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# Structure of normal twisted group rings

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**Abstract.** Let $K_\lambda G$ be the twisted group ring of a group $G$ over a commutative ring $K$ with 1, and let $\lambda$ be a factor set (2-cocycle) of $G$ over $K$. Suppose $f : G \to U(K)$ is a map from $G$ onto the group of units $U(K)$ of the ring $K$ satisfying $f(1) = 1$. If $x = \sum_{g\in G} \alpha_g u_g \in K_\lambda G$ then we denote $\sum_{g\in G} \alpha_g f(g) u_g^{-1}$ by $x^f$ and assume that the map $x \to x^f$ is an involution of $K_\lambda G$. In this paper we describe those groups $G$ and commutative rings $K$ for which $K_\lambda G$ is $f$-normal, i.e. $xx^f = x^f x$ for all $x \in K_\lambda G$.

## 1. Introduction

Let $G$ be a group and $K$ a commutative ring with unity. Suppose that the elements of the set

$$\Lambda = \{\lambda_{a,b} \in U(K) \mid a, b \in G\}$$

satisfy the condition

$$\lambda_{a,b}\lambda_{ab,c} = \lambda_{b,c}\lambda_{a,bc} \tag{1}$$

for all $a, b, c \in G$. Then $\Lambda$ will be called a *factor system* (2-cocycle) of the group $G$ over the ring $K$. The twisted group ring $K_\lambda G$ of $G$ over the commutative ring $K$ is an associative $K$-algebra with basis $\{u_g \mid g \in G\}$

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and with multiplication defined distributively by $u_g u_h = \lambda_{g,h} u_{gh}$, where $g, h \in G$ and

$$\lambda_{g,h} \in \Lambda = \{\lambda_{a,b} \in U(K) \mid a, b \in G\}.$$

Note that if $\lambda_{g,h} = 1$ for all $g, h \in G$, then $K_\lambda G \cong KG$, where $KG$ is the group ring of the group $G$ over the ring $K$.

Properties of twisted group algebras and their groups of units were studided by many authors, see, for instance, the paper by S. V. Mihovski and J. M. Dimitrova [1]. Our aim is to describe the structure of $f$-normal twisted group rings. This result for group rings was obtained in [2, 3].

We shall refer to two twisted group rings $K_\lambda G$ and $K_\mu G$ as being diagonally equivalent if there exists a map $\theta : G \to U(K)$ such that

$$\lambda_{a,b} = \theta(a)\theta(b)\mu_{a,b}(\theta(ab))^{-1}.$$

We say that a factor system $\Lambda$ is normalized if it satisfies the condition

$$\lambda_{a,1} = \lambda_{1,b} = \lambda_{1,1} = 1$$

for all $a, b \in G$.

Hence, given $K_\mu G$ there always exists a diagonally equivalent twisted group ring $K_\lambda G$ with factor system $\Lambda$ defined by $\lambda_{a,b} = \mu_{1,1}^{-1}\mu_{a,b}$ such that $\Lambda$ is normalized. From now on, all the factor systems considered are supposed to be normalized.

The map $\phi$ from the ring $K_\lambda G$ onto $K_\lambda G$ is called *an involution*, if it satisfies the conditions

(i) $\phi(a+b) = \phi(a) + \phi(b)$; (ii) $\phi(ab) = \phi(b)\phi(a)$; (iii) $\phi^2(a) = a$ for all $a, b \in K_\lambda G$.

Let $f : G \to U(K)$ be a map from the group $G$ onto the group of units $U(K)$ of the commutative ring $K$, satisfying $f(1) = 1$. For an element $x = \sum_{g\in G} \alpha_g u_g \in K_\lambda G$ we define $x^f = \sum_{g\in G} \alpha_g f(g) u_g^{-1} \in K_\lambda G$.

Let $x \to x^f$ be an involution of the twisted group ring $K_\lambda G$. The twisted group ring $K_\lambda G$ is called *$f$-normal* if

$$xx^f = x^f x \tag{2}$$

for all $x \in K_\lambda G$.

Recall that a $p$-group is called *extraspecial* (see [4], Definition III.13.1) if its centre, commutator subgroup and Frattini subgroup are equal and have order $p$.

**Theorem.** *Let $x \to x^f$ be an involution of the twisted group ring $K_\lambda G$. If the ring $K_\lambda G$ is $f$-normal then the group $G$ and the ring $K$ satisfy one of the following conditions:*

1) *$G$ is abelian and the factor system is symmetric, i.e. $\lambda_{a,b} = \lambda_{b,a}$ for all $a, b \in G$;*
2) *$G$ is an abelian group of exponent 2 and the factor system satisfies*

$$(\lambda_{a,b} - \lambda_{b,a})(1 + f(b)\lambda_{b,b}^{-1}) = 0 \tag{3}$$

*for all $a, b \in G$;*

3) *$G = H \rtimes C_2$ is a semidirect product of an abelian group $H$ of exponent not equal to 2 and $C_2 = \langle a \mid a^2 = 1\rangle$ with $h^a = h^{-1}$ for all $h \in H$, the factor system of $H$ is symmetric, $f(a) = -\lambda_{a,a}$ and*

$$\lambda_{a,h} = f(h)\lambda_{h,h^{-1}}^{-1}\lambda_{h^{-1},a}, \quad \lambda_{h,a} = f(h)\lambda_{h,h^{-1}}^{-1}\lambda_{a,h^{-1}}; \tag{4}$$

4) *$G$ is a hamiltonian 2-group and the factor system satisfies*
   4.i) *for all noncommuting $a, b \in G$*

$$\lambda_{a,b} = f(a)\lambda_{a,a^{-1}}^{-1}\lambda_{b,a^{-1}} = f(b)\lambda_{b,b^{-1}}^{-1}\lambda_{b^{-1},a}; \tag{5}$$

   4.ii) *$\lambda_{g,h} = \lambda_{h,g}$ for any $h \in C_G(\langle g\rangle)$ and $f(c) = \lambda_{c,c}$ for every $c$ of order 2;*

5) *$G = \Gamma \curlyvee C_4$ is a central product of a hamiltonian 2-group $\Gamma$ and a cyclic group $C_4 = \langle d \mid d^4 = 1\rangle$ with $\Gamma' = \langle d^2\rangle$. The factor system satisfies (5) and*

$$\lambda_{b,a}\lambda_{ba,d} + f(d)\lambda_{d,d^{-1}}^{-1}\lambda_{a,b}\lambda_{ab,d^{-1}} = 0, \tag{6}$$

*where $a, b \in \Gamma$, $a^4 = b^4 = 1$ and $[a, b] \neq 1$;*

6) *$G$ is either $E \times W$ or $(E \curlyvee C_4) \times W$, where $E$ is an extraspecial 2-group, $E \curlyvee C_4$ is the central product of $E$ and $C_4 = \langle c \mid c^4 = 1\rangle$ with $E' = \langle c^2\rangle$ and $\exp(W)|2$. The factor system satisfies:*
   6.i) *If $a \in G$ has order 4 then $\lambda_{a,h} = \lambda_{h,a}$ for all $h \in C_G(\langle a\rangle)$;*
   6.ii) *if $\langle a, b\rangle$ is a quaternion subgroup of order 8 of $G$ then the properties (5) and (6) are satisfied for every $d \in C_G(\langle a, b\rangle)$ of order 4, and $f(v) = \lambda_{v,v}$ for all $v \in C_G(\langle a, b\rangle)$ of order 2;*

6.iii) *if* $\langle a, b \mid a^4 = b^2 = 1\rangle$ *is the dihedral subgroup of order* 8, *then* $f(b) = -\lambda_{b,b}$ *and the properties* (4) *and* (6) *are satisfied for every* $d \in C_G(\langle a, b\rangle)$ *of order* 4.

*Moreover, the conditions* 1)–5) *are also sufficient for* $K_\lambda G$ *to be* $f$-*normal. The condition* 6) *is sufficient if* $K$ *is an integral domain of characteristic* 2.

## 2. Lemmas

Let $C_4$, $Q_8$ and $D_8$ be a cyclic group of order 4, a quaternion group of order 8 and a dihedral group of order 8, respectively. As usual, $x^y = y^{-1}xy$, $\exp(G)$ and $C_G(\langle a, b\rangle)$ denote the exponent of $G$ and the centralizer of the subgroup $\langle a, b\rangle$ in $G$.

It is easy to see that $\lambda_{g,g^{-1}} = \lambda_{g^{-1},g}$ and $u_g^{-1} = \lambda_{g,g^{-1}}^{-1} u_{g^{-1}}$ hold for all $g \in G$.

**Lemma 1.** *The map* $x \to x^f$ *is an involution of the ring* $K_\lambda G$ *if and only if*

$$f(gh)\lambda_{g,h}^2 = f(g)f(h)$$

*for all* $g, h \in G$.

Proof. Let the map $x \to x^f$ be an involution of the ring $K_\lambda G$. If $g, h \in G$, then $(u_g u_h)^f = u_h^f u_g^f$. Thus

$$\begin{aligned}\lambda_{g,h} f(gh) u_{gh}^{-1} = (\lambda_{g,h} u_{gh})^f = (u_g u_h)^f &= f(g)f(h)u_h^{-1}u_g^{-1}\\ &= f(g)f(h)(\lambda_{g,h}^{-1} u_{gh})^{-1}\end{aligned}$$

and $f(gh)\lambda_{g,h}^2 = f(g)f(h)$ for all $g, h \in G$. □

Clearly, if $K_\lambda G$ is a group ring, then the map $x \to x^f$ is an involution of the group ring $KG$ if and only if $f$ is a homomorphism from $G$ to $U(K)$.

**Lemma 2.** *If the ring* $K_\lambda G$ *is* $f$-*normal then the group* $G$ *satisfies one of the conditions* 1)–6) *of Theorem 1.*

Proof. Let $K_\lambda G$ be an $f$-normal twisted group ring. If $a, b \in G$ and $x = u_a + u_b \in K_\lambda G$, then $x^f = f(a)u_a^{-1} + f(b)u_b^{-1}$ and by (2)

$$\begin{aligned}&f(a)\lambda_{a,a^{-1}}^{-1}\lambda_{a^{-1},b}u_{a^{-1}b} + f(b)\lambda_{b,b^{-1}}^{-1}\lambda_{b^{-1},a}u_{b^{-1}a}\\ &= f(a)\lambda_{a,a^{-1}}^{-1}\lambda_{b,a^{-1}}u_{ba^{-1}} + f(b)\lambda_{b,b^{-1}}^{-1}\lambda_{a,b^{-1}}u_{ab^{-1}}.\end{aligned} \tag{7}$$

Now put $y = u_a(u_1 + u_b)$. Then $y^f = (u_1 + f(b)u_b^{-1})f(a)u_a^{-1}$ and by (2)

$$\lambda_{a,b}u_{ab} + f(b)\lambda_{b,b^{-1}}^{-1}\lambda_{a,b^{-1}}u_{ab^{-1}} = \lambda_{b,a}u_{ba} + f(b)\lambda_{b,b^{-1}}^{-1}\lambda_{b^{-1},a}u_{b^{-1}a}. \tag{8}$$

We shall treat two cases.

I. Let $[a,b] \neq 1$ for $a, b \in G$ and $a^2 \neq 1$, $b^2 \neq 1$. Then by (8) $b^a = b^{-1}$ and by (7) $a^2 = b^2$. The factor system satisfies

$$\begin{cases} \lambda_{a,b} = f(a)\lambda_{a,a^{-1}}^{-1}\lambda_{b,a^{-1}} = f(b)\lambda_{b,b^{-1}}^{-1}\lambda_{b^{-1},a}; \\ \lambda_{b,a} = f(a)\lambda_{a,a^{-1}}^{-1}\lambda_{a^{-1},b} = f(b)\lambda_{b,b^{-1}}^{-1}\lambda_{a,b^{-1}}. \end{cases} \tag{9}$$

II. Let $[a,b] \neq 1$ for $a, b \in G$ and $a^2 = 1$, $b^2 \neq 1$. Then by (8) we have $b^a = b^{-1}$ and by (7), $f(a) = -\lambda_{a,a}$. The factor system satisfies

$$\begin{cases} \lambda_{a,b} = f(b)\lambda_{b,b^{-1}}^{-1}\lambda_{b^{-1},a}; \\ \lambda_{b,a} = f(b)\lambda_{b,b^{-1}}^{-1}\lambda_{a,b^{-1}}. \end{cases}$$

Let $G$ be a nonabelian group and let $W = \{g \in G \mid g^2 \neq 1\}$.

First we consider the case when the elements of $W$ commute. Then $\langle w \mid w \in W\rangle$ is an abelian subgroup and if $b \in W$ and $a \in G \setminus \langle W\rangle$ then $a^2$=1 and $(ab)^2 = 1$. Therefore, $b^a = b^{-1}$ for all $b \in W$. Let $c \in C_G(\langle W\rangle) \setminus \langle W\rangle$. Then $c^2 = 1$, $(cb)^2 = 1$ and $cb \notin \langle W\rangle$. But $(cb)^2 = c^2b^2 = 1$ and $b^2 = 1$, which is impossible. Therefore, $C_G(\langle W\rangle) = \langle W\rangle$ and $H = \langle W\rangle$ is a subgroup of index 2. This implies that $G = H \rtimes \langle a\rangle$ and $h^a = h^{-1}$ for all $h \in H$.

Now suppose that in $W$ there exist elements $a, b$ such that $[a,b] \neq 1$. Since $a^2 \neq 1$ and $b^2 \neq 1$, by (I) we have $a^2 = b^2$ and $b^a = b^{-1}$. Then $b^2 = ab^2a^{-1} = b^{-2}$ and the elements $a, b$ are of order 4. Clearly, the subgroup $\langle a, b\rangle$ is a quaternion group of order 8. Let $c \in C_G(\langle a, b\rangle)$. If $c^2 \neq 1$ and $(ac)^2 \neq 1$ then (I) implies that $(ac)^b = (ac)^{-1}$ and $c^2 = 1$, which is impossible. Therefore, if $c \in C_G(\langle a, b\rangle)$ then either $c^2 = 1$ or $c^2 = a^2$.

Let $Q = \langle a, b\rangle$ be a quaternion subgroup of order 8 of $G$. Then we will prove that $G = Q \cdot C_G(Q)$. Suppose $g \in G \setminus C_G(Q)$. Pick the elements $a, b \in Q$ of order 4 such that $a^g = a^{-1}$ and $b^g = b^{-1}$. Then $(ab)^g = ab$ and $d = gab \in C_G(Q)$. It follows that $g = d(ab)^{-1}$ and $G = Q \cdot C_G(Q)$. Similary as in [3] we obtain that $G$ satisfies the conditions 4) or 5) of the Theorem. □

## 3. Proof of Theorem

*Necessity.* Let $K_\lambda G$ be $f$-normal. Then by Lemma 2 $G$ satisfies one of the conditions 1)–5) of the Theorem.

First, suppose that $G$ is abelian of exponent greater than 2 and $a, b \in G$. If $b^2 \neq 1$ then by (8) we have $\lambda_{a,b} = \lambda_{b,a}$.

Let $a$, $b$ be elements of order two and assume that there exists $c$ with $c^2 = a$. Then by (1) we have

$$\lambda_{c^2,b}\lambda_{c,c} = \lambda_{c,cb}\lambda_{c,b} \quad \text{and} \quad \lambda_{b,c^2}\lambda_{c,c} = \lambda_{bc,c}\lambda_{b,c}. \tag{10}$$

Since $c^2 \neq 1$, we have $\lambda_{c,cb} = \lambda_{bc,c}$ and $\lambda_{c,b} = \lambda_{b,c}$. Then (10) implies $\lambda_{c^2,b} = \lambda_{b,c^2}$ and $\lambda_{a,b} = \lambda_{b,a}$.

Let $a^2 = b^2 = 1$ such that neither $a$ nor $b$ is the square of any element of $G$. Then there exists $c$ such that $(ca)^2 \neq 1$. Thus,

$$\lambda_{ca,b}\lambda_{c,a} = \lambda_{c,ab}\lambda_{a,b}, \quad \lambda_{b,ac}\lambda_{a,c} = \lambda_{ba,c}\lambda_{b,a}. \tag{11}$$

Since $\lambda_{b,ac} = \lambda_{ac,b}$ and $\lambda_{c,a} = \lambda_{a,c}$ from (11) we have $\lambda_{a,b} = \lambda_{b,a}$ for all $a, b \in G$. Therefore, if $G$ is abelian and $G^2 \neq 1$ then the factor system is symmetric and $K_\lambda G$ is commutative.

Now, let $\exp(G) = 2$. Then by (8) $\lambda_{a,b} + f(b)\lambda_{b,b}^{-1}\lambda_{a,b} = \lambda_{b,a} + f(b)\lambda_{b,b}^{-1}\lambda_{b,a}$ for all $a, b \in G$. Therefore, $(\lambda_{a,b} - \lambda_{b,a})(1 + f(b)\lambda_{b,b}^{-1}) = 0$.

Next, let $G = H \rtimes C_2$ be a semidirect product of an abelian group $H$ with $\exp(H) \neq 2$ and $C_2 = \langle a \mid a^2 = 1\rangle$, and with $h^a = h^{-1}$ for all $h \in H$. Clearly, $K_\lambda H$ is $f$-normal and the factor system of $H$ is symmetric. Put $x = u_h + u_a$ for $h \in H$. Since $K_\lambda G$ is $f$-normal, we have $S_f(x) = xx^f - x^f x = 0$ and

$$\begin{aligned} &f(a)\lambda_{a,a}^{-1}\lambda_{h,a}u_{ha} + f(h)\lambda_{h,h^{-1}}^{-1}\lambda_{a,h^{-1}}u_{ah^{-1}} \\ &\quad -f(h)\lambda_{h,h^{-1}}^{-1}\lambda_{h^{-1},a}u_{h^{-1}a} - f(a)\lambda_{a,a}^{-1}\lambda_{a,h}u_{ah} = 0. \end{aligned} \tag{12}$$

We will prove $u_a u_h = u_h^f u_a$ for every $h \in H$.

First, let $h^2 \neq 1$. Because $h^a = h^{-1}$, by (12) we have

$$u_a^f u_h + u_h^f u_a = 0 \tag{13}$$

and

$$\text{(14)} \qquad \begin{cases} f(a)\lambda_{a,a}^{-1}\lambda_{a,h} + f(h)\lambda_{h,h^{-1}}^{-1}\lambda_{h^{-1},a} = 0; \\ f(a)\lambda_{a,a}^{-1}\lambda_{h,a} + f(h)\lambda_{h,h^{-1}}^{-1}\lambda_{a,h^{-1}} = 0. \end{cases}$$

Now, let $h^2 = 1$. Then there exists $b \in H$ with $b^2 \neq 1$ and $(hb)^2 \neq 1$. Put $x = u_a + u_h u_b$. Because $(hb)^a = (hb)^{-1}$ and $S_f(x) = xx^f - x^f x = 0$ we have

$$\text{(15)} \qquad u_a^f u_h u_b + (u_h u_b)^f u_a = 0.$$

Since $[u_h, u_b] = 1$, by (15) and (13) we have $u_a^f(u_h u_b) = u_a^f u_b u_h = -u_b^f u_a u_h$ and $u_a^f(u_h u_b) = -(u_h u_b)^f u_a = -u_b^f u_h^f u_a$. Therefore, $u_a u_h = u_h^f u_a$ for all $h \in H$ and this implies

$$\begin{cases} \lambda_{a,h} = f(h)\lambda_{h,h^{-1}}^{-1}\lambda_{h^{-1},a}; \\ \lambda_{h,a} = f(h)\lambda_{h,h^{-1}}^{-1}\lambda_{a,h^{-1}}, \end{cases}$$

and, by (14), $f(a) = -\lambda_{a,a}$.

Let $G$ be a hamiltonian 2-group. It is well known (see [5], Theorem 12.5.4) that $G = Q_8 \times W$, where $Q_8$ is a quaternion group and $\exp(W)|2$. If $a, b \in G$ are noncommuting elements of order 4, then $a^b = a^{-1}$ and by (8) we have 4.i) of the theorem. If $c, d \in G$ are involutions, then $c$ and $d$ commute with all $a \in G$ of order 4. Then $H = \langle a, d, c\rangle$ is abelian of exponent greater than 2 and $K_\lambda H$ is $f$-normal. By the condition 1) of the theorem, the factor system of $H$ is symmetric, and $u_a$ and $u_b$ commute with $u_c$.

Now prove $f(c) = \lambda_{c,c}$ for all involutions $c \in G$. Choose the elements $a$, $b$ of order 4 such that $b^a = b^{-1}$. Put $x = u_c u_a + u_b$. Since $\lambda_{a,c} = \lambda_{c,a}$ and $\lambda_{b,c} = \lambda_{c,b}$ by (2), for $x$ we obtain

$$\begin{aligned} S_f(x) = (&f(b)u_a u_b^{-1} + f(a)f(c)\lambda_{c,c}^{-1}u_b u_a^{-1} \\ &- f(b)u_b^{-1}u_a - f(a)f(c)\lambda_{c,c}^{-1}u_a^{-1}u_b)u_c = 0 \end{aligned}$$

and $f(b)\lambda_{b,b^{-1}}^{-1}\lambda_{a,b^{-1}} = f(c)f(a)\lambda_{c,c}^{-1}\lambda_{a,a^{-1}}^{-1}\lambda_{a^{-1},b}$. From this property and (9) we deduce $f(c) = \lambda_{c,c}$.

Now, suppose that either $G = E \times W$ or $G = (E \curlyvee C_4) \times W$, where $E$ is an extraspecial 2-group, $\exp(W)|2$ and $E \curlyvee C_4$ is the central product of $E$ and $C_4 = \langle c\rangle$ with $E' = \langle c^2\rangle$.

Let $a$ be an element of order 4 and $h \in C_G(\langle a\rangle)$. Then by the condition 1) of the theorem $\lambda_{a,h} = \lambda_{h,a}$.

Let $\langle a, b \mid a, b \in G\rangle$ be the quaternion subgroup of order 8. Then by 4) we obtain (5).

Now, let $G = \langle a, b\rangle \curlyvee \langle d \mid d^4 = 1\rangle$ be a subgroup of $G$ and $d^2 = a^2$. Then $a^b = a^{-1}$, and $\langle a, d\rangle$ and $\langle b, d\rangle$ are abelian subgroups of exponent not equal to 2 and by the condition 1) of the theorem, $\lambda_{a,d} = \lambda_{d,a}$ and $\lambda_{b,d} = \lambda_{d,b}$. Put $x = u_b + u_a u_d$. Since $K_\lambda G$ is $f$-normal, we obtain

$$
\begin{aligned}
&f(b)\lambda^{-1}_{b,b^{-1}}\lambda_{a,b^{-1}}u_{ab^{-1}}u_d + f(d)f(a)\lambda^{-1}_{a,a^{-1}}\lambda^{-1}_{d,d^{-1}}\lambda_{b,a^{-1}}u_{ba^{-1}}u_{d^{-1}}\\
&= f(d)f(a)\lambda^{-1}_{a,a^{-1}}\lambda^{-1}_{d,d^{-1}}\lambda_{a^{-1},b}u_{a^{-1}b}u_{d^{-1}} + f(b)\lambda^{-1}_{b,b^{-1}}\lambda_{b^{-1},a}u_{b^{-1}a}u_d
\end{aligned}
$$

and by (5)

$$
\begin{aligned}
&\lambda_{b,a}\lambda_{ab^{-1},d}u_{ab^{-1}d} + f(d)\lambda^{-1}_{d,d^{-1}}\lambda_{a,b}\lambda_{ba^{-1},d^{-1}}u_{ba^{-1}d^{-1}}\\
&= f(d)\lambda^{-1}_{d,d^{-1}}\lambda_{b,a}\lambda_{a^{-1}b,d^{-1}}u_{a^{-1}bd^{-1}} + \lambda_{a,b}\lambda_{b^{-1}a,d}u_{b^{-1}ad}.
\end{aligned}
$$

Since $d^2 \in G'$ and $a^2 = b^2$, we have $a^{-1}bd^{-1} = abd$, $ab^{-1}d = ba^{-1}d^{-1}$ and

$$
\lambda_{b,a}\lambda_{ba,d} + f(d)\lambda^{-1}_{d,d^{-1}}\lambda_{a,b}\lambda_{ab,d^{-1}} = 0.
$$

Therefore, we proved 6.i).

If $\langle a, b \mid a^4 = b^2 = 1\rangle$ is the dihedral subgroup of order 8 of $G$, then by 3) of the theorem we have (4) and $f(b) = -\lambda_{b,b}$.

Let $L = D_8 \curlyvee C_4 = \langle a, b \mid a^4 = b^2 = 1\rangle \curlyvee \langle c\rangle$. Then any $x \in K_\lambda L$ can be written as $x = x_0 + x_1 u_c$, where $x_0, x_1 \in K_\lambda D_8$. Since $K_\lambda G$ is $f$-normal, $K_\lambda L$ is $f$-normal, too, and $(x_0^f x_1 - x_1 x_0^f)u_c = (x_0 x_1^f - x_1^f x_0)u_c^f$. By the $f$-normality of $K_\lambda D_8$ $(x_0 + x_1)(x_0 + x_1)^f = (x_0 + x_1)^f(x_0 + x_1)$ and we have

$$
(x_0^f x_1 - x_1 x_0^f)u_c - (x_0 x_1^f - x_1^f x_0)u_c^f = (x_0^f x_1 - x_1 x_0^f)(u_c - u_c^f).
$$

If $x_0^f x_1 - x_1 x_0^f$ can be written as a sum of elements of form $u_a^f u_b - u_b u_a^f$ then

$$
\begin{aligned}
(x_0^f x_1 - x_1 x_0^f)(u_c - u_c^f) &= (\lambda_{b,a}\lambda_{ba,c} + f(c)\lambda^{-1}_{c,c^{-1}}\lambda_{a,b}\lambda_{ab,c^{-1}})u_{bac}\\
&\quad -(\lambda_{a,b}\lambda_{ab,c} + f(c)\lambda^{-1}_{c,c^{-1}}\lambda_{b,a}\lambda_{ba,c^{-1}})u_{abc} = 0
\end{aligned}
$$

and we have (6).

*Sufficiency.* We wish to prove that $S_f(x) = xx^f - x^f x$ is equal to 0 for all $x \in KG$. Let $x = \sum_{g\in G} \alpha_g u_g \in K_\lambda G$. It is easy to see that $S_f(x)$ is a sum of elements of the form

$$S_f(g,h) = \alpha_g\alpha_h(f(h)\lambda^{-1}_{h,h^{-1}}\lambda_{g,h^{-1}}u_{gh^{-1}} + f(g)\lambda^{-1}_{g,g^{-1}}\lambda_{h,g^{-1}}u_{hg^{-1}} \\ - f(h)\lambda^{-1}_{h,h^{-1}}\lambda_{h^{-1},g}u_{h^{-1}g} - f(g)\lambda^{-1}_{g,g^{-1}}\lambda_{g^{-1},h}u_{g^{-1}h}).$$

First, let $G$ be abelian of exponent greater than 2, and assume that the factor system of $G$ is symmetric. Then $K_\lambda G$ is commutative, and therefore, $f$-normal.

Next, suppose that $G$ is of exponent 2 and the factor system satisfies $(\lambda_{g,h} - \lambda_{h,g})(1 + f(h)\lambda^{-1}_{h,h}) = 0$ for all $g, h \in G$.
This implies $(\lambda_{g,h} - \lambda_{b,h})(f(g)\lambda^{-1}_{g,g} - f(h)\lambda^{-1}_{h,h}) = 0$ for all $g, h \in G$. Then

$$S_f(g,h) = \alpha_g\alpha_h(f(h)\lambda^{-1}_{h,h}\lambda_{g,h}u_{gh} + f(g)\lambda^{-1}_{g,g}\lambda_{h,g}u_{hg} - f(h)\lambda^{-1}_{h,h}\lambda_{h,g}u_{hg} \\ -f(g)\lambda^{-1}_{g,g}\lambda_{g,h}u_{gh}) = \alpha_g\alpha_h(f(h)\lambda^{-1}_{h,h} - f(g)\lambda^{-1}_{g,g})(\lambda_{g,h} - \lambda_{h,g})u_{gh} = 0$$

and $S_f(x) = 0$, thus, $K_\lambda G$ is $f$-normal.

Now, let $G = H \rtimes C_2$, where $H$ is an abelian group of exponent not equal to 2 and $C_2 = \langle a\rangle$ with $h^a = h^{-1}$ for all $h \in H$. Using the properties of the factor system we obtain

$$\begin{aligned} f(a)u_a^{-1}u_h &= -f(h)u_h^{-1}u_a, \qquad & f(a)u_hu_a^{-1} &= -f(h)u_au_h^{-1}, \\ u_a^f y &= -y^f u_a, \qquad & yu_a^f &= -u_ay^f \end{aligned} \tag{16}$$

for any $h \in H$ and $y \in K_\lambda H$. If $x = x_1 + x_2u_a \in K_\lambda G$ where $x_1, x_2 \in K_\lambda H$, then $x^f = x_1^f + f(a)u_a^{-1}x_2^f$ and

$$xx^f = x_1x_1^f + f(a)x_1u_a^{-1}x_2^f + x_2u_ax_1^f + f(a)x_2x_2^f.$$

Because in $K_\lambda H$ the factor system is symmetric and $K_\lambda H$ is commutative, by (16) we have

$$xx^f = x_1x_1^f + (x_2x_1 - x_1x_2)u_a + f(a)x_2x_2^f = x_1x_1^f + f(a)x_2x_2^f.$$

Similarly, $x^f x = x_1^f x_1 + f(a)x_2^f x_2$ and we conclude that $S_f(x) = 0$ and $K_\lambda G$ is $f$-normal.

Next, let $G$ be a hamiltonian 2-group. Then $G = Q_8 \times W$, where $Q_8 = \langle a, b\rangle$ is a quaternion group and $\exp(W)|2$. Suppose that the conditions 4.i)–4.ii) of the theorem are satisfied. If $H = \langle a^2, W\rangle$ then any element $x \in K_\lambda G$ can be written as

$$x = x_0 + x_1 u_a + x_2 u_b + x_3 u_{ab},$$

where $x_i \in K_\lambda H$, $(i = 0, \dots, 3)$. Since $\langle a\rangle \times H$ and $\langle b\rangle \times H$ are abelian groups of exponent 4, by the condition 1) of the theorem the elements $x_0$, $x_1$, $x_2$, $x_3$ commute with $u_a$, $u_b$ and $u_{ab}$. Since $K_\lambda H$ is $f$-normal, we have $x_i x_j^f - x_i^f x_j = x_j^f x_i - x_j x_i^f$. Using these properties we obtain

$$\begin{aligned} S_f(x) &= (x_1 x_2^f - x_1^f x_2)(\lambda_{b,a} u_{ba} - \lambda_{a,b} u_{ab}) \\ &\quad + (x_1 x_3^f - x_1^f x_3)(\lambda_{ab,a} u_b - \lambda_{a,ab} u_{b^3}) \\ &\quad + (x_2 x_3^f - x_2^f x_3)(\lambda_{ab,b} u_{a^3} - \lambda_{b,ab} u_a). \end{aligned}$$

Clearly, the element $x_i x_j^f - x_i^f x_j$ can be written as a sum of elements of form

$$S_f(c, d) = \gamma_{c,d}(f(d) u_c u_d^{-1} - f(c) u_c^{-1} u_d),$$

where $c, d \in H$. Since $H$ is an elementary 2-subgroup, by the condition 4.ii) $f(d) = \lambda_{d,d}$, $f(c) = \lambda_{c,c}$, and we obtain

$$S_f(c, d) = \gamma_{c,d}(f(d)\lambda_{d,d}^{-1}\lambda_{c,d} u_{cd} - f(c)\lambda_{c,c}^{-1}\lambda_{c,d} u_{cd}) = 0.$$

Therefore, $S_f(x) = 0$ and $K_\lambda G$ is $f$-normal.

Next, let $G = H \times W$, where $H$ is an extraspecial 2-group and $\exp(W)|2$. Since $G$ is a locally finite group, it suffices to establish the $f$-normality of all finite subgroups $H$ of $G$. Let $G$ be a finite group and $G = H \times W$, where $H$ is a finite extraspecial 2-group and $\exp(W)|2$. We know (see [4], Theorem III.13.8) that $H$ is a central product of $n$ copies of dihedral groups of order 8 or a central product of a quaternion group of order 8 and $n-1$ copies of dihedral groups of order 8. We can write $H_n = H$. Then $G = H_n \times W$ and by induction on $n$ we prove the $f$-normality of $K_\lambda G$.

If $n = 1$ then either $H_1 = Q_8$ or $H_1 = D_8$ or $H_1 = Q_8 \curlyvee C_4$. In the first and second cases the $f$-normality $K_\lambda G$ is implied by the conditions 3) or 4) of the theorem.

Let $G = Q_8 \curlyvee C_4$. Then any element $x \in K_\lambda G$ can be written as $x = x_0 + x_1 u_c$, where $x_i \in K_\lambda Q_8$, $c \in C_4$ and $c^2 \in Q_8$. From the $f$-normality of $K_\lambda Q_8$ we obtain $x_0^f x_1 - x_1 x_0^f = x_1^f x_0 - x_0 x_1^f$ and $S_f(x) = (x_0^f x_1 - x_1 x_0^f)(u_c - u_c^f)$. The element $x_0^f x_1 - x_1 x_0^f$ can be written as a sum of elements of form $\alpha(u_a^f u_b - u_b u_a^f)$, where $\alpha \in K$, $a, b \in Q_8$. We will prove $S_f(a,b) = (u_a^f u_b - u_b u_a^f)(u_c - u_c^f) = 0$ for all $a, b \in Q_8$.

If $a, b \in Q_8$ does not generate $Q_8$ then $u_a u_b = u_b u_a$ and $S_f(a,b) = 0$. Let $\langle a, b\rangle = Q_8$. Then by (5)

$$\begin{aligned} S_f(a,b) &= (\lambda_{b,a} u_{ba} - \lambda_{a,b} u_{ab})(u_c - u_c^f) \\ &= (\lambda_{b,a}\lambda_{ba,c} + f(c)\lambda_{c,c^{-1}}^{-1}\lambda_{a,b}\lambda_{ab,c^{-1}})u_{bac} \\ &\quad + (\lambda_{a,b}\lambda_{ab,c} + f(c)\lambda_{c,c^{-1}}^{-1}\lambda_{b,a}\lambda_{ba,c^{-1}})u_{abc} \end{aligned}$$

and from (6) $S_f(a,b) = 0$.

It is easy to see $D_8 \curlyvee D_8 \cong Q_8 \curlyvee Q_8$, and $H_n$ $(n > 1)$ can be written as $Q_8 \curlyvee H_{n-1}$.

Let $Q_8 = \langle a, b\rangle$ and $L = W \times H_{n-1}$. Then any element $x \in K_\lambda G$ can be written as

$$x = x_0 + x_1 u_a + x_2 u_b + x_3 u_a u_b,$$

where $x_i \in K_\lambda L$. By 6.i) the $x_i$ commute with $u_a$ and $u_b$. Since $\langle a, b\rangle$ is a quaternion group of order 8, by the condition 6.ii) of the theorem we have $u_a u_b = u_b^f u_a = u_b u_a^f$. Hence,

$$\begin{aligned} S_f(x) = {} & (x_0x_1^f - x_1^f x_0)u_a^f + (x_0x_2^f - x_2^f x_0)u_b^f + (x_0x_3^f - x_3^f x_0)u_b^f u_a^f \\ & + (x_1x_0^f - x_0^f x_1)u_a + (x_1x_2^f - x_1^f x_2)u_a u_b^f + (x_1x_3^f - x_1^f x_3)u_b f(a) \\ & + (x_2x_0^f - x_0^f x_2)u_b + (x_2x_1^f - x_2^f x_1)u_a u_b + (x_2x_3^f - x_2^f x_3)u_a^f f(b) \\ & + (x_3x_0^f - x_0^f x_3)u_a u_b + (x_3x_1^f - x_3^f x_1)u_a u_{ab} \\ & + (x_3x_2^f - x_3^f x_2)u_a f(b). \end{aligned} \tag{17}$$

Since by induction $K_\lambda L$ is $f$-normal, $(x_i + x_j)(x_i + x_j)^f = (x_i + x_j)^f (x_i + x_j)$ implies $x_i x_j^f - x_i^f x_j = x_j^f x_i - x_j x_i^f$ and $x_i x_j^f - x_j^f x_i =$

$x_i^f x_j - x_j x_i^f$. Therefore, by (17)

$$\begin{aligned}
S_f(x) &= (x_0x_1^f - x_1^f x_0)(u_a^f - u_a) + (x_0x_2^f - x_2^f x_0)(u_b^f - u_b)\\
&\quad + (x_0x_3^f - x_3^f x_0)(u_a^f - u_a)u_b + (x_1x_2^f - x_1^f x_2)u_a(u_b^f - u_b)\\
&\quad + (x_1x_3^f - x_1^f x_3)u_a(u_b - u_b^f)f(a) + (x_2x_3^f - x_2^f x_3)(u_a^f - u_a)f(b).
\end{aligned}$$

Clearly, the element $x_i x_j^f - x_j^f x_i$ can be written as a sum of elements of form $S_f(c,d) = \gamma_{c,d}(u_c u_d^f - u_d^f u_c)$, where $c, d \in L$, $\gamma_{c,d} \in K$. We will prove $S_f(c,d,a) = (u_c u_d^f - u_d^f u_c)(u_a - u_a^f) = 0$ for any $c, d \in L$.

We consider the following cases:

*Case 1).* Let $[c,d] = 1$. Then $L = \langle c, d, a\rangle$ is abelian with $\exp(L) \neq 2$, and by 6.i) the factor system is symmetric and $S_f(c,d,a) = 0$.

*Case 2).* Let $\langle c, d\rangle = Q_8$. Then by 6.ii) (5) holds and $(u_c u_d^f - u_d^f u_c)(u_a - u_a^f) = (\lambda_{d,c}\lambda_{dc,a} + f(a)\lambda_{a,a^{-1}}^{-1}\lambda_{c,d}\lambda_{cd,a^{-1}})u_{dca} - (\lambda_{c,d}\lambda_{cd,a} + f(a)\lambda_{a,a^{-1}}^{-1}\lambda_{d,c}\lambda_{a^{-1},dc})u_{cda}$. Now by 6.ii) the property (6) is satisfied and we conclude $S_f(c,d,a) = 0$.

*Case 3).* Let $\langle c, d\rangle = D_8$ and $c^4 = d^2 = 1$. Then by 6.iii) $f(d) = -\lambda_{d,d}$ and by (4)

$$\begin{aligned}
(u_c u_d^f - u_d^f u_c)(u_a - u_a^f) &= (\lambda_{d,c}u_{dc} - \lambda_{c,d}u_{cd})(u_a - u_a^f)\\
&= (\lambda_{c,d}\lambda_{cd,a} + f(a)\lambda_{a,a^{-1}}^{-1}\lambda_{dc,a^{-1}}\lambda_{d,c})u_{cda}\\
&\quad + (\lambda_{d,c}\lambda_{dc,a} + f(a)\lambda_{a,a^{-1}}^{-1}\lambda_{cd,a^{-1}}\lambda_{c,d})u_{dca}.
\end{aligned}$$

Now by 6.ii) we have (6) and we conclude $S_f(c,d,a) = 0$.

*Case 4).* Let $\langle c, d\rangle = D_8$ and $d^4 = c^2 = 1$. Then by (4)

$$\begin{aligned}
u_c u_d^f - u_d^f u_c &= f(d)\lambda_{d,d^{-1}}^{-1}\lambda_{c,d^{-1}}u_{dc} - f(d)\lambda_{d,d^{-1}}^{-1}\lambda_{d^{-1},c}u_{cd}\\
&= \lambda_{d,c}u_{dc} - \lambda_{c,d}u_{cd}.
\end{aligned}$$

Similarly to the case 3) we have $S_f(c,d,a) = 0$.

*Case 5).* Let $\langle c, d\rangle = D_8$ and $d^2 = c^2 = 1$. Then by 6.iii) $f(d) = -\lambda_{d,d}$. In $\langle c, d\rangle$ we choose a new generator system $\{a_1, b_1 \mid a_1^4 = b_1^2 =$

$1, a_1^{b_1} = a_1^{-1}\}$ such that $c = b_1$ and $d = a_1^i b_1$, where $i = 1$ or $3$. Then $a^2 = a_1^2$ and

$$(u_c u_d^f - u_d^f u_c)(u_a - u_a^f) = (u_d u_c - u_c u_d)(u_a - u_a^f)$$
$$= \lambda_{a_1^i, b_1}^{-1}(u_{a_1^i} u_{b_1} - u_{b_1} u_{a_1^i})(u_a - u_a^f)u_{b_1}.$$

As in the Case 3) it is easy to see $(u_{a_1^i} u_{b_1} - u_{b_1} u_{a_1^i})(u_a - u_a^f) = 0$ and $S_f(c, d, a) = 0$.

Analogously, the element $x_i x_j^f - x_i^f x_j$ can be written as a sum of elements of form $\gamma_{c,d}(u_c u_d^f - u_c^f u_d)$, where $c, d \in L$. Let us prove that if $c, d \in L$, then $S_f(c, d, a) = (u_c u_d^f - u_c^f u_d)(u_a - u_a^f) = 0$.

Let $z \in L$, $a \in Q_8$ be commuting elements of order 4 with $z^2 = a^2$. First, we will prove that $K$ is of characteristic 2, then $(u_z + u_z^f)(u_a + u_a^f) = 0$.

Indeed,

$$(u_z + u_z^f)(u_a + u_a^f) = (\lambda_{z,a} + f(z)\lambda_{z,z^{-1}}^{-1} f(a)\lambda_{a,a^{-1}}^{-1}\lambda_{z^{-1},a^{-1}})u_{za}$$
$$+(f(a)\lambda_{a,a^{-1}}^{-1}\lambda_{z,a^{-1}} + f(z)\lambda_{z,z^{-1}}^{-1}\lambda_{z^{-1},a})u_{za^3}.$$

First let $za$ be a noncentral element of order 2. Then by 6.iii) $f(za) = \lambda_{za,za}$. Since $((u_z u_a)u_a)u_{a^3} = u_z(u_a(u_a u_{a^3}))$ we conclude that

$$\lambda_{z,a}\lambda_{za,a}\lambda_{za^2,a^3} = \lambda_{z,a}\lambda_{a,1}\lambda_{a,a^{-1}}$$

and $\lambda_{a,a^{-1}}^{-1} = \lambda_{z^3,a^3}^{-1}\lambda_{za,a}^{-1}$. Clearly, $f(z)f(a) = f(za)\lambda_{z,a}^2 = \lambda_{za,za}\lambda_{z,a}^2$ and

$$\begin{aligned}
&\lambda_{z,a} + f(z)\lambda_{z,z^{-1}}^{-1} f(a)\lambda_{a,a^{-1}}^{-1}\lambda_{z^{-1},a^{-1}} \\
&\qquad = \lambda_{z,a}(1 + (\lambda_{za,az}\lambda_{a,z})\lambda_{z,z^{-1}}^{-1}\lambda_{a,a^{-1}}^{-1}\lambda_{z^{-1},a^{-1}}) \\
&\qquad = \lambda_{z,a}(1 + \lambda_{z,za^2}\lambda_{a,az}\lambda_{a,a^{-1}}^{-1}\lambda_{z,za^2}^{-1}\lambda_{z^{-1},a^{-1}} \\
&\qquad = \lambda_{z,a}(1 + (\lambda_{a,az}\lambda_{zaa,a^{-1}})\lambda_{a,a^{-1}}^{-1}) \\
&\qquad = \lambda_{z,a}(1 + \lambda_{za,aa^{-1}}\lambda_{a,a^{-1}}\lambda_{a,a^{-1}}^{-1}) = 2\lambda_{z,a} = 0.
\end{aligned} \tag{18}$$

By (1) we have

$$(\lambda_{z,a^{-1}}\lambda_{za^{-1},za^{-1}})\lambda_{z^{-1},a} = \lambda_{z,a^{-1}za^{-1}}\lambda_{a^{-1},z^{-1}a}\lambda_{z^{-1},a}$$
$$= \lambda_{z,z^{-1}}(\lambda_{a^{-1},az^{-1}}\lambda_{a,z^{-1}}) = \lambda_{z^{-1},z}\lambda_{aa^{-1},z^{-1}}\lambda_{a,a^{-1}} = \lambda_{z^{-1},z}\lambda_{a,a^{-1}}$$

and since $az^{-1}$ has order 2, $f(az^{-1}) = \lambda_{az^{-1},az^{-1}}$, and we obtain

$$\begin{aligned}
&f(a^{-1})^{-1}f(a^{-1})(f(a)\lambda^{-1}_{a,a^{-1}}\lambda_{z,a^{-1}} + f(z)\lambda^{-1}_{z,z^{-1}}\lambda_{z^{-1},a})\\
&\quad= f(a^{-1})^{-1}(\lambda^{2}_{a,a^{-1}}\lambda^{-1}_{a,a^{-1}}\lambda_{z,a^{-1}} + f(az^{-1})\lambda^{2}_{a^{-1},z}\lambda^{-1}_{z,z^{-1}}\lambda_{z^{-1},a})\\
&\quad= f(a^{-1})^{-1}(\lambda_{a,a^{-1}}\lambda_{z,a^{-1}} + \lambda_{z,a^{-1}}(\lambda_{z,a^{-1}}\lambda_{az^{-1},az^{-1}}\lambda_{z^{-1},a})\lambda^{-1}_{z,z^{-1}}) \qquad (19)\\
&\quad= f(a^{-1})^{-1}(\lambda_{z,a^{-1}}(\lambda_{a^{-1},a} - \lambda_{z,z^{-1}}\lambda_{a^{-1},a}\lambda^{-1}_{z^{-1},z}\lambda_{z^{-1},a})\\
&\quad= 2f(a^{-1})^{-1}\lambda_{z,a^{-1}}\lambda_{a,a^{-1}} = 0.
\end{aligned}$$

Clearly, if $[c,d]=1$ then $S_f(c,d,a)$ can be written as

$$\begin{aligned}
S_f(c,d,a) &= (u_c u_d^f + (u_d^f u_c)^f)(u_a - u_a^f)\\
&= f(d)\lambda_{d,d^{-1}}\lambda_{c,d^{-1}}(u_{cd^{-1}} - u^f_{cd^{-1}})(u_a - u_a^f).
\end{aligned} \qquad (20)$$

Similarly, the element $x_i x_j^f - x_i^f x_j$ can be written as a sum of elements of form $\gamma_{c,d}(u_c u_d^f - u_c^f u_d)$, where $c,d \in L$. Now let us prove $S_f(c,d,a) = (u_c u_d^f - u_c^f u_d)(u_a - u_a^f) = 0$, where $c,d \in L$.

We consider the following cases:

*Case 1).* Let $[c,d]=1$, $c^2=d^2=1$ and $c,d \notin \zeta(G)$. Then $S = \langle c,d,a\rangle$ is abelian of exponent greater that 2 and by 6.i) the factor system of $S$ is symmetric. We know that in $L$ every element of order 2 is either central or coincides with a noncentral element of some dihedral subgroup of order 8. Since $c,d \notin \zeta(G)$, we have $f(c) = \lambda_{c,c}$ and $f(d) = \lambda_{d,d}$ and

$$S_f(c,d,a) = \lambda_{c,d}(f(d)\lambda^{-1}_{d,d} - f(c)\lambda^{-1}_{c,c})u_{cd}(u_a - u_a^f) = 0.$$

*Case 2).* Let $[c,d]=1$, $c^2=d^2=1$ and $c,d \in \zeta(G)$. Then $c=d=a^2$ and $S_f(c,d,a)=0$.

*Case 3).* Let $[c,d]=1$, $c^2=d^2=1$ and $c \in \zeta(G)$, $d \notin \zeta(G)$. Then $f(d) = \lambda^{-1}_{d,d}$, $c=a^2$ and

$$\begin{aligned}
S_f(c,d,a) &= -u_d(u_{a^2} + u^f_{a^2})(u_a - u_a^f)\\
&= -u_d(\lambda_{a,a^2}u_{a^{-1}} - f(a)\lambda^{-1}_{a,a^{-1}}u_a)(1 + f(a^2)\lambda^{-1}_{a^2,a^2}).
\end{aligned}$$

Since $K$ is an integral domain of characteristic 2 and $f^2(a^2) = \lambda^2_{a^2,a^2}f(a^4) = \lambda^2_{a^2,a^2}$, we conclude $f(a^2) = \pm\lambda_{a^2,a^2}$ and $S_f(c,d,a)=0$.

*Case 4).* Let $[c,d]=1$, $d^2=1$ and suppose that $c$ has order 4. Then $dc$ has order 4 and by (20) $S_f(c,d,a)=0$.

*Case 5).* Let $[c,d]=1$ with $c,d$ of order 4. Then $d^2=c^2=a^2$,

$$S_f(c,d,a)=(f(d)\lambda^{-1}_{d,d^{-1}}\lambda_{c,d^{-1}}+f(c)\lambda^{-1}_{c,c^{-1}}\lambda_{c^{-1},d})u_{cd^{-1}}(u_a-u_a^f),$$

and by (19) we have $S_f(c,d,a)=0$.

*Case 6).* Let $\langle c,d\rangle$ be a quaternion group of order 8. Then by 6.ii) (5) holds and

$$\begin{aligned}u_cu_d^f-u_c^fu_d&=(f(d)\lambda^{-1}_{d,d^{-1}}\lambda_{c,d^{-1}}-f(c)\lambda^{-1}_{c,c^{-1}}\lambda_{c^{-1},d})u_{c^{-1}d}\\&=(\lambda_{d,c}-\lambda_{d,c})u_{c^{-1}d}=0.\end{aligned}$$

*Case 7).* Let $\langle c,d\rangle$ be a dihedral group of order 8. If $c^2\neq 1$ then $f(d)=\lambda_{d,d}$ and

$$\begin{aligned}S_f(c,d,a)&=(\lambda_{c,d}u_{cd}+f(c)\lambda^{-1}_{c,c^{-1}}\lambda_{c^{-1},d}u_{dc})(u_a-u_a^f)\\&=(\lambda_{c,d}u_{cd}+\lambda_{d,c}u_{dc})(u_a-u_a^f)=(\lambda_{c,d}\lambda_{cd,a}+f(a)\lambda_{a,a^{-1}}\lambda_{d,c}\lambda_{dc})u_{acd}\\&\quad-(\lambda_{d,c}\lambda_{dc,a}+f(a)\lambda_{a,a^{-1}}\lambda_{c,d}\lambda_{cd,a^{-1}})u_{adc}.\end{aligned}$$

By (6) we obtain $S_f(c,d,a)=0$.

*Case 8).* Let $\langle c,d\rangle$ be a dihedral group of order 8 and $c^2=d^2=1$. Then $f(d)=\lambda_{d,d}$, $f(c)=\lambda_{c,c}$ and $S_f(c,d,a)=2u_cu_d(u_a-u_a^f)=0$. □

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