l$, $1 \leq l \leq k$ such that $(\tilde{W}_1, \ldots, \tilde{W}_l) = g_l(k)(W_1, \ldots, W_l)$.*

Proof. The case $k = 1$ is vacuous. Assume that the statement is true for $k \geq 1$, let us prove it for $k + 1$. We may set $g_l(k + 1) := g_l(k)$, $1 \leq l \leq k$, it remains to construct $g_{k+1}(k + 1)$ and W_{k+1}.

We wish to apply Lemma 4.3 in this induction step. It provides a measurable bijection $s : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ such that $s^m(x^1, \ldots, x^{k+1}) = x^m$, $1 \leq m \leq k$ and $W_{k+1} := s^{k+1}(\tilde{W}_1, \ldots, \tilde{W}_{k+1})$ is independent of $(\tilde{W}_1, \ldots, \tilde{W}_k)$ and hence of

$$(W_1, \ldots, W_k) = g_k(k)^{-1}(\tilde{W}_1, \ldots, \tilde{W}_k).$$

Define $a : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ by

$$
\begin{aligned}
a(x^1, \ldots, x^{k+1}) &:= (g_k(k)^{-1}(x^1, \ldots, x^k), s^{k+1}(x^1, \ldots, x^{k+1})) \\
&= s(g_k(k)^{-1}(x^1, \ldots, x^k), x^{k+1}),
\end{aligned}
$$

a is clearly a measurable bijection. Notice that $a(\tilde{W}_1, \ldots, \tilde{W}_{k+1}) = (W_1, \ldots, W_{k+1})$. Set $g_{k+1}(k + 1) := a^{-1}$. This finishes the proof of the induction step and hence concludes the proof. $\qquad\square$

Proof of Proposition 4.1. Apply Corollary 4.4 with the choice $k := TN$ and

$$\tilde{W}_{(t-1)N+l} := \tilde{Z}_t^l, \quad l = 1, \ldots, N$$

and $t = 1, \ldots, T$. By the construction in Corollary 4.4, taking $Z_t^l := W_{(t-1)N+l}$, one has $(\tilde{Z}_1, \ldots, \tilde{Z}_t) = g_{tN}(TN)(Z_1, \ldots, Z_t)$ hence indeed $\mathcal{G}_t = \sigma(Z_1, \ldots, Z_t)$ for $t = 1, \ldots, T$. $\qquad\square$

5 Conclusions

In this work we have proved two types of results concerning the optimal portfolio problem for a behavioural investor in a discrete-time setting with power-like utility and distortion functions. Namely, well-posedness and the existence of optimal strategies were shown under two different conditions on the parameters. We managed to remove some restrictive assumptions of [2]. The evolution of the stock price is allowed to be very general, our results cover a myriad of models in discrete-time incomplete markets, see section 8 of [2] for examples.

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