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Covering theorems for Artinian rings

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Abstract. The covering properties of Artinian rings which depend on their additive structure only, are investigated.

1. Introduction

For simplicity, it is convenient to introduce the following notation. A set S is said to be the proper union of the sets $S_1, \dots, S_n$ if

$$\bigcup_{i=1}^{n} S_i = S \quad \text{and} \quad \bigcup_{i \neq k} S_i \neq S,$$

for all $k = 1, \dots, n$. Generalizing some earlier results of [1] and [2], like a field is not a proper union of subfields, Ôhori [3] proved that if a unitary ring A contains a unitary subring B such that $B/\mathfrak{J}(B)$, where $\mathfrak{J}(B)$ is the Jacobson radical of B, is an infinite (left) Artinian simple ring then A is not a proper union of rings. As it was remarked by the reviewer of [3] (see [4]) the word "Artinian" can be deleted by using a theorem of Lewin [5].

The purpose of this note is to point out that the covering properties of Artinian rings depend on their additive structure and in case of fields the multiplicative structure can be treated as well.

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2. Results

Theorem 1. *An Artinian ring is not a proper union of additive subgroups if and only if its additive group is a direct sum of a divisible group and a finite cyclic group.*

Corollary 1. *An Artinian ring is not a proper union of cosets if and only if its additive group is a divisible group.*

Theorem 2. *A ring with minimal condition for principal left ideals is not a proper union of cosets if and only if its additive group is a direct sum of a divisible group and a torsion group which has no subgroup of finite index.*

Corollary 2. *A ring with minimal condition for principal left ideals is not a proper union of additive subgroups if and only if its additive group is a direct sum of a divisible group, and a torsion group such that every finite factorgroup of it is cyclic.*

Theorem 3. *Let R be an infinite skew field and $\{H_1, H_2, \dots, H_t\}$ be a family of its proper subskew fields. Then*

(i) *the additive group of R cannot be covered by finitely many cosets of the additiv subgroups of $H_1, \dots, H_t$;*

(ii) *the group of units of R cannot be covered by finitely many cosets of the unit subgroups of $H_1, \dots, H_t$.*

Theorem 4. *The group of units of a field is a proper union of subsemigroups if and only if the field is not an algebraic extension of a finite field.*

Remark. *As it was pointed out by I. Ruzsa the polynomial ring $\mathbb{Z}[x]$ is a proper union of the following three rings:*

$$S_1 = \{f(x) \in \mathbb{Z}[x] \mid f(0) \text{ is even}\}, \quad S_2 = \{f(x) \in \mathbb{Z}[x] \mid f(1) \text{ is even}\},$$
$$S_3 = \{f(x) \in \mathbb{Z}[x] \mid f(0) + f(1) \text{ is even}\}.$$

3. Preliminaries

Lemma 1. *Let $H_1, H_2, \dots, H_t$ be subgroups of the group G. If G is covered by finite number of cosets of the H_i then at least one of these subgroups has finite index.*

Proof. We use induction on the number of the subgroups. The statement is evident if $t = 1$ and assume its truth for $t - 1$.

We may suppose that H_t has infinite index. Then there exists a coset $H_t g$ which is not in the cover. Hence $H_t g$ is covered by finite number of cosets of $H_1, H_2, \dots, H_{t-1}$. If these cosets are multiplied by g^{-1}, a cover of H_t is obtained. Thus we can construct a new cover of G with finite number of cosets of $H_1, H_2, \dots, H_{t-1}$, and by the inductive hypothesis Lemma 1 follows. $\square$

Lemma 2 ([6], [7]). *A group is the additive group of an Artinian ring if and only if it has the form*

$$\bigoplus_{\mathfrak{M}} \mathbb{Q} \oplus \bigoplus_{\text{finite}} C_{p_i^\infty} \oplus \bigoplus_{\mathfrak{N}} C_{q_j^{k_j}},$$

where p_i, q_i are prime numbers, $\mathfrak{N}$, and $\mathfrak{M}$ are arbitrary cardinals and the factors $q_j^{k_j}$ are divisors of a fixed natural number m.

Lemma 3 ([8], [9]). *A group is the additive group of a ring with minimal condition for principal left ideals if and only if its additive group is a direct sum of a divisible group and a torsion group.*

Lemma 4. *Let $\{G_\gamma \mid \gamma \in \Gamma\}$ be a family of abelian groups. If G_γ is not a proper union of finitely many cosets for every γ, then $G = \oplus_{\gamma\in\Gamma} G_\gamma$ is also not a proper union of finitely many cosets.*

Proof. To prove it by transfinite induction we have two cases to distinguish. If Γ is not a limit ordinal, that is, $\Gamma = \Gamma' + 1$ with some Γ' and for Γ' the statement is true. Then we get $G = G_\gamma \oplus G'$, where $G' = \bigoplus_{\gamma'\in\Gamma'} G_{\gamma'}$. Let S be a coset of G with respect to a subgroup H such that $b + G_\gamma \subseteq S$ with some $b \in G'$. Then $G_\gamma \subset H$ and S has the form $S = G_\gamma + S'$, where S' is a proper coset of G'.

Suppose that G is a proper union of the cosets $S_1, \cdots, S_n$. If S_l contains a coset of the form $b + G_\gamma$ then it can be written as $G_\gamma + S_l'$; otherwise, S_l' is the empty set. By induction

$$\bigcup_{l=1}^{n} S_l' \neq G',$$

therefore, there is a $d \in G'$, such that $d + G_\gamma$ is not contained in any S_l'. Moreover, if $(d + G_\gamma) \cap S_l$ is not empty then it contains an $r_l + d$ and $S_l = r_l + d + G_l$, where $r_l \in G_\gamma$ and G_l is a subgroup of G. The relations

$$S_l \cap (d + G_\gamma) = (r_l + d + G_l) \cap (r_l + d + G_\gamma) = (r_l + d) + G_l \cap G_\gamma$$

and

$$d + G_\gamma = \bigcup_{l=1}^{n} S_l \cap (d + G_\gamma)$$

imply that G_γ is a proper union of some of the cosets $r_l + (G_l \cap G_\gamma)$, which contradicts.

In the second case Γ is a limit ordinal. For a $\Gamma' < \Gamma$ set

$$G_{\Gamma'} = \bigoplus_{\alpha \in \Gamma'} G_\alpha.$$

Assuming G is a proper union of the cosets $T_1, \cdots, T_k$ we obtain

$$G_{\Gamma'} = \bigcup_{l=1}^{k} (G_\Gamma \cap T_l).$$

Since $G_{\Gamma'} \cap T_l$ is also a coset in G_γ, this union cannot be a proper one, that is, for every $\Gamma' < \Gamma$, $G_{\Gamma'}$ belongs to one of the cosets T_l, $1 \leq l \leq k$, which is obviously impossible. $\square$

Proof of Theorem 1. Let A be the additive group of an Artinian ring. According to Lemma 2, if the non-divisible part $\oplus_{\mathfrak{N}} C_{p_i^{k_i}}$ of A contains a direct summand $C_{p_i^k}$ at least twice, then $A = L \oplus C_{p^k} \oplus C_{p^l}$ and if $k \leq l$, $C_{p^k} = \langle a \rangle$, $C_{p^l} = \{b_1, b_2, \ldots, b_{p^l}\}$ and $A_i = \langle L, ab_i \rangle$, therefore, A is a proper union of the subgroups $A_1, A_2, \dots, A_{p^l}$. Furthermore, $\oplus_{\mathfrak{N}} C_{p_i^{k_i}}$ is a finite cyclic group.

A quasycyclic group and the additive group of $\mathbb{Q}$ have no maximal subgroups, hence by Lemma 1 they are not a proper union of cosets. Applying Lemma 2 we may assume that the maximal divisible subgroup B of A is not a proper union of cosets. Clearly, the finite cyclic group C is not a proper union of subgroups. It yields that $A = B \oplus C$ is also not a proper union of subgroups. $\square$

One can repeat the argument detailed above to prove Theorem 2.

Proof of Theorem 3. (i) If R is covered by finitely many cosets of the additive subgroups $H_1, H_2, \dots, H_t$ then by Lemma 1 there exists a subgroup $H = H_i$ of finite index in the additive group of R. Let

$$R = a_1 + H \cup a_2 + H \cup \ldots \cup a_s + H$$

be a decomposition of R with respect to H. Then H is an infinite subskew field and in the infinite set $\{a_i + a_j\lambda \mid 0 \neq \lambda \in H\}$ there exist two different $a_i + a_j\lambda_1$ and $a_i + a_j\lambda_2$, which belong to the same coset $a_k + H$. Then $a_j(\lambda_1 - \lambda_2) \in H$ and $a_j \in H$, which is impossible.

(ii) Let the group of units $U(R)$ be covered by finitely many cosets of the multiplicative subgroups $U(H_1), U(H_2), \dots, U(H_t)$. Then by Lemma 1 there exists a subgroup $H = U(H_i)$ of finite index in the group of units $U(R)$ and we have the decomposition

$$U(R) = a_1H \cup a_2H \cup \ldots \cup a_sH.$$

Then H is an infinite subskew field and in the infinite set $\{a_i + a_j\lambda \mid 0 \neq \lambda \in H\}$ there exist two different $a_i + a_j\lambda_1$ and $a_i + a_j\lambda_2$, which belong to the same coset a_kH. Therefore, $a_i + a_j\lambda_2 = (a_i + a_j\lambda_1)\lambda_3$ and we obtain

$$a_i(1 - \lambda_3) = a_j(\lambda_1\lambda_3 - \lambda_2),$$

and $1 - \lambda_3, \lambda_1\lambda_3 - \lambda_2 \in H$, which is impossible. $\square$

Proof of Theorem 4. Let F be a field with multiplicative group $U(F)$. If F is an algebraic extension of a finite field F_0 and $U(F)$ is a proper union of the subsemigroups $M_1, \dots, M_n$, then there are elements $m_i \in M_i$ with

$$m_i \notin \bigcup_{l \neq i} M_l,$$

where $i = 1, \dots, n$ furthermore, the multiplicative group of $F_0(m_1, \dots, m_n)$ is a proper union of the groups $M_i \cap F_0(m_1, \dots, m_n)$, $(i = 1, \dots, n)$. However, $F_0(m_1, \dots, m_n)$ is a finite field having cyclic multiplicative group, which cannot be a proper union.

If F is not an algebraic extension of a finite field then $U(F)$ contains two multiplicatively independent elements denoted by z_1 and z_2. Indeed, if $\mathrm{char}(F) = 0$ then one can take $z_1 = 2$ and $z_2 = 3$, say; and if F has a transcendental element τ (over a finite ground field contained in F), then put $z_1 = \tau$ and $z_2 = \tau + 1$. Let G be a multiplicatively independent generating set for $U(F)$ containing z_1 and z_2. Moreover, for a $z \in U(F)$ let $e_i(z)$ $(i = 1, 2)$ denote the exponent of z_i $(i = 1, 2)$ in the expression of z as a product of generators from G. The lattice $\mathbb{Z} \oplus \mathbb{Z}$ is a proper union of the lattices L_1, L_2 and L_3 spanned by

$$\{(1,0),(1,2)\},\ \{(0,1),(2,1)\},\ \{(1,1),(-1,1)\}$$

respectively, hence $U(F)$ is a proper union of the subsemigroups

$$\{z \in U(F) \mid (e_1(z), e_2(z)) \in L_i\},$$

where $i = 1, 2, 3$. $\square$

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