

GENERALIZED ISOMETRIES OF THE SPECIAL UNITARY GROUP

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ABSTRACT. In this paper we determine the structure of all so-called generalized isometries of the special unitary group which are transformations that respect any member of a large collection of generalized distance measures.

In the paper [4] the first author has determined the structure of all isometries of the special unitary group $\mathbb{S}\mathbb{U}_n$ with respect to the spectral norm (operator norm) what we denote by $\|\cdot\|$. In this short paper we extend that result significantly and describe the structure of all transformations which respect any member of a large collection of generalized distance measures on $\mathbb{S}\mathbb{U}_n$.

In what follows \mathbb{M}_n denotes the algebra of all $n \times n$ complex matrices, \mathbb{U}_n stands for the group of all unitary elements in \mathbb{M}_n , and $\mathbb{S}\mathbb{U}_n$ denotes the special unitary group, i.e., the subgroup of \mathbb{U}_n consisting of those unitary matrices which have determinant 1. Furthermore, \mathbb{H}_n stands for the linear space of all Hermitian matrices in \mathbb{M}_n and $\mathbb{H}_{0,n}$ denotes its subspace consisting of all traceless (or, in other words, trace zero) Hermitian matrices.

By a generalized distance measure on a set X we mean a function $d : X \times X \rightarrow [0, \infty[$ with the sole property that $d(x, y) = 0$ holds for the elements $x, y \in X$ if and only if $x = y$. Such d can be viewed as a generalized metric, we do not assume either that it is symmetric or that the triangle inequality holds.

Let $N(\cdot)$ be a norm on \mathbb{M}_n and $f : \mathbb{T} \rightarrow \mathbb{C}$ be a continuous function having zero exactly at 1. Define

$$(1) \quad d_{N,f}(U, V) = N(f(UV^{-1})), \quad U, V \in \mathbb{U}_n.$$

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Clearly, by spectral mapping theorem $d_{N,f}$ is a generalized distance measure on \mathbb{U}_n . In particular, when $f(z) = z - 1$, $z \in \mathbb{T}$, we obtain $d_{N,f}(U, V) = N(U - V)$, $U, V \in \mathbb{U}_n$, which is just the usual norm distance with respect to $N(\cdot)$.

Consider the following conditions concerning a continuous function $f : \mathbb{T} \rightarrow \mathbb{C}$

- (a1) we have $f(y) = 0$ if and only if $y = 1$;
- (a2) there exists a number $K > 1$ such that

$$|f(y^2)| \geq K|f(y)|$$

holds for all $y \in \mathbb{T}$ from a neighborhood of 1.

It is rather easy to see that if f has the property (a1) and the limit $\lim_{z \rightarrow 1} \frac{f(z) - f(1)}{z - 1}$ exists and differs from zero, then (a2) is also satisfied.

After this preparation we can formulate our statement which is a far reaching generalization of the result in [4].

Theorem 1. *Let $N(\cdot)$ be a unitarily invariant norm on \mathbb{M}_n . Assume $f : \mathbb{T} \rightarrow \mathbb{C}$ is a continuous function with the properties (a1) and (a2). If $\phi : \mathbb{S}\mathbb{U}_n \rightarrow \mathbb{S}\mathbb{U}_n$ is a generalized isometry with respect to the generalized distance measure $d_{N,f}$, i.e., ϕ satisfies*

$$(2) \quad d_{N,f}(\phi(U), \phi(V)) = d_{N,f}(U, V), \quad U, V \in \mathbb{S}\mathbb{U}_n,$$

then there exists a pair $U, U' \in \mathbb{U}_n$ of unitary matrices with $\det UU' = 1$ such that ϕ is of one of the following forms:

- (b1) $\phi(V) = UVU'$, $V \in \mathbb{S}\mathbb{U}_n$;
- (b2) $\phi(V) = UV^{-1}U'$, $V \in \mathbb{S}\mathbb{U}_n$;
- (b3) $\phi(V) = UV^t U'$, $V \in \mathbb{S}\mathbb{U}_n$;
- (b4) $\phi(V) = U\bar{V}U'$, $V \in \mathbb{S}\mathbb{U}_n$.

Observe that the above four possible forms of generalized isometries of $\mathbb{S}\mathbb{U}_n$ are just the same as those of the isometries of \mathbb{U}_n obtained in Theorem 3, [7]. The proof of the present result will follow after executing small modifications in the proofs of Theorems 6-8 in [8] (we also refer to the arguments in the paper [7]). In our opinion proofs presented in papers should in principal be readable without troubling the readers by asking them to consult other sources (papers, books) and hence we usually give complete and easily accessible proofs in our works, but this time we feel we must make an exception. Namely, the complete proof with all details worked out would be too long and would mainly be comprised of repetitions of arguments presented in our previous paper [8]. Hence we have chosen to give a short proof, basically a sort of sketch, in which we clearly point out the distinctions between the proofs in that former paper and the present arguments.

Proof. Let $\phi : \mathbb{S}\mathbb{U}_n \rightarrow \mathbb{S}\mathbb{U}_n$ be a generalized isometry, i.e., a map which satisfies (2) and consider the transformation $V \mapsto \phi(V)\phi(I)^{-1}$. In that way we obtain a generalized isometry which sends I to I . Therefore, without loss of generality we can assume that our original transformation is unital, $\phi(I) = I$.

We show that ϕ is in fact surjective. To do this, we first need the following observation: for any sequence $V_k \in \mathbb{S}\mathbb{U}_n$ and element $V \in \mathbb{S}\mathbb{U}_n$ we have

$$(3) \quad d_{N,f}(V_k, V) \rightarrow 0 \text{ if and only if } V_k \rightarrow V \text{ in the spectral norm.}$$

In fact, this is a consequence of the following argument. Since $N(\cdot)$ is equivalent to the spectral norm (recall that on a finite dimensional vector space all norms are equivalent), hence $N(f(VW^{-1}))$ is small enough if and only if $\|f(VW^{-1})\|$ is small enough which (by the continuity of f , property (a1), and the spectral mapping theorem) happens exactly when the eigenvalues of VW^{-1} are close enough to 1. But this is equivalent to that the quantity $\|V - W\| = \|VW^{-1} - I\|$ is small enough. In particular, we obtain that the generalized isometry ϕ is continuous with respect to the spectral norm. Let us see why ϕ is surjective. Since $\mathbb{S}\mathbb{U}_n$ is compact in the spectral norm topology, so is $\phi(\mathbb{S}\mathbb{U}_n)$. Assume there exists $W \in \mathbb{S}\mathbb{U}_n$ which is not in $\phi(\mathbb{S}\mathbb{U}_n)$. We have

$$\inf\{\|\phi(V) - W\| : V \in \mathbb{S}\mathbb{U}_n\} > 0,$$

and by the argument above it implies that

$$\inf\{d_{N,f}(\phi(V), W) : V \in \mathbb{S}\mathbb{U}_n\} = \rho > 0.$$

Let $W_0 = W$ and define $W_k = \phi(W_{k-1})$, $k \in \mathbb{N}$. Using (2) we obtain that

$$d_{N,f}(W_k, W_l) = d_{N,f}(\phi^k(W_0), \phi^l(W_0)) = d_{N,f}(\phi^{k-l}(W), W) \geq \rho$$

holds for all $k, l \in \mathbb{N}$, $k > l$. It follows that $\|W_k - W_l\|$ is also bounded away from zero and hence one cannot choose a norm convergent subsequence of (W_k) which contradicts the compactness of $\mathbb{S}\mathbb{U}_n$. Therefore, ϕ is a surjective generalized isometry.

In [8, Proposition 20] we presented a general Mazur-Ulam type theorem for surjective generalized isometries between groups equipped with generalized distance measures that are compatible with the group operations. Just as in the proof of [8, Theorem 7] we can show that that general Mazur-Ulam type result applies for our present map ϕ and obtain that ϕ preserves the inverted Jordan triple product, i.e., we have

$$\phi(VW^{-1}V) = \phi(V)\phi(W)^{-1}\phi(V), \quad V, W \in \mathbb{S}\mathbb{U}_n.$$

As we have already pointed out ϕ is norm continuous. Since ϕ is assumed to be unital, it follows easily that ϕ is a continuous Jordan triple automorphism of $\mathbb{S}\mathbb{U}_n$, i.e., we have

$$\phi(VWV) = \phi(V)\phi(W)\phi(V), \quad V, W \in \mathbb{S}\mathbb{U}_n.$$

We can now literally follow the proof of [8, Lemma 21] (cf. Lemma 6 in [7]) to prove that ϕ is a Lipschitz function. The only thing that needs some more attention is to see that every element of $\mathbb{S}\mathbb{U}_n$ is the square of another element of $\mathbb{S}\mathbb{U}_n$. But this is a simple consequence of the fact that any element $V \in \mathbb{S}\mathbb{U}_n$ is of the form $V = e^{iA}$ with some $A \in \mathbb{H}_{0,n}$.

Next, following the proof of [8, Lemma 22] (cf. the proof of Lemma 7 in [7] and the part of the proof of Theorem 1 there after the displayed formula (2)) we have a bijective linear transformation $f : \mathbb{H}_{0,n} \rightarrow \mathbb{H}_{0,n}$ such that

$$(4) \quad \phi(e^{itA}) = e^{itf(A)}, \quad A \in \mathbb{H}_{0,n}$$

and f preserves commutativity in both directions. Clearly, $\mathbb{H}_n = \mathbb{H}_{0,n} \oplus \mathbb{R}I$. We can extend f to \mathbb{H}_n in a trivial way, namely, we define the transformation $F : \mathbb{H}_n \rightarrow \mathbb{H}_n$ by

$$F(A + \lambda I) = f(A) + \lambda I, \quad A \in \mathbb{H}_{0,n}.$$

Apparently, F is a bijective linear map of \mathbb{H}_n which preserves commutativity and also preserves the trace.

Assume that $n \geq 3$. In this case the structure of commutativity preserving nonsingular linear maps on \mathbb{H}_n is known. For example, Theorem 2 in [3] tells us that every such map is a nonzero real scalar multiple of a unitary similarity transformation possibly composed by the transposition plus a linear functional on \mathbb{H}_n multiplied by the identity matrix. Hence, we have a nonzero $c \in \mathbb{R}$, a unitary matrix $U \in \mathbb{U}_n$ and an element $S \in \mathbb{H}_n$ such that either

$$F(T) = cUTU^* + \text{Tr}(TS)I, \quad T \in \mathbb{H}_n$$

or

$$F(T) = cUT^tU^* + \text{Tr}(TS)I, \quad T \in \mathbb{H}_n.$$

Suppose we have the first possibility. Using the trace preserving property of F we infer that $\text{Tr } T = c \text{Tr } T + n \text{Tr}(TS)$ and hence that $\text{Tr}(T(1-c)I) = \text{Tr}(TnS)$, $T \in \mathbb{H}_n$. This implies that $(1-c)I = nS$, i.e., S is a scalar multiple of the identity. It follows that for any $A \in \mathbb{H}_{0,n}$ we have $f(A) = F(A) = cUAU^*$ and, consequently, $\phi(e^{itA}) = Ue^{itcA}U^*$.

Since the Jordan triple automorphism ϕ is unital, it preserves the square operation and hence it sends symmetries (i.e., unitaries with square equal to I) to symmetries. Consequently, for any $A \in \mathbb{H}_{0,n}$ if e^{itA} is

a symmetry, then so is e^{itcA} . Pick mutually orthogonal rank-one projections $P, Q \in \mathbb{M}_n$. Since $e^{i\pi(P-Q)}$ is a symmetry, so is $e^{i\pi c(P-Q)}$. This gives us that $e^{i\pi c} \in \{-1, 1\}$ implying that c is necessarily an integer. That means that we have $k \in \mathbb{Z}$ such that $\phi(V) = UV^kU^*$, $V \in \mathbb{S}\mathbb{U}_n$. Since ϕ is a Jordan triple automorphism, it easily implies that for any $V, W \in \mathbb{U}_n$ we have $(VWV)^k = V^k W^k V^k$. Applying [8, Proposition 18] (cf. the first paragraph on p. 3526 in [7]) we deduce that this can happen only if k is either 1 or -1. This gives us that either we have $\phi(V) = UVU^{-1}$, $V \in \mathbb{S}\mathbb{U}_n$ or we have $\phi(V) = UV^{-1}U^{-1}$, $V \in \mathbb{S}\mathbb{U}_n$.

In the second possibility with the transpose appearing in the form of F above, one can argue just in the same way. So we are done in the case where $n \geq 3$.

Assume now that $n = 2$. Pick mutually orthogonal rank-one projections $P, Q \in \mathbb{M}_2$. Since the only symmetry in $\mathbb{S}\mathbb{U}_2$ different from the identity is $-I$, it follows that $-I = \phi(e^{i\pi(P-Q)}) = e^{i\pi f(P-Q)}$. This implies that the eigenvalues of the traceless Hermitian matrix $\pi f(P-Q)$ are $-(2k+1)\pi, (2k+1)\pi$ for some integer k . Due to the continuity of f and to the continuity of the spectrum, this k does not depend on P, Q . Hence, setting $g = (1/(2k+1))f$, we have $g : \mathbb{H}_{0,2} \rightarrow \mathbb{H}_{0,2}$ is a bijective linear map with the property that for every pair $P, Q \in \mathbb{M}_2$ of mutually orthogonal rank-one projections we have another pair $P', Q' \in \mathbb{M}_2$ of mutually orthogonal rank-one projections such that $g(P-Q) = P' - Q'$. Slightly differently, we can write $g(2P - I) = 2P' - I$. We extend g to a bijective linear map G on \mathbb{H}_2 by the formula

$$G(\alpha(2P - I) + \beta I) = g(\alpha(2P - I)) + \beta I = \alpha(2P' - I) + \beta I, \quad \alpha, \beta \in \mathbb{R}.$$

We clearly have that $G(P) = P'$, $G(0) = 0$, $G(I) = I$. This means that G maps projections to projections which implies that G is a Jordan *-automorphism of \mathbb{M}_2 (see, e.g., Theorem A.4 in [6]). Those maps are well known to be equal to unitary similarity transformations possibly composed the transposition. Consequently, there exists a unitary matrix $U \in \mathbb{U}_2$ such that g is either of the form $g : A \mapsto UAU^*$ or of the form $g : A \mapsto UA^tU^*$. By (4) we deduce that either we have $\phi : V \mapsto UV^{2k+1}U^*$ or we have $\phi : V \mapsto UV^{2k+1t}U^*$. One can verify that this implies $2k+1 \in \{-1, 1\}$ just as above in the case where $n \geq 3$. This completes the proof of the theorem. \square

In the above theorem we have obtained that all considered generalized isometries of $\mathbb{S}\mathbb{U}_n$ are of one of the forms (b1)-(b4). It is a natural problem to investigate if the converse is also true, i.e., if all four formulas (b1)-(b4) always define generalized isometries on $\mathbb{S}\mathbb{U}_n$ (we mean for any unitarily

invariant norm $N(\cdot)$ and continuous function $f : \mathbb{T} \rightarrow \mathbb{C}$ with the properties (a1) and (a2)). To answer this question we begin with the following. Since $N(\cdot)$ is unitarily invariant and we have $f(UVU^{-1}) = Uf(V)U^{-1}$ for all $U \in \mathbb{U}_n$ and $V \in \mathbb{S}\mathbb{U}_n$, (b1) really always gives a generalized isometry. Furthermore, for any $V, W \in \mathbb{S}\mathbb{U}_n$ we have that $V^t(W^t)^{-1} = (W^{-1}V)^t$ is unitarily similar to $W^{-1}V$ (indeed, by spectral theorem any normal matrix is unitary similar to a diagonal one which is symmetric). Next, $W^{-1}V = W^{-1}(VW^{-1})W$ which means that $W^{-1}V$ is unitarily similar to VW^{-1} finally giving that $V^t(W^t)^{-1}$ is unitarily similar to VW^{-1} . Consequently, (b3) also always defines a generalized isometry. As for (b2) and (b4), since $\overline{V} = V^{-1t}$, $V \in \mathbb{S}\mathbb{U}_n$, anyone of them gives a generalized isometry if and only if so does the other one. Therefore, we need only to investigate when the inverse operation on $\mathbb{S}\mathbb{U}_n$ is a generalized isometry. It is easy to verify (see the discussion below on unitarily invariant norms and symmetric gauge functions) that this is the case when the corresponding continuous function $f : \mathbb{T} \rightarrow \mathbb{C}$ has the symmetry property $|f(\lambda)| = |f(1/\lambda)|$, $\lambda \in \mathbb{T}$. Observe that on the full group \mathbb{U}_n this is not just a sufficient but also a necessary condition for the inverse operation to be a generalized isometry. As we see below, concerning $\mathbb{S}\mathbb{U}_n$ the situation is rather different.

Let now $N(\cdot)$ be a unitarily invariant norm on \mathbb{M}_n and $f : \mathbb{T} \rightarrow \mathbb{C}$ be a continuous function satisfying (a1). We assert that the map $V \mapsto V^{-1}$ on $\mathbb{S}\mathbb{U}_n$ is a generalized isometry with respect to $d_{N,f}$ if and only if $d_{N,f}$ is symmetric (i.e., $d_{N,f}(V, W) = d_{N,f}(W, V)$, $V, W \in \mathbb{S}\mathbb{U}_n$) which holds exactly when $N(f(V^{-1})) = N(f(V))$, $V \in \mathbb{S}\mathbb{U}_n$. (Observe that the latter equality holds trivially if $n = 2$ and hence in that case our problem is meaningless). To see this, for any $V, W \in \mathbb{S}\mathbb{U}_n$ we compute

$$\begin{aligned} d_{N,f}(V^{-1}, W^{-1}) &= N(f(V^{-1}W)) = N(W^{-1}f(WV^{-1})W) \\ &= N(f(WV^{-1})) = d_{N,f}(W, V) \end{aligned}$$

and

$$\begin{aligned} d_{N,f}(W, V) &= N(f(WV^{-1})) = N(f((VW^{-1})^{-1})), \\ d_{N,f}(V, W) &= N(f(VW^{-1})). \end{aligned}$$

By a famous theorem of von Neumann, there is a one-to-one correspondence between unitarily invariant norms on \mathbb{M}_n and so-called symmetric gauge functions on \mathbb{C}^n (i.e., absolute norms on \mathbb{C}^n which are invariant under permutations of the components), see, e.g., 7.4.24. Theorem in [5]. In fact, the unitarily invariant norm of a matrix equals the corresponding gauge norm of its singular values.

In the next proposition we prove that for a relatively large class of unitarily invariant norms the inverse operation is a generalized isometry

(hence the formulas (b2), (b4) define generalized isometries) if and only if f is symmetric by which we mean that it satisfies $|f(\lambda)| = |f(1/\lambda)|$, $\lambda \in \mathbb{T}$.

Proposition 2. *Let $n \geq 3$ and $N(\cdot)$ be a unitarily invariant norm with associated symmetric gauge function G . Assume that G has the property that for any $\alpha, \beta, \beta' \in [0, \infty[$,*

$$(5) \quad G(\alpha, \beta, \beta, \dots, \beta) = G(\alpha, \beta', \beta', \dots, \beta') \implies \beta = \beta'.$$

Then for any continuous function $f : \mathbb{T} \rightarrow \mathbb{C}$ with property (a1) we have that the inverse operation on $\mathbb{S}\mathbb{U}_n$ is a generalized isometry with respect to $d_{N,f}$ (or, equivalently, $N(f(V^{-1})) = N(f(V))$ holds for all $V \in \mathbb{S}\mathbb{U}_n$) if and only if $|f(\lambda)| = |f(1/\lambda)|$, $\lambda \in \mathbb{T}$.

Observe that the above condition (5) is satisfied by the Schatten p -norms, $1 \leq p < \infty$, by the Ky Fan k -norms, $1 < k \leq n$, or more generally, by the (p, k) -norms with $1 \leq p < \infty$, $1 < k \leq n$ and also by the (c, p) -norms which are not scalar multiples of the operator norm, see [1], [2].

Proof. Only the 'only if' part needs verification. So, assume we have $N(f(V^{-1})) = N(f(V))$ for all $V \in \mathbb{S}\mathbb{U}_n$. Pick any n th root of unit, say, λ_1 . Considering the element $V = \text{diag}[\lambda_1, \dots, \lambda_1]$ of $\mathbb{S}\mathbb{U}_n$ we clearly have $|f(\lambda_1)| = |f(1/\lambda_1)|$. Next choose and $(n-1)$ th root of λ_1 , say λ_2 , and consider the unitary $\text{diag}[\lambda_1, (1/\lambda_2), \dots, (1/\lambda_2)]$ which belongs to $\mathbb{S}\mathbb{U}_n$. By the property (5) we have $|f(\lambda_2)| = |f(1/\lambda_2)|$. Continue the process with λ_2 in the place of λ_1 . In that way, we see that $|f(\lambda)| = |f(1/\lambda)|$ holds for all $n(n-1)^{k-1}$ th roots of unit ($k \in \mathbb{N}$) which set is dense in \mathbb{T} . By the continuity of f it follows that the same holds for all $\lambda \in \mathbb{T}$, too. \square

In the last proposition we prove that the condition (5) in the above result is indispensable. In fact, we show that for the spectral norm there is a continuous function $f : \mathbb{T} \rightarrow \mathbb{C}$ which satisfies (a1) and (a2), but f is not symmetric (i.e., does not satisfy $|f(\lambda)| = |f(1/\lambda)|$, $\lambda \in \mathbb{T}$) and the inverse operation is still a generalized isometry with respect to $d_{\|\cdot\|, f}$.

Proposition 3. *Let $n \geq 3$ be an integer. Then there is a continuous non-symmetric function $f : \mathbb{T} \rightarrow \mathbb{C}$ satisfying (a1), (a2) such that*

$$d_{\|\cdot\|, f}(V^{-1}, W^{-1}) = d_{\|\cdot\|, f}(V, W), \quad V, W \in \mathbb{S}\mathbb{U}_n.$$

Proof. Recall that the spectral norm (operator norm) $\|A\|$ for $A \in \mathbb{M}_n$ coincides with the maximum value of the singular values of A . In what follows we shall construct a nonnegative function f . Clearly, we have $f(V)$ is positive semidefinite for every $V \in \mathbb{S}\mathbb{U}_n$. In this case $\|f(V)\|$ coincides with the maximum value of the eigenvalues of $f(V)$. The eigenvalues of $f(V)$ are the images of the eigenvalues of V under f . Thus $\|f(V)\|$ is the

maximum value of f on the spectrum of V for any $V \in \mathbb{S}\mathbb{U}_n$. It follows that we only need to guarantee that

$$\max_k \{f(\alpha_k)\} = \max_k \{f(\overline{\alpha_k})\}$$

holds for every set $\{\alpha_k\}$ of n elements (counting the multiplicity) α_k in \mathbb{T} with $\prod_{k=1}^n \alpha_k = 1$.

Let ε be a small positive number, say, $0 < \varepsilon < \frac{\pi}{n}$. Set $I = \{e^{i\theta} : -\frac{\varepsilon}{n} \leq \theta \leq \frac{\varepsilon}{n}\}$ and $J = \{e^{i\theta} : \frac{\pi-\varepsilon}{n} < \theta < \frac{\pi+\varepsilon}{n}\}$. Let \bar{J} denote the conjugate of J , $\bar{J} = \{\bar{z} \in \mathbb{T} : z \in J\}$. Let $f : \mathbb{T} \rightarrow \mathbb{C}$ be defined in the following way

$$f(e^{i\theta}) = \begin{cases} \frac{2n}{\varepsilon}\theta, & 0 \leq \theta \leq \frac{\varepsilon}{n}, \\ \frac{n}{2\varepsilon-\pi}\theta + 1 + \frac{\pi-\varepsilon}{\pi-2\varepsilon}, & \frac{\varepsilon}{n} \leq \theta \leq \frac{\pi-\varepsilon}{n}, \\ 1, & \frac{\pi-\varepsilon}{n} \leq \theta \leq \frac{\pi+\varepsilon}{n} \\ \theta + 1 - \frac{\pi+\varepsilon}{n}, & \frac{\pi+\varepsilon}{n} \leq \theta \leq \pi \end{cases}$$

for $0 \leq \theta \leq \pi$ and $f(e^{i\theta}) = f(e^{-i\theta}) + g(e^{i\theta})$ for $0 \geq \theta \geq -\pi$, where g is a continuous function on $\{e^{i\theta} : 0 \geq \theta \geq -\pi\}$ such that

$$g(e^{i\theta}) = \begin{cases} 0, & 0 \geq \theta \geq -\frac{\pi-\varepsilon}{n}, -\frac{\pi+\varepsilon}{n} \geq \theta \geq -\pi \\ \text{negative} > -1, & -\frac{\pi-\varepsilon}{n} > \theta > -\frac{\pi+\varepsilon}{n}. \end{cases}$$

Clearly, the function f is continuous, non-symmetric and it satisfies (a1) and (a2).

We show that $\max_k \{f(\alpha_k)\} = \max_k \{f(\overline{\alpha_k})\}$ also holds for every set $\{\alpha_k\}_k$ of n points (counting the multiplicity) in \mathbb{T} with $\prod_{k=1}^n \alpha_k = 1$. What we essentially need in the proof are the following: outside $J \cup \bar{J}$ the function f symmetric, outside $I \cup J \cup \bar{J}$ its values are greater than 1, on J it is constant 1 and on \bar{J} it is less than 1.

Suppose that there exists $w \in \{\alpha_k\}_k \cap (\mathbb{T} \setminus (J \cup \bar{J} \cup I))$. Then $f(w) = f(\bar{w})$ is greater than 1 and 1 is the maximum value of f over $J \cup \bar{J}$. This means that the images of the values from $\{\alpha_k\}_k \cap (J \cup \bar{J})$ under f do not affect $\max_k \{f(\alpha_k)\}$ and $\max_k \{f(\overline{\alpha_k})\}$. Moreover f is symmetric on I , hence we have $\max_k \{f(\alpha_k)\} = \max_k \{f(\overline{\alpha_k})\}$.

We consider the case $\{\alpha_k\}_k \subset (J \cup \bar{J} \cup I)$. If $\{\alpha_k\}_k \subset I$, we are done since f is symmetric on I . Assume $\{\alpha_k\}_k \not\subset I$ but $\{\alpha_k\}_k \subset (J \cup I)$. One can check that it is impossible since in this case $\prod_{k=1}^n \alpha_k$ cannot be 1. Similarly, $\{\alpha_k\}_k \not\subset I$, $\{\alpha_k\}_k \subset (\bar{J} \cup I)$ cannot happen either. It follows that $\{\alpha_k\}_k \cap J \neq \emptyset$ and $\{\alpha_k\}_k \cap \bar{J} \neq \emptyset$. By the properties of f we again have $\max_k \{f(\alpha_k)\} = \max_k \{f(\overline{\alpha_k})\}$. This completes the proof. \square

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