A SIMPLE PROOF OF THE LEBESGUE DECOMPOSITION THEOREM

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The aim of this short note is to present an elementary, self-contained, and direct proof for the classical Lebesgue decomposition theorem. In fact, I will show that the absolutely continuous part just measures the squared semidistance of the characteristic functions from a suitable subspace.

This approach also gives a decomposition in the finitely additive case, but it differs from the Lebesgue-Darst decomposition [1], because the involved absolute continuity concepts are different.

Notations. Let \( A \) be a \( \sigma \)-algebra over \( X \neq \emptyset \), and consider the finite measures \( \mu, \nu : A \to \mathbb{R}_+ \) on it. The measure \( \mu \) is \( \nu \)-absolutely continuous (\( \mu \ll \nu \), in symbols) if \( \nu(A) = 0 \) implies \( \mu(A) = 0 \) for all \( A \in A \). Singularity of \( \mu \) and \( \nu \) (denoted by \( \mu \perp \nu \)) means that the only measure dominated by both \( \mu \) and \( \nu \) is the zero measure. As it is known, this is equivalent with the existence of a measurable set \( P \in A \) such that \( \mu(P) = \nu(X \setminus P) = 0 \).

Theorem. Let \( \mu \) and \( \nu \) be finite measures on \( A \). Then \( \mu \) splits uniquely into \( \mu_{ac} \ll \nu \) and \( \mu_s \perp \nu \).

Proof. Consider the real vector space \( \mathcal{E} \) of real valued \( A \)-measurable step-functions and let \( \mathcal{N} \) be the linear subspace generated by the characteristic functions of those measurable sets \( A \) such that \( \nu(A) = 0 \). Define the set function \( \mu_{ac} \) by

\[
\mu_{ac}(A) := \inf_{\psi \in \mathcal{A}} \int_X |\mathbb{1}_A - \psi|^2 \, d\mu \quad (A \in A).
\]

It is clear that \( \mu_{ac} \leq \mu \) (\( \psi := \mathbb{1}_\emptyset \)), and that \( \nu(A) = 0 \) implies \( \mu_{ac}(A) = 0 \) (\( \psi := \mathbb{1}_A \)). Furthermore, trivial verification shows that if \( A \) and \( B \) are disjoint elements of \( \mathcal{A} \), then

\[
\inf_{\psi \in \mathcal{A}} \int_X |\mathbb{1}_{A \cup B} - \psi|^2 \, d\mu = \inf_{\psi \in \mathcal{A}} \int_X |\mathbb{1}_A - \psi|^2 \, d\mu + \inf_{\psi \in \mathcal{N}} \int_X |\mathbb{1}_B - \psi|^2 \, d\mu.
\]

Since \( \mu_{ac} \) is nonnegative, additive, and dominated by the measure \( \mu \), we infer that \( \mu_{ac} \) is a measure itself.

What is left is to show that \( \mu_s := \mu - \mu_{ac} \) and \( \nu \) are singular, and that the decomposition is unique. Both follow immediately from the fact that \( \mu_{ac} \) is maximal among those measures \( \vartheta \) such that \( \vartheta \leq \mu \) and \( \vartheta \ll \nu \). Indeed, let \( \vartheta \) be such a measure, \( \psi \in \mathcal{N} \), and observe that

\[
\vartheta(A) = \int_X |\mathbb{1}_A|^2 \, d\vartheta = \int_X |\mathbb{1}_A - \psi|^2 \, d\vartheta \leq \int_X |\mathbb{1}_A - \psi|^2 \, d\mu.
\]

Taking the infimum over \( \mathcal{N} \) we obtain that \( \vartheta \leq \mu_{ac} \).

Now, let \( \eta \) be a measure, such that \( \eta \leq \nu \) and \( \eta \leq \mu - \mu_{ac} \). In this case, \( \mu_{ac} + \eta \leq \mu \) and \( \mu_{ac} + \eta \ll \nu \), thus \( \eta = 0 \). If \( \mu = \mu_1 + \mu_2 \), where \( \mu_1 \ll \nu \) and \( \mu_2 \perp \nu \), then \( \mu_{ac} - \mu_1 \) is a measure, which is simultaneously \( \nu \)-absolutely continuous and \( \nu \)-singular. This yields that \( \mu_1 = \mu_{ac} \). \( \square \)

References


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