Publ. Math. Debrecen
57 / 1-2 (2000), 231–239

On elements in algebras having finite number of conjugates

By VICTOR BOVDI (Debrecen)

Dedicated to Professor Kálmán Győry on his 60th birthday

Abstract. Let R be a ring with unity and $U(R)$ its group of units. Let $\Delta U = \{a \in U(R) \mid [U(R) : C_{U(R)}(a)] < \infty\}$ be the FC-radical of $U(R)$ and let $\nabla(R) = \{a \in R \mid [U(R) : C_{U(R)}(a)] < \infty\}$ be the FC-subring of R.

An infinite subgroup H of $U(R)$ is said to be an ω-subgroup if the left annihilator of each nonzero Lie commmutator $[x, y]$ in R contains only finite number of elements of the form $1 - h$, where $x, y \in R$ and $h \in H$. In the case when R is an algebra over a field F, and $U(R)$ contains an ω-subgroup, we describe its FC-subalgebra and the FC-radical. This paper is an extension of [1].

1. Introduction

Let R be a ring with unity and $U(R)$ its group of units. Let

$$\Delta U = \{a \in U(R) \mid [U(R) : C_{U(R)}(a)] < \infty\},$$

and

$$\nabla(R) = \{a \in R \mid [U(R) : C_{U(R)}(a)] < \infty\},$$

which are called the FC-radical of $U(R)$ and FC-subring of R, respectively. The FC-subring $\nabla(R)$ is invariant under the automorphisms of R and contains the center of R.

Mathematics Subject Classification: Primary: 16U50, 16U60, 20C05; Secondary: 16N99.
Key words and phrases: units, FC-elements, finite conjugacy.
The research was supported by OTKA T 025029 and T 029132.

The investigation of the FC-radical ΔU and the FC-subring $\nabla(R)$ was proposed by S. K. Sehgal and H. Zassenhaus [8]. They described the FC-subring of a $\mathbb{Z}$-order as a unital ring with a finite $\mathbb{Z}$-basis and a semisimple quotient ring.

Definition. An infinite subgroup H of $U(R)$ is said to be an ω-subgroup if the left annihilator of each nonzero Lie commmutator $[x, y] = xy - yx$ in R contains only a finite number of elements of the form $1 - h$, where $h \in H$ and $x, y \in R$.

The groups of units of the following infinite rings R contain ω-subgroups, of course:

1. Let A be an algebra over an infinite field F. Then the subgroup $U(F)$ is an ω-subgroup.
2. Let $R = KG$ be the group ring of an infinite group G over the ring K. It is well-known (see [6], Lemma 3.1.2, p. 68) that the left annihilator of any $z \in KG$ contains only a finite number of elements of the form $g - 1$, where $g \in G$. Thus G is an ω-subgroup.
3. Let $R = F_\lambda G$ be an infinite twisted group algebra over the field F with an F-basis $\{u_g \mid g \in G\}$. Then the subgroup $\overline{G} = \{\lambda u_g \mid \lambda \in U(F), g \in G\}$ is an ω-subgroup.
4. If A is an algebra over a field F, and A contains a subalgebra D such that $1 \in D$ and D is either an infinite field or a skewfield, then every infinite subgroup of $U(D)$ is an ω-subgroup.

2. Results

In this paper we study the properties of the FC-subring $\nabla(R)$ when R is an algebra over a field F and $U(R)$ contains an ω-subgroup. We show that the set of algebraic elements A of $\nabla(R)$ is a locally finite algebra, the Jacobson radical $\mathfrak{J}(A)$ is a central locally nilpotent ideal in $\nabla(R)$ and $A/\mathfrak{J}(A)$ is commutative. As a consequence, we describe the FC-radical ΔU, which is a solvable group of length at most 3, and the subgroup $t(\Delta U)$ is nilpotent of class at most 2. If F is an infinite field then any algebraic unit over F belongs to the centralizer of $\nabla(R)$, and, as a consequence, we obtain that $t(\Delta U)$ is abelian and ΔU is nilpotent of class at most 2. These results are extensions of the results obtained by the author in [1] for groups of units of twisted group algebras.

By the Theorem of B. H. Neumann [5], elements of finite order in ΔU form a normal subgroup, which we denote by $t(\Delta U)$, and the factor group $\Delta U/t(\Delta U)$ is a torsion free abelian group. If x is a nilpotent element of the ring R, then the element $y = 1 + x$ is a unit in R, which is called the unipotent element of $U(R)$.

Let $\zeta(G)$ be the center of G and $(g,h) = g^{-1}h^{-1}gh$, where $g, h \in G$.

Lemma 1. *Assume that $U(R)$ has an ω-subgroup. Then all nilpotent elements of the subring $\nabla(R)$ are central in $\nabla(R)$.*

Proof. Let x be a nilpotent element of $\nabla(R)$. Then $x^k = 0$, and by induction on k we shall prove that $vx = xv$ for all $v \in \nabla(R)$.

Choose an infinite ω-subgroup H of $U(R)$. By Poincare's Theorem the centralizer S of the subset $\{v, x\}$ in H is a subgroup of finite index in H. Since H is infinite, S is infinite and $fx = xf$ for all $f \in S$. Then xf is nilpotent and $1 + xf$ is a unit in $U(R)$. Since $v \in \nabla(R)$, the set $\{(1 + xf)^{-1}v(1 + xf) \mid f \in S\}$ is finite. Let $v_1, \dots, v_t$ be all the elements of this set and

$$W_i = \{f \in S \mid (1 + xf)^{-1}v(1 + xf) = v_i\}.$$

Obviously, $S = \bigcup W_i$ and there exists an index j such that W_j is infinite. Fix an element $f \in W_j$. Then any element $q \in W_j$ such that $q \neq f$ satisfies

$$(1 + xf)^{-1}v(1 + xf) = (1 + xq)^{-1}v(1 + xq)$$

and

$$v(1 + xf)(1 + xq)^{-1} = (1 + xf)(1 + xq)^{-1}v.$$

Then

$$v\{(1 + xq) + (xf - xq)\}(1 + xq)^{-1} = \{(1 + xq) + (xf - xq)\}(1 + xq)^{-1}v,$$

$$v(1 + x(f - q)(1 + xq)^{-1}) = (1 + x(f - q)(1 + xq)^{-1})v$$

and

$$vx(f - q)(1 + xq)^{-1} = x(f - q)(1 + xq)^{-1}v. \tag{1}$$

Let $xv \neq vx$ and $k = 2$. Then $x^2 = 0$ and $(1 + xq)^{-1} = 1 - xq$. Since f and q belong to the centralizer of the subset $\{x, v\}$, from (1) we have

$$(f - q)vx(1 - xq) = (f - q)x(1 - xq)v,$$

whence $(f-q)(vx-vx^2q-xv+x^2qv)=0$ and evidently $(f-q)(vx-xv)=0$. Therefore, $(q^{-1}f-1)(vx-xv)=0$ for any $q\in W_j$. Since $q^{-1}W_j$ is an infinite subset of the ω-subgroup H, we obtain a contradiction, and thus $vx=xv$.

Let $k>2$. If $i\geq 1$ then x^{i+1} is nilpotent of index less than k, thus applying an induction on k, first we obtain that $x^{i+1}v=vx^{i+1}$ and then

$$x(f-q)x^iq^iv=(f-q)x^{i+1}q^iv=(f-q)vx^{i+1}q^i=vx(f-q)x^iq^i.$$

Hence

$$\begin{aligned} &vx(f-q)(1-xq+x^2q^2+\cdots+(-1)^{k-1}x^{k-1}q^{k-1})\\ &=x(f-q)((1-xq)v+(x^2q^2+\cdots+(-1)^{k-1}x^{k-1}q^{k-1})v). \end{aligned}$$

and $(f-q)(vx-xv)=0$. As before, we have a contradiction in the case $xv\neq xv$.

Thus nilpotent elements of $\nabla(R)$ are central in $\nabla(R)$. □

Lemma 2. *Let R be an algebra over a field F such that the group of units $U(R)$ contains an ω-subgroup. Then the radical $\mathfrak{J}(A)$ of every locally finite subalgebra A of $\nabla(R)$ consists of central nilpotent elements of the subalgebra $\nabla(R)$, and $A/\mathfrak{J}(A)$ is a commutative algebra.*

Proof. Let $x\in\mathfrak{J}(A)$, then $x\in L$ for some finite dimensional subalgebra L of A. Since L is left Artinian, Proposition 2.5.17 in [7] (p. 185) ensures that $L\cap\mathfrak{J}(A)\subseteq\mathfrak{J}(L)$, moreover $\mathfrak{J}(L)$ is nilpotent. Now $x\in\mathfrak{J}(L)$ implies that x is nilpotent and the application of Lemma 1 gives that x belongs to the center of $\nabla(R)$. Then Theorem 48.3 in [4] (p. 209) will enable us to verify the existence of the decomposition into the direct sum

$$L=Le_1\oplus\cdots\oplus Le_n\oplus N,$$

where Le_i is a finite dimensional local F-algebra (i.e. $Le_i/\mathfrak{J}(Le_i)$ is a division ring), N is a commutative artinian radical algebra, and $e_1,\dots,e_n$ are pairwise orthogonal idempotents. Since nilpotent elements of $\nabla(R)$ belong to the center of $\nabla(R)$, by Lemma 13.2 of [4] (p. 57) any idempotent e_i is central in L and the subring Le_i of $\nabla(R)$ is also an FC-ring, whence $\mathfrak{J}(Le_i)$ is a central nilpotent ideal.

Suppose that $Le_i/\mathfrak{J}(Le_i)$ is a noncommutative division ring. Then $1+\mathfrak{J}(Le_i)$ is a central subgroup and

$$U(Le_i)/(1+\mathfrak{J}(Le_i)) \cong U(Le_i/\mathfrak{J}(Le_i)).$$

Applying Herstein's Theorem [2] we establish that a noncentral unit of $Le_i/\mathfrak{J}(Le_i)$ has an infinite number of conjugates, which is impossible. Therefore, $L/\mathfrak{J}(L)$ is a commutative algebra and from $\mathfrak{J}(L) \subseteq \mathfrak{J}(A)$ and $\mathfrak{J}(L)$ is nil (actualy nilpotent) in A, it follows that $A/\mathfrak{J}(A)$ is a commutative algebra. □

Theorem 1. *Let R be an algebra over a field F such that the group of units $U(R)$ contains an ω-subgroup, and let $\nabla(R)$ be the FC-subalgebra of R. Then the set of algebraic elements A of $\nabla(R)$ is a locally finite algebra, the Jacobson radical $\mathfrak{J}(A)$ is a central locally nilpotent ideal in $\nabla(R)$ and $A/\mathfrak{J}(A)$ is commutative.*

Proof. Since any nilpotent element of $\nabla(R)$ is central in $\nabla(R)$ by Lemma 1, one can see immediately that the set of all nilpotent elements of $\nabla(R)$ form an ideal I, and the factor algebra $\nabla(R)/I$ contains no nilpotent elements. Obviously, I is a locally finite subalgebra in $\nabla(R)$, and all idempotents of $\nabla(R)/I$ are central in $\nabla(R)/I$.

Let $x_1, x_2, \dots, x_s$ be algebraic elements of $\nabla(R)/I$. We shall prove that the subalgebra generated by $x_1, x_2, \dots, x_s$ is finite dimension.

For every x_i the subalgebra $\langle x_i\rangle_F$ of the factor algebra $\nabla(R)/I$ is a direct sum of fields

$$\langle x_i\rangle_F = F_{i1} \oplus F_{i2} \oplus \dots \oplus F_{ir_i},$$

where F_{ij} is a field and is finite dimensional over F. Choose F-basis elements u_{ijk} ($i=1,\dots,s$, $j=1,\dots,r_i$, $k=1,\dots,[F_{ij}:F]$) in F_{ij} over F and denote by $w_{ijk} = 1-e_{ij}+u_{ijk}$, where e_{ij} is the unit element of F_{ij}. Obviously, w_{ijk} is a unit in $\nabla(R)/I$. We collect in the direct summand all these units w_{ijk} for each field F_{ij} ($i=1,\dots,s$, $j=1,\dots,r_i$) and this finite subset in the group $U(\nabla(R)/I)$ is denoted by W.

Let H be the subgroup of $U(\nabla(R)/I)$ generated by W. The subgroup H of $\nabla(R)/I$ is a finitely generated FC-group, and as it is well-known, a natural number m can be assigned to H such that for any $u, v \in H$ the elements u^m, v^m are in the center $\zeta(H)$, and $(uv)^m = u^m v^m$ (see [5]).

Since H is a finitely generated group, the subgroup $S = \{v^m \mid v \in H\}$ has a finite index in H and $\{w^m \mid w \in W\}$ is a finite generated system for S. Let $t_1, t_2, \dots, t_l$ be a transversal to S in H.

Let H_F be the subalgebra of $\nabla(R)/I$ spanned by the elements of H over F. Clearly, the commutative subalgebra S_F of H_F, generated by central algebraic elements w^m $(w \in W)$, is finite dimensional over F and any $u \in H_F$ can be written as

$$u = u_1 t_1 + u_2 t_2 + \cdots + u_l t_l,$$

where $u_i \in S_F$. Since $t_i t_j = \alpha_{ij} t_{r(ij)}$ and $\alpha_{ij} \in S_F$, it yields that the subalgebra H_F is finite dimensional over F. Recall that

$$x_i = \sum_{j,k} \beta_{jk} w_{ijk} - \sum_{j,k} \beta_{jk}(1 - e_{ij}),$$

where $\beta_{jk} \in F$ and e_{ij} are central idempotents of $\nabla(R)/I$. The subalgebra T generated by e_{ij} $(i = 1, 2, \dots, s,\ j = 1, 2, \dots, r_i)$ is finite dimensional over F and T is contained in the center of $\nabla(R)/I$. Therefore, x_i belongs to the sum of two subspaces H_F and T and the subalgebra of $\nabla(R)/I$ generated by H_F and T is finite dimensional over F. Since $\langle x_1, \dots, x_s \rangle_F$ is a subalgebra of $\langle H_F, T \rangle_F$, is also finite dimensional over F. We established that the set of algebraic elements of $\nabla(R)/I$ is a locally finite algebra. One can see that all the algebraic elements of $\nabla(R)$ form a locally finite algebra A (see [3], Lemma 6.4.1, p. 162). Since the radical of an algebraic algebra is a nil ideal, according to Lemma 1 we have that $\mathfrak{J}(A)$ is a central locally nilpotent ideal in $\nabla(R)$, and $A/\mathfrak{J}(A)$ is commutative by Lemma 2. $\square$

Recall that by Neumann's Theorem [5] the set $t(\Delta U)$ of ΔU containing all elements of finite order of ΔU is a subgroup.

Theorem 2. *Let R be an algebra over a field F such that the group of units $U(R)$ contains an ω-subgroup. Then*

1. *the elements of the commutator subgroup of $t(\Delta U)$ are unipotent and central in ΔU;*
2. *if all elements of $\nabla(R)$ are algebraic then ΔU is nilpotent of class 2;*

3. *ΔU is a solvable group of length at most 3, and the subgroup $t(\Delta U)$ is nilpotent of class at most 2.*

PROOF. It is easy to see that $\Delta U \subseteq \nabla(R)$, and any element of $t(\Delta U)$ is algebraic. According to Theorem 1 the set A of algebraic elements of $\nabla(R)$ is a subalgebra, the Jacobson radical $\mathfrak{J}(A)$ is a central locally nilpotent ideal in $\nabla(R)$, and $A/\mathfrak{J}(A)$ is commutative. The isomorphism

$$U(A)/(1+\mathfrak{J}(A)) \cong U(A/\mathfrak{J}(A)),$$

implies that $\big(t(\Delta U)(1+\mathfrak{J}(A))\big)/(1+\mathfrak{J}(A))$ is abelian, the commutator subgroup of $t(\Delta U)$ is contained in $1+\mathfrak{J}(A)$ and consists of unipotent elements.

By Neumann's Theorem $\Delta U/t(\Delta U)$ is abelian, therefore ΔU is a solvable group of length at most 3. □

Let R be an algebra over a field F. Let m be the order of the element $g \in U(R)$ and assume that the element $1-\alpha^m$ is a unit in F for some $\alpha \in F$. It is well-known that $g-\alpha \in U(R)$ and

$$(g-\alpha)^{-1} = (1-\alpha^m)^{-1} \sum_{i=0}^{m-1} \alpha^{m-1-i} g^i.$$

We know that the number of solutions of the equation $x^m - 1 = 0$ in F does not exceed m. If F is an infinite field, then it follows that, there exists an infinite set of elements $\alpha \in F$ such that $g-\alpha$ is a unit. We will show that this is true for any algebraic unit.

Lemma 3. *Let $g \in U(R)$ be an algebraic element over an infinite field F. Then there are infinitely many elements α of the field F such that $g-\alpha$ is a unit.*

PROOF. Since g is an algebraic element over F, $F[g]$ is a finite dimensional subalgebra over F. Let T be the radical of $F[g]$. There exists an orthogonal system of idempotents $e_1, e_2, \dots, e_s$ such that

$$F[g] = F[g]e_1 \oplus F[g]e_2 \oplus \cdots \oplus F[g]e_s,$$

and Te_i is a nilpotent ideal such that $F[g]e_i/Te_i$ is a field. It is well-known that $F[g]e_i$ is a local ring, and all elements of $F[g]e_i$, which do not belong to Te_i, are units. Moreover, if $\alpha \in F$, then

$$g-\alpha = (ge_1 - \alpha e_1) + (ge_2 - \alpha e_2) + \cdots + (ge_s - \alpha e_s). \tag{2}$$

Clearly, ge_i is a unit and $ge_i \notin Te_i$ for every i. Put

$$L_i = \{ge_i - \alpha e_i \mid \alpha \in F\}.$$

Suppose that $ge_i - \beta e_i$ and $ge_i - \gamma e_i$ belong to Te_i for some $\alpha, \beta \in F$. Then

$$(ge_i - \beta e_i) - (ge_i - \gamma e_i) = (\gamma - \beta)e_i \in Te_i,$$

which is impossible for $\beta \neq \gamma$. Therefore, Te_i contains at most one element from L_i. Since $F[g]e_i$ is a local ring, all elements of the form $ge_i - \alpha e_i$ with $ge_i - \alpha e_i \notin Te_i$ are units, and there are infinitely many units of the form (2). □

Lemma 4. *Let $g \in U(R)$ and $a \in R$. If $g - \alpha$, $g - \beta$ are units for some $\alpha, \beta \in F$ and $ag \neq ga$, then*

$$(g - \alpha)^{-1} a (g - \alpha) \neq (g - \beta)^{-1} a (g - \beta).$$

Proof. Suppose that $(g - \alpha)^{-1} a (g - \alpha) = (g - \beta)^{-1} a (g - \beta)$. Then

$$(g - \alpha - (\beta - \alpha))(g - \alpha)^{-1} a = a(g - \alpha - (\beta - \alpha))(g - \alpha)^{-1}$$

and $(1 - (\beta - \alpha)(g - \alpha)^{-1})a = a(1 - (\beta - \alpha)(g - \alpha)^{-1})$.

Hence

$$(\beta - \alpha)(g - \alpha)^{-1} a = a(\beta - \alpha)(g - \alpha)^{-1}$$

and $(g - \alpha)^{-1} a = a(g - \alpha)^{-1}$, which provides the contradiction $ag = ga$. □

Theorem 3. *Let R be an algebra over an infinite field F. Then*

1. *any algebraic unit over F belongs to the centralizer of $\nabla(R)$;*
2. *if R is generated by algebraic units over F, then $\nabla(R)$ belongs to the center of R.*

Proof. Let $a \in \nabla(R)$, and $g \in U(R)$ be an algebraic element over F. Then by Lemma 3 there are infinitely many elements $\alpha \in F$ such that $g - \alpha$ is a unit for every α. If $[a, g] \neq 0$, then by Lemma 4 the elements of the form $(g - \alpha)^{-1} a (g - \alpha)$ are different, and a has an infinite number of conjugates, which is impossible. Therefore, g belongs to the centralizer of $\nabla(R)$.

Now, suppose that R is generated by algebraic units $\{a_j\}$ over F. Since every $w \in U(R)$ can be written as a sum of elements of the form $\alpha_i a_{i_1}^{\gamma_{i_1}} \dots a_{i_s}^{\gamma_{i_s}}$, where $\alpha_j \in F$, $\gamma_{i_j} \in \mathbb{Z}$, by the first part of this theorem w commute with elements of $\nabla(R)$. Hence $\nabla(R)$ is central in R. □

Corollary. *Let R be an algebra over an infinite field F. Then*

1. *$t(\Delta U)$ is abelian and ΔU is a nilpotent group of class at most 2;*
2. *if every unit of R is an algebraic element over F, then ΔU is central in $U(R)$.*

Proof. Clearly, all elements from $t(\Delta U)$ are algebraic and by Theorem 3 every algebraic unit belongs to the centralizer of $\nabla(R)$. Since $t(\Delta U) \subseteq \nabla(R)$, it follows that $t(\Delta U)$ is central in $\nabla(R)$. Since $\Delta U/t(\Delta U)$ is abelian, by Neumann's Theorem ΔU is a nilpotent group of class at most 2.

Let $a \in \Delta U$ and $g \in U(R)$ be an algebraic element over F. Then by Theorem 3 we get $[a, g] = 0$. Hence, if every unit of R is an algebraic element over F, then ΔU is central in $U(R)$. $\square$

References

[1] V. Bovdi, Twisted group rings whose units form an FC-group, *Canad. J. Math.* **47** (1995), 247–289.

[2] I. N. Herstein, Conjugates in division ring, *Proc. Amer. Soc.* **7** (1956), 1021–1022.

[3] I. N. Herstein, Noncommutative rings, *Math. Association of America*, *John Wiley and Sons*, 1968.

[4] A. Kertész, Lectures on artinian rings, *Akadémiai Kiadó*, *Budapest*, 1987.

[5] B. H. Neumann, Groups with finite clasess of conjugate elements, *Proc. London Math. Soc.* **1** (1951), 178–187.

[6] D. S. Passman, The algebraic structure of group rings, *A Wiley-Interscience*, *New York – Sydney – Toronto*, 1977.

[7] L. H. Rowen, Ring Theory, vol. 1, *Academic Press*, *Boston, MA*, 1988, 538.

[8] S. K. Sehgal and H. J. Zassenhaus, On the supercentre of a group and its ring theoretical generalization., *Integral Represantations and Applications. Proc. Conf., Oberwolfach, 1980, Lect. Notes Math.* **882** (1981), 117–144.

VICTOR BOVDI
INSTITUTE OF MATHEMATICS AND INFORMATICS
UNIVERSITY OF DEBRECEN
H–4010 DEBRECEN, P.O. BOX 12
HUNGARY

E-mail: vbovdi@math.klte.hu

(Received February 23, 2000; revised May 23, 2000)