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Extensions of the representation modules of a prime order group ☆

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Abstract

For the ring R of integers of a ramified extension of the field of p-adic numbers and a cyclic group G of prime order p we study the extensions of the additive groups of R-representations modules of G by the group G.

Let $\mathfrak{T}$ be the field of fractions of a principal ideal domain R, F a field which contains R, let G be a finite group and Γ a matrix R-representation of G. Let M be an RG-module, which affords the R-representation Γ of G, and $FM = F \otimes_R M$ the smallest linear space over F which contains M and $\widehat{M} = FM^+/M$, the factor group of the additive group of the space FM by the additive group of M. Clearly, the group $\widehat{M}$ and the space FM are RG-modules. Put $\widehat{F} = F^+/R$.

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Let $\mathfrak{f}: G \to \widehat{M}$ be a 1-cocycle of G with values in $\widehat{M}$, i.e.

$$\mathfrak{f}(xy) = x\mathfrak{f}(y) + \mathfrak{f}(x) \quad (x, y \in G).$$

Define $[g, x]$ by $\left(\begin{smallmatrix} g & x \\ 0 & 1 \end{smallmatrix}\right)$ and set

$$\mathfrak{Crys}(G, M, \mathfrak{f}) = \big\{[g, x] \mid g \in G,\ x \in \mathfrak{f}(g)\big\},$$

where x runs over the cosets $\mathfrak{f}(g) \in \widehat{M}$ for any $g \in G$.

Clearly, $\mathfrak{Crys}(G, M, \mathfrak{f})$ is a group, where the multiplication is the usual matrix multiplication. Of course $K_1 = \{[e, x] \mid e$ is the unit element of $G, x \in \mathfrak{f}(e)\}$ is a normal subgroup of $\mathfrak{Crys}(G, M, \mathfrak{f})$ such that $K_1 \cong M^+$ and $\mathfrak{Crys}(G, M, \mathfrak{f})/K_1 \cong G$. The group $\mathfrak{Crys}(G, M, \mathfrak{f})$ is an extension of the additive group of the RG-module M by G.

We are using the terminology of the theory of group representations [1].

A 1-cocycle $\mathfrak{f}: G \to \widehat{M}$ is called *coboundary*, if there exists an $x \in FM$ such that $\mathfrak{f}(g) = (g - 1)x + M$ for every $g \in G$. The 1-cocycles $\mathfrak{f}_1: G \to \widehat{M}$ and $\mathfrak{f}_2: G \to \widehat{M}$ are called *cohomologous* if $\mathfrak{f}_1 - \mathfrak{f}_2$ is a coboundary. Let $H^1(G, \widehat{M})$ be the first cohomology group. Clearly, each element of $H^1(G, \widehat{M})$ defines a class of equivalence of groups.

If the 1-cocycles $\mathfrak{f}_1, \mathfrak{f}_2$ are cohomologous, then $\mathfrak{Crys}(G, M, \mathfrak{f}_1)$ and $\mathfrak{Crys}(G, M, \mathfrak{f}_2)$ are isomorphic. This isomorphism is called *equivalence* and these groups are called equivalent. In particular, the group $\mathfrak{Crys}(G, M, \mathfrak{f})$ is split (i.e. $\mathfrak{Crys}(G, M, \mathfrak{f}) = M \rtimes G$) if and only if $\mathfrak{f}$ is coboundary.

The *dimension* of the group $\mathfrak{Crys}(G, M, \mathfrak{f})$ is called the R-rank of the R-module M. (Note that M is a free R-module of finite rank.) The group $\mathfrak{Crys}(G, M, \mathfrak{f})$ is called *irreducible* (*indecomposable*), if M is an irreducible (indecomposable) RG-module and the 1-cocycle $\mathfrak{f}$ is not cohomologous to zero.

The group $\mathfrak{Crys}(G, M, \mathfrak{f})$ is *non-split*, if the 1-cocycle $\mathfrak{f}$ defines a non-zero element of $H^1(G, \widehat{M})$.

Note that the properties of the group $\mathfrak{Crys}(G, M, \mathfrak{f})$ were studied in [5,6,8], in the cases when R is either the ring of rational integers $\mathbb{Z}$, or the p-adic integers $\mathbb{Z}_p$, or the localization $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$ at p.

Let $G = \langle a \mid a^p = 1\rangle$ be the cyclic group of prime order p, R the ring of integers of the ramified finite extension T of the field of p-adic numbers. We calculate the group $H^1(G, \widehat{M})$ for some module M of an indecomposable R-representation of G.

Let $\Phi_p(x) = x^{p-1} + \cdots + x + 1$ be a cyclotomic polynomial of degree p and let $\eta(x)$ be a divisor of $\Phi_p(x)$ over the field $\mathfrak{T}$ with $\deg(\eta(x)) < p - 1$ (provided that such non-trivial polynomial exists).

Lemma 1. *Let M_1 and M_2 be RG-modules which afford an R-representation Γ of $G = \langle a \mid a^p = 1\rangle$.*

(i) *If $M_1 \cong M_2$ then $H^1(G, \widehat{M_1}) \cong H^1(G, \widehat{M_2})$.*
(ii) *If the matrix $\Gamma(a)$ does not have 1 as eigenvalue, then $H^1(G, \widehat{M_1})$ is trivial.*

Proof. See [1]. □

Theorem 1. *Let $G = \langle a \mid a^p = 1\rangle$ and $M_\eta = \eta(a)RG$. Then the RG-module M_η is indecomposable and*

$$H^1\big(G, \widehat{M_\eta}\big) \cong R\big/\big(\eta(1)R\big),$$

where $R/(\eta(1)R)$ is the additive group of the factor ring of R by the ideal $\eta(1)R$.

Proof. Let $t \in R$ be a prime element and $\overline{R} = R/(tR)$. Then in $\overline{R}$ we have that

$$x^p - 1 = (x-1)^p, \qquad \eta(x) = (x-1)^n, \qquad \big(x^p - 1\big)\eta^{-1}(x) = (x-1)^{p-n}, \quad (1)$$

where $n = \deg(\eta(x))$.

Put $\eta_1(x) = (x^p - 1)\eta^{-1}(x)$. Then M_η and $RG/(\eta_1(a)RG)$ are isomorphic as RG-modules. If $\overline{M_\eta} = M_\eta/(tM_\eta)$, then by (1) follows that $\overline{M_\eta}$ is a root subspace of the linear operator a over $\overline{R}$. It is easy to see that $\overline{M_\eta}$ is not decomposable into a direct sum of invariant subspaces. It follows that M_η is an indecomposable RG-module. Clearly $FM_\eta = \eta(a)FG = \eta(a)F + (a-1)FM_\eta$ and the group $\widehat{M_\eta} = FM_\eta^+/M_\eta$ is isomorphic to a direct sum of groups $\eta(a)(F^+/R) + (a-1)\widehat{M_\eta}$. This means that in the class of 1-cocycles there is a cocycle $\mathfrak{f} : G \to \widehat{M_\eta}$ such that

$$\mathfrak{f}(a) = \lambda\eta(a) + M_\eta \quad (\lambda \in F).$$

Moreover, from $\mathfrak{f}(a^p) = 0$ (in $\widehat{M_\eta}$) it follows that if $\omega = a^{p-1} + \cdots + a + 1$, then

$$\omega \cdot \mathfrak{f}(a) = \lambda \cdot \eta(1) \cdot \omega \in M_\eta$$

if and only if $\lambda\eta(1) \in R$. Therefore, $H^1(G, \widehat{M_\eta})$ is isomorphic to the subgroup $\{\lambda + R \mid \lambda \in F,\ \lambda \cdot \eta(1) \in R\}$ of F/R and

$$\big\{\lambda + R \in F/R \bigm| \lambda \cdot \eta(1) \in R\big\} \cong R\big/\big(\eta(1)R\big). \qquad \square$$

Corollary 1. *Let $\alpha \in R$ and $\eta(1)R = t^s R$, where t is a prime element of R. Put*

$$K_\alpha(G, M_\eta) = \left\langle \begin{pmatrix} e & m \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & \alpha t^{-s}\eta(a) \\ 0 & 1 \end{pmatrix} \biggm| m \in M_\eta \right\rangle,$$

where α runs over the representative elements of the cosets of $R/(t^s R)$. Up to equivalence, the groups $K_\alpha(G, M_\eta)$ give all extensions of the additive group of the RG-module M_η by the group G.

Suppose $p = t^d\theta$, where $d > 1$ is the ramification index and θ is a unit in R. Set

$$\mathfrak{X}_{ji} = t^j RG + (a-1)^i RG \quad (1 \leqslant j < d,\ 1 \leqslant i < p).$$

Theorem 2. *The module $\mathfrak{X}_{ji}$ is an RG-module affording an indecomposable R-representation of G and*

$$H^1\big(G, \widehat{\mathfrak{X}}_{ji}\big) \cong R\big/\big(t^{d-j}R\big).$$

Proof. Suppose that the RG-module $\mathfrak{X}_{ji}$ is decomposable into a direct sum of RG-submodules. Then $t^j = u_1 + u_2$, where u_1, u_2 are non-zero elements of RG with $u_1u_2 = 0$. Thus $e_1 = t^{-j}u_1$ is an idempotent. Since the trace $\mathrm{tr}(e_1)$ of e_1 is a rational number (see [4, Theorem 3.5, p. 21]) of the form rp^{-1} ($1 \leqslant r \leqslant p$), we get $t^j rp^{-1} \in R$, which is impossible for $j < d$. This contradiction proves the indecomposability of $\mathfrak{X}_{ji}$.

Clearly $F\mathfrak{X}_{ji} = FG = F + (a-1)FG$. Therefore, in each class of 1-cocycles there is a cocycle $\mathfrak{f}: G \to \widehat{\mathfrak{X}}_{ji}$ such that $\mathfrak{f}(a) = \lambda + \mathfrak{X}_{ji}$, where $\lambda \in F$ with $\lambda\omega \in \mathfrak{X}_{ji}$. It follows that $\lambda p = t^j\alpha$, where $\alpha \in R$ and

$$H^1\big(G, \widehat{\mathfrak{X}}_{ji}\big) \cong \big\{\lambda + R \mid \lambda \in F,\ \lambda t^{d-j} \in R\big\}$$

is a subgroup of F/R. $\square$

Set

$$K_\alpha(G, \mathfrak{X}_{ji}) = \left\langle \begin{pmatrix} e & m \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & \alpha t^{j-d} \\ 0 & 1 \end{pmatrix} \,\middle|\, m \in \mathfrak{X}_{ji} \right\rangle,$$

where α runs over the representative elements of the cosets of $R/(t^{d-j}R)$.

Corollary 2. *The groups $K_\alpha(G, \mathfrak{X}_{ji})$ give all extensions of the additive group of the RG-module $\mathfrak{X}_{ji}$ by G.*

Lemma 2. *The set $\{\mathfrak{X}_{ji} \mid j = 1, \ldots, d-1;\ i = 1, \ldots, (p-1)/2\}$ consists of pairwise non-isomorphic modules.*

Proof. Let us consider an indecomposable $\overline{R}G$-module $V_i = \overline{R}G/((a-1)^i\overline{R}G)$, where $\overline{R} = R/(tR)$ and $1 \leqslant i \leqslant p$. It is easy to check that the elements

$$u_1 = t^j, \quad \ldots, \quad u_i = t^j(a-1)^{i-1}, \quad u_{i+1} = (a-1)^i, \quad \ldots, \quad u_p = (a-1)^{p-1} \tag{2}$$

form an R-basis in $\mathfrak{X}_{ji}$ and

$$\Phi_p(x) - (x-1)^{p-1} = p\theta(x), \tag{3}$$

where $\theta(x) \in \mathbb{Z}[x]$, $\deg(\theta(x)) \leqslant p-2$. Note that since $\theta(1) = 1$, it follows that $\theta(a)$ is a unit in the group ring RG. Using the identity

$$xy - 1 = (x-1)(y-1) + (x-1) + (y-1),$$

from (3) we obtain that

$$(a-1)^p = p(a-1)\cdot\bigl(\alpha_0+\alpha_1(a-1)+\cdots+\alpha_{p-2}(a-1)^{p-2}\bigr), \tag{4}$$

where $\alpha_0,\alpha_1,\dots,\alpha_{p-2}\in\mathbb{Z}$. Since $p=t^d\theta=t(t^{d-1}\theta)$, from (4) we get

$$(a-1)^p=(a-1)u_p=tm, \tag{5}$$

where $m\in\mathfrak{X}_{ji}$. According to (2), $(a-1)u_i=t^ju_{i+1}$, and from (5) we obtain that the RG-module $\overline{\mathfrak{X}}_{ji}=\mathfrak{X}_{ji}/(t\mathfrak{X}_{ji})$ is isomorphic to a direct sum $V_i\oplus V_{p-i}$ of indecomposable $\overline{R}G$-modules, so by Theorem 2 and Lemma 1 the proof is complete. □

Let $n>1$ be the degree of a divisor of $\Phi_p(x)$ which is irreducible over R. We consider the following RG-modules:

$$\mathfrak{U}_{ji}=t^j(a-1)RG+(a-1)^{s+1}RG \quad (1\leqslant j<d,\ 1\leqslant s<n).$$

It is easy to check that the RG-module $\mathfrak{U}_{ji}$ satisfies the condition (ii) of Lemma 1, so $H^1(G,\widehat{\mathfrak{U}_{ji}})=0$.

Let $\mathfrak{Z}_{js}$ be a submodule on the free module $RG^{(2)}=\{(x,y)\mid x,y\in RG\}$ of rank 2, which consists of the solutions (x,y) of the equality

$$t^j(a-1)x+(a-1)^{s+1}y=0. \tag{6}$$

Lemma 3. *Let $\omega=\Phi_p(a)$ and set $u_1=[0,\omega]$, $u_2=[(a-1)^s,-t^j]$ and $u_3=[t^{-j}(\omega-(a-1)^{p-1}),(a-1)^{p-s-1}]$. Then $\mathfrak{Z}_{js}$ is an RG-module generated by u_1,u_2,u_3.*

Proof. Clearly, $u_1,u_2,u_3\in\mathfrak{Z}_{js}$. Let $u=[x,y]$ be an arbitrary element of $\mathfrak{Z}_{js}$. If $x=0$ then $u=u_1$. Suppose $x\neq 0$. By substraction of the elements of RGu_3 from u we obtain that $y=\gamma_0+\gamma_1(a-1)+\cdots+\gamma_{p-s-2}(a-1)^{p-s-2}$ ($\gamma_r\in R$). By (6)

$$t^j(a-1)x+\bigl(\gamma_0+\gamma_1(a-1)+\cdots+\gamma_{p-s-2}(a-1)^{p-s-2}\bigr)\cdot(a-1)^{s+1}=0,$$

which is possible if and only if $\gamma_0\equiv\cdots\equiv\gamma_{p-s-2}\equiv 0 \pmod{t^j}$. Now, since u is an element of RGu_2, we obtain that $y=0$. Then $t^j(a-1)x=0$, which implies $x=\alpha\omega$ ($\alpha\in R$) and $u=\alpha(t^ju_3-(a-1)^{p-s-1}u_2)$. □

Theorem 3. *The RG-module $\mathfrak{Z}_{js}$ is indecomposable. Moreover,*

$$H^1(G,\widehat{\mathfrak{Z}}_{js})\cong R/\bigl(t^dR\bigr)\oplus R/\bigl(t^{d-j}R\bigr)$$

and the RG-modules $\mathfrak{X}_{js}$ are pairwise non-isomorphic.

S0021-8693(05)00393-5/FLA AID:10668 Vol.●●●(●●●) yjabr10668 [DTD5] P.6 (1-11)
YJABR:m1 v 1.46 Prn:13/09/2005; 12:00 by:JOL p. 6

Proof. It is easy to see that

$$u_1 = t^j(a-1), \quad \ldots, \quad u_{i-1} = t^j(a-1)^{i-1},$$
$$u_i = (a-1)^i, \quad \ldots, \quad u_{p-1} = (a-1)^{p-1}$$

form an R-basis in the RG-module $\mathfrak{U}_{js}$ and

$$\overline{\mathfrak{U}}_{js} = \mathfrak{U}_{js}/(t\mathfrak{U}_{js}) \cong V_s \oplus V_{p-s-1}.$$

Since $s < n$, it follows that the RG-module $\mathfrak{U}_{js}$ is indecomposable. Moreover, it follows that the RG-modules $\mathfrak{U}_{js}$ are pairwise non-isomorphic and RG-modules $\mathfrak{Z}_{js}$, $\mathfrak{U}_{js}$ and RG^2 form an exact sequence

$$0 \to \mathfrak{Z}_{js} \to RG^{(2)} \to \mathfrak{U}_{js} \to 0.$$

Therefore, $\mathfrak{Z}_{js}$ is the kernel of a minimal projective covering of the indecomposable RG-module $\mathfrak{U}_{js}$, so $\mathfrak{Z}_{js}$ is also indecomposable. □

Lemma 4. *Let $\widehat{\mathfrak{Z}}_{js} = (F\mathfrak{Z}_{js})^+/\mathfrak{Z}_{js}$, $\widehat{F} = F^+/R$ and $M = (a-1)\widehat{\mathfrak{Z}}_{js}$. Then*

$$\widehat{\mathfrak{Z}}_{js}/M = \widehat{F}\nu_1 + \widehat{F}\nu_2,$$

where $\nu_1 = [0,\omega] + M$, $\nu_2 = [\omega, 0] + M$ and $a\nu_1 = \nu_1$, $a\nu_2 = \nu_2$.

Proof. Clearly, $ax = x$ $(x \in \widehat{\mathfrak{Z}}_{js}/M)$ and $\widehat{F}\nu_1 = \widehat{F}[0,\omega] + M \in \widehat{\mathfrak{Z}}_{js}/M$. Moreover,

$$\omega\widehat{F}u_3 + M = \widehat{F}\big[t^{-1}p\omega, 0\big] + M = \widehat{F}\big(tp^{-1}\big)\big[t^{-1}p\omega, 0\big] + M = \widehat{F}[\omega, 0] + M.$$

By analogy

$$\omega\widehat{F}u_2 + M = \widehat{F}\big[0, -t^j\omega\big] + M = \widehat{F}[0,\omega] = \widehat{F}\nu_1.$$

Therefore $\widehat{\mathfrak{Z}}_{js}/M = \widehat{F}\nu_1 + \widehat{F}\nu_2$. □

From Lemma 4 it follows that each class of 1-cocycles of the group G with values in the group $\widehat{\mathfrak{Z}}_{js} = (F\mathfrak{Z}_{js})^+/\mathfrak{Z}_{js}$ contains a 1-cocycle $\mathfrak{f}$ such that

$$\mathfrak{f}(a) = \alpha[0,\omega] + \beta[\omega,0] + \mathfrak{Z}_{js},$$

where $\alpha, \beta \in F$ and $\omega(\alpha[0,\omega] + \beta[\omega,0]) \in \mathfrak{Z}_{js}$. This condition holds if and only if $\alpha p, \beta p \in R$. Moreover,

$$\alpha[0,\omega] + \beta[\omega,0] \in \mathfrak{Z}_{js} + (a-1)\widehat{\mathfrak{Z}}_{js}$$

S0021-8693(05)00393-5/FLA AID:10668 Vol.●●●(●●●) yjabr10668 [DTD5] P.7 (1-11)
YJABR:m1 v 1.46 Prn:13/09/2005; 12:00 by:JOL p. 7

if and only if $\alpha \in R$ and $\beta \in t^{-j}R$. Using properties of the 1-cocycle $\mathfrak{f}$ it is easy to show that the two 1-cocycles $\mathfrak{f}_j$ $(j = 1, 2)$:

$$\mathfrak{f}_1(a) = \alpha_1[0,\omega] + \beta_1[\omega,0] + \mathfrak{Z}_{js}, \qquad \mathfrak{f}_2(a) = \alpha_2[0,\omega] + \beta_2[\omega,0] + \mathfrak{Z}_{js}$$

are cohomologous if and only if

$$p\alpha_1 \equiv p\alpha_2 \pmod{t^d} \qquad \text{and} \qquad p\beta_1 \equiv p\beta_2 \pmod{t^{d-j}},$$

where $\alpha_j, \beta_j \in F$, $p\alpha_j, p\beta_j \in R$. Note that $p = t^d\theta$.

It follows that the map $\mathfrak{f} \mapsto (p\alpha + t^dR, p\beta + t^{d-j}R)$ gives the isomorphism

$$H^1(G, \widehat{\mathfrak{Z}}_{js}) \cong R/(t^dR) \oplus R/(t^{d-j}R).$$

Therefore, according to (ii) of Lemma 1, the RG-modules $\mathfrak{Z}_{js}$ $(1 \leqslant j < d)$ are pairwise non-isomorphic. □

Now, using the description of 1-cocycles it is easy to prove the following

Corollary 3. *Put*

$$K_{\alpha,\beta}(G, \mathfrak{Z}_{js}) = \left\langle \begin{pmatrix} e & m \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & \alpha t^{-d}[0,\omega] + \beta t^{-d}[\omega,0] \\ 0 & 1 \end{pmatrix} \;\middle|\; m \in Z_{js} \right\rangle,$$

where α and β independently run over the representative elements of the cosets $R/(t^dR)$ and $R/(t^{d-j}R)$, respectively. Up to equivalence, the groups $K_{\alpha,\beta}(G, \mathfrak{Z}_{js})$ give all extensions of the additive group of the RG-module $\mathfrak{Z}_{js}$ by the group G.

If R is the quadratic extension of the ring of p-adic integers, then the R-representations of G were described by P.M. Gudivok (see [7]). Finally, we have the following result.

Theorem 4. *Let $\Phi_p(x)$ be decomposable into the product of at least two irreducible polynomials over R. Then the dimensions of the non-split indecomposable groups $\mathfrak{Crys}(G, M, \mathfrak{f})$ are unbounded.*

Proof. Let $\Phi_p(x) = \eta_1(x) \cdots \eta_k(x)$ $(k > 2)$ be a decomposition into a product of polynomials irreducible over R and suppose that

$$\eta_1(x) = x^n - \alpha_{n-1}x^{n-1} - \cdots - \alpha_1 x - \alpha_0 \in R[x].$$

Note that $\deg(\eta_1(x)) = \deg(\eta_2(x)) = \cdots = \deg(\eta_k(x)) = n$ and $kn = p - 1$.

We will use the technique of integral representation of finite groups, which was developed by S.D. Berman and P.M. Gudivok in [2,3,7].

Let ε be a primitive pth root of unity such that $\eta_1(\varepsilon) = 0$ and let r_j be a natural number, such that $\varepsilon_j = \varepsilon^{r_j}$ is a root of the polynomial $\eta_j(x)$, where $r_1 = 1$ and $j = 1, \ldots, k$. Let

$$\tilde{\varepsilon} = \begin{pmatrix} 0 & \cdots & 0 & \alpha_0 \\ 1 & \cdots & 0 & \alpha_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & \alpha_{n-1} \end{pmatrix}$$

be the comparing matrix of $\eta_1(x)$.

The following R-representations of $G = \langle a \mid a^p = 1 \rangle$ are irreducible:

$$\delta_0 : a \mapsto 1, \qquad \delta_1 : a \mapsto \tilde{\varepsilon}, \qquad \delta_j : a \mapsto \tilde{\varepsilon_j} = \tilde{\varepsilon_j}^{r_j} \quad (j = 2, \ldots, k).$$

Note that the module which affords representation δ_1 is $R[\varepsilon]$ with R-basis $1, \varepsilon, \ldots, \varepsilon^{n-1}$. $\square$

Let $m \in \mathbb{N}$. Define the following R-representation of $G = \langle a \rangle$ of degree $(3n + 1)m$:

$$\Gamma_m : a \mapsto \begin{pmatrix} \Delta_{1m}(a) & U_m(a) \\ 0 & \Delta_{2m}(a) \end{pmatrix},$$

where

- $\Delta_{1m}(a) = \delta_0^{(m)}(a) + \delta_1^{(m)}(a) = \left(\begin{smallmatrix} E_m \otimes \delta_0(a) & 0 \\ 0 & E_m \otimes \delta_1(a) \end{smallmatrix}\right)$;
- $\Delta_{2m}(a) = \delta_2^{(m)}(a) + \delta_3^{(m)}(a) = \left(\begin{smallmatrix} E_m \otimes \delta_2(a) & 0 \\ 0 & E_m \otimes \delta_3(a) \end{smallmatrix}\right)$;
- $U_m(a) = \left(\begin{smallmatrix} E_m \otimes u & J_m(1) \otimes u \\ E_m \otimes \bar{u} & E_m \otimes \bar{u} \end{smallmatrix}\right)$;
- $u = (0, 0, \ldots, 0, 1)$ defines a non-zero element of $\mathrm{Ext}(\delta_0, \delta_j)$;
- $J_m(\lambda)$ is a Jordan block of degree m with λ in the main diagonal;
- $\bar{u}$ is a matrix in which the first row is $(0, \ldots, 0, 1)$ and all other rows are zero. The matrix $\bar{u}$ defines a non-zero element of the group $\mathrm{Ext}(\delta_1, \delta_j)$, where $j = 2, 3$;
- E_m is the unity matrix of degree m.

Lemma 5 (see [2,3]). *Γ_m is an indecomposable R-representation of G.*

Let $\mathfrak{W}_m = R^l$ be a module of l-dimension vectors over R affording the R-representation Γ_m. Put $\widehat{F} = F^+/R$, $\widehat{\mathfrak{W}}_m = F\mathfrak{W}_m^+/\mathfrak{W}_m$. Clearly $\widehat{F}^l \cong \widehat{\mathfrak{W}}_m$. Define $\tau : F \to F^n$ by

$$\tau(w) = w(\alpha_0, \alpha_0 + \alpha_1, \alpha_0 + \alpha_1 + \alpha_2, \ldots, \alpha_0 + \cdots + \alpha_{n-2}, 1), \tag{7}$$

where the α_j are coefficients of $\eta_1(x)$ and $w \in F$.

S0021-8693(05)00393-5/FLA AID:10668 Vol.●●●(●●●) yjabr10668 [DTD5] P.9 (1-11)
YJABR:m1 v 1.46 Prn:13/09/2005; 12:00 by:JOL p. 9

Lemma 6.

(i) *Each* 1*-cocycle of* $G = \langle a \mid a^p = 1\rangle$ *at* $\widehat{\mathfrak{W}}_m$ *is cohomologous to a cocycle* $\mathfrak{f}$*, such that*

$$\mathfrak{f}(a) = (X, 0, \dots, 0) + \mathfrak{W}_m,$$

where $X \in F^m$ *and* $pX = 0$ *in* $\widehat{F}^m$ *(i.e.* $pX \in R^m$*).*

(ii) *Let* $z \in F^n$ *such that* $(\tilde{\varepsilon} - E_n)z = 0$ *in* $\widehat{F}^n$*. Then* $z = \tau(w) \pmod{R^n}$*, with* $w \in F$ *such that* $\eta_1(1)w = 0$ *in* $\widehat{F}$*.*

(iii) *If* $V = R/(\frac{p}{\eta(1)}R)$ *is the residual of ring* R *by the ideal* $(\frac{p}{\eta(1)}R)$*, then* $H^1(G, \widehat{\mathfrak{W}}_n) \cong V^m$*.*

Proof. (i) follows from (ii) of Lemma 1. (ii) is easy to check.

(iii) By (i) we can put $\mathfrak{f}(a) = (X, 0, 0, 0)$ and $\mathfrak{g}(a) = (Y, 0, 0, 0)$, where $X = (x_1, \dots, x_n)$, $Y = (y_1, \dots, y_n)$ and $pX = pY = 0$. Note that all the equalities considered here are understood modulo the group R. Suppose that these 1-cocycles are cohomologous and $Z \in F^l$ is such that

$$\big(\Gamma_m(a) - E_l\big)Z + \mathfrak{f}(a) = \mathfrak{g}(a). \tag{8}$$

Put $Z = (Z_1, Z_2, Z_3, Z_4)$, where $Z_1 \in F^m$ and Z_2, Z_3, Z_4 are m-dimensional vectors, with i-components belong to F^n, and denoted by Z_2^i, Z_3^i and Z_4^i, respectively. By (8) we get

$$(E_m \otimes u)Z_3 + (J_m \otimes u)Z_4 + X = Y, \tag{9}$$

$$\big(E_m \otimes (\tilde{\varepsilon} - E_n)\big)Z_2 + (E_m \otimes \bar{u})(Z_3 + Z_4) = 0, \tag{10}$$

$$\big(E_m \otimes (\tilde{\varepsilon}_2 - E_n)\big)Z_3 = 0, \qquad \big(E_m \otimes (\tilde{\varepsilon}_3 - E_n)\big)Z_4 = 0. \tag{11}$$

From (11) and by (ii) we have

$$Z_3 = \big(\tau(v_1), \dots, \tau(v_m)\big), \qquad Z_4 = \big(\tau(u_1), \dots, \tau(u_m)\big), \tag{12}$$

where $u_j, v_j \in F$, τ is from (7) and

$$\eta_1(1)u_j = \eta_1(1)v_j = 0. \tag{13}$$

Clearly, the equality (10) consists of m matrix equalities of the form

$$(\tilde{\varepsilon} - E_n)Z_2^i + \bar{u}\tau(w) = 0, \tag{14}$$

where $Z_2^i \in F^n$ is the ith component of Z_2, $i = 1, \dots, m$, and $w \in F$. Since $u\tau(w) = w$ and $\bar{u}\tau(w) = (w, 0, \dots, 0)$, when all the rows of (14) are added together we obtain

$$-\eta(1)Z_2^n + w = 0, \tag{15}$$

where Z_2^n is the last component of the vector Z_2. According to (12) and (15), (10) gives the equalities

$$-\eta(1)z_j + v_j + u_j = 0 \quad (j = 1, \dots, m), \tag{16}$$

where z_j are some components of Z_2. From (9)

$$\begin{gathered} v_j + u_j + u_{j+1} + x_j = y_j \quad (j = 1, \dots, m-1), \\ v_m + u_m + x_m = y_m, \end{gathered} \tag{17}$$

where $X = (x_1, \dots, x_n)$, $Y = (y_1, \dots, y_n)$ and $pX = pY = 0$. Multiplying (17) by $\eta_1(1)$ and using (16) we obtain for the components of X and Y

$$\eta_1(1)x_j = \eta_1(1)y_j \quad (j = 1, \dots, m). \tag{18}$$

Therefore, if the 1-cocycles $\mathfrak{f}$ and $\mathfrak{g}$ are cohomologous then (18) holds.

Conversely, suppose that (18) holds. Then it is not difficult to construct vectors Z_2, Z_3, Z_4 that satisfy (9) and (10), which is equivalent to (8), i.e. the 1-cocycles $\mathfrak{f}$ and $\mathfrak{g}$ are cohomologous. It follows that by going from a cocycle to an element of the cohomology group, we need to change each component in X by $\beta = \alpha \cdot p^{-1}$ modulo the group R, where $\alpha \in R$. Moreover, if $\eta_1 \cdot \beta \in R$, then must change β to 0. □

Theorem 5. *Let $\varepsilon \in R$, where $\varepsilon^p = 1$ and $p > 2$. Then the description of the non-split indecomposable groups $\mathfrak{Crys}(G, M, \mathfrak{f})$ is a wild type problem.*

Proof. For arbitrary matrices $A, B \in M(m, R)$ the map

$$\Gamma_{A,B} : a \mapsto \begin{pmatrix} E & 0 & E & A & E \\ & \varepsilon E & E & E & B \\ & & \varepsilon^2 E & 0 & 0 \\ & & & \varepsilon^3 E & 0 \\ & & & & \varepsilon^4 E \end{pmatrix}$$

is an R-representation of G of degree $l = 5m$. The R-representations $\Gamma_{A,B}$ and Γ_{A_1,B_1} are R-equivalent if and only if

$$C^{-1}AC \equiv A_1 \quad \bigl(\mathrm{mod}\ (1-\varepsilon)\bigr), \qquad C^{-1}BC \equiv B_1 \quad \bigl(\mathrm{mod}\ (1-\varepsilon)\bigr)$$

for some invertible matrix C. It follows that the description of the R-representations $\Gamma_{A,B}$ of G is a wild type problem.

For the module affording the representation $\Gamma_{A,B}$ of G we put R^l. Let X be an m-dimensional vector over F with $pX \in R^m$. Then there is a 1-cocycle $\mathfrak{f}_X : G \to \widehat{R}^l$, such that $\mathfrak{f}_X(a) = (X, 0, \dots, 0) + R^l$. The 1-cocycles $\mathfrak{f}_X$ and $\mathfrak{f}_Y$ are cohomologous if and only if

$$(1-\varepsilon)(X - Y) \in R^m.$$

Putting $X = (p^{-1}, 0, \dots, 0)$ we obtain that $H^1(G, \widehat{R}^l) \neq 0$. □

S0021-8693(05)00393-5/FLA AID:10668 Vol.●●●(●●●) yjabr10668 [DTD5] P.11 (1-11)
YJABR:m1 v 1.46 Prn:13/09/2005; 12:00 by:JOL p. 11

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