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Whose Logic is Three-Valued Logic?

Abstract. 'It would be unfair to judge that I use a three-valued logic or that I abandon the principle of *tertium non datur*", writes Imre Ruzsa in his (Ruzsa 1991, 11.). For Ruzsa a truth value gap (the "third" truth value) arises only from the "defects" of our expressions (for example when a definite description does not denote anything) and not because there are "gaps" in reality. In the first part of the paper we explain in some detail how the truth value gaps arise and how they are transmitted in Ruzsa's system. In the second part we will argue that there may be sentences which in a sense reflects real gaps, in other words, that the third truth value is a real truth value.

1 THE SEMANTICS OF SEMANTIC VALUE GAPS

One of Imre Ruzsa's main achievements in logic is his system of intensional logic with semantic value gaps. A semantic value gap arises when a well formed expression of our (natural or artificial) language fails to denote anything. The simplest case is perhaps a definite description without a denotation (e.g. 'the present king of France'). In Ruzsa's system there *are* denotations even in such cases—these are the artificial entities "filling" the gaps. The individual denoted by 'the present king of France' is not a real individual: Ruzsa's choice is the set U, the set of "real" (actual and possible) individuals, simply because evidently $U \notin U$. A bit more precisely: the type ι of individuals has the domain $D(\iota) = U \cup \{U\}$, and $\Theta(\iota) = U$ is the type's *zero entity*—the "object" denoted by e.g. the empty descriptions.

Systems of intensional logics have in general two kinds of semantic values: the extensions—in Ruzsa's terminology, factual values—and the intensions. In the type o of sentences the factual values are the truth values, the intensions are functions from worlds and times (technically, from the set $I = W \times T$, where W is the set of possible worlds and T is the linearly ordered set of time moments) to the factual values. The truth values are represented by $3 = \{0, 1, 2\}$, $\Theta(o) = 2$ being the zero entity representing the truth value gap. If a sentence p has 2 as factual value (in a world w in a given moment t) than we say it has no "real" truth value (in the world w in the moment t). The "real" truth values are of course 0 (representing falsity) and 1 (representing truth).

From the basic types ι and o, we get the other (functor) types. For example, predicates are expressions of type $o(\iota)$. The domain of this type is the set of all functions $f : D(o) \longrightarrow D(\iota)$ for which $f(U) = 2$. The zero entity of this type is the constant function having 2 for all the arguments. A predicate is partial if its interpretation $\sigma(P) : D(\iota) \longrightarrow 3$ maps more than one individual to 2.

In what follows $|A|_{vi} =$ denotes the factual value of A at the *index* $i = \langle w, t\rangle \in W \times T$ according to the valuation v. If i is an index, $d(i) \subseteq D(\iota)$ is the set of the "actual" individuals at i, that is, the actual individuals in the world w at the time moment t. If x is a variable of type ι, than the value $v(x)$ is always an element of $D(\iota)$; if $v(x) \notin d(i)$, then $|x|_{vi} = \Theta(\iota)$ and similarly for constants of type ι. It could happen that a value of variable in the world w at the moment t is an individual not belonging to the domain of w at t—in such cases the factual value of the variable is the zero entity of the type.

Definite descriptions are handled as it is expected. If F is of type $o(\iota)$ then the factual value of $\mathbb{I}F$ ('the F') is $|\mathbb{I}F|_{vi} = u_i$ if $\{u \in d(i) : |F|_{vi}(u) = 1\} = \{u_i\}$, and in all other cases $|\mathbb{I}F|_{vi} = \Theta(\iota)$. If there is exactly one F in the world w at the moment t, then $|\mathbb{I}F|_{vi}$ is *this* object; and if the set of the Fs is empty or has more than one element, $|\mathbb{I}F|_{vi}$ is the zero entity.

The identities are (of course) expressions of type o. If A and B are of the same type α, then

$$|A = B|_{vi} = \begin{cases} 2 & \text{if } |A|_{vi} = \Theta(\alpha) \text{ or } |B|_{vi} = \Theta(\alpha) \\ 1 & \text{if } |A|_{vi} = |B|_{vi} \neq \Theta(\alpha) \\ 0 & \text{otherwise} \end{cases}$$

According to this rule, if on one side of an identity stands an expression having the zero entity of its type as its factual value then the identity's factual value will be automatically 2. It has the (somewhat strange) consequence that non-existent individuals cannot be identical even with themselves. For example the sentence 'the present king of France' = 'the present king of France' falls in the truth value gap.

Ruzsa—following an idea of Tarski—defines the propositional connectives in terms of λ and $=$. The technical details (see (Ruzsa 1991, 39—41)) do not concern us, only the truth tables governing the connectives. The truth tables for negation, conjunction, and alternation are the following:

p	$\sim p$
1	0
0	1
2	2

$p \wedge q$	1	0	2
1	1	0	2
0	0	0	2
2	2	2	2

$p \vee q$	1	0	2
1	1	1	2
0	1	0	2
2	2	2	2

The tables for the conditional and the biconditional (the latter is simply the = in the type *o*):

$p \supset q$	1	0	2
1	1	0	2
0	1	1	2
2	2	2	2

$p \equiv q$	1	0	2
1	1	0	2
0	0	1	2
2	2	2	2

These tables are the weak Kleene tables. The connectives working according to them always transmit the truth value gap from the part to the whole. This is not true for the strong Kleene connectives for which the truth tables are the following:

p	$\neg p$
1	0
0	1
2	2

$p \,\&\, q$	1	0	2
1	1	0	2
0	0	0	0
2	2	0	2

$p \vee q$	1	0	2
1	1	1	1
0	1	0	2
2	1	2	2

$p \supset q$	1	0	2
1	1	0	2
0	1	1	1
2	1	2	2

$p \equiv q$	1	0	2
1	1	0	2
0	0	1	2
2	2	2	2

The strong Kleene conjunction, alternation, and conditional does not transmit the truth value gap: for example, if p is true then the value of $p \vee q$ is 1 (true) even if the truth value of q is 2. The strong Kleene connectives are in better harmony with "the logic of empirical investigations", the conception that Ruzsa calls *epistemic*. In such a logic 2 denotes the value "yet unknown".

Why did Ruzsa decide in favor of the weak versions? The question has (at least) three answers. One is a bit personal: the epistemic conception reminds him of the "so called" intuitionist logic; and this logic, according to his conception, does not even deserve the name 'logic'.[1] The second, and less personal, answer is that using Ruzsa's system's temporal operators and introducing an epistemic operator, one could probably succeed in modeling some aspects of the "epistemic conception".

[1]He once told us the following story. In a conference (probably in the sixties) when he used the arrow for the conditional, the chair asked him: "So you are an intuitionist, aren't you?' At the very moment he decided to use the horseshoe symbol: let there be no mistake.

But Ruzsa's most prominent reason is the importance of the transmittal of the semantic value gaps, a phenomenon we have already seen in the definitions of the domain of functor types and that of the factual values of the identites. A general theorem of his system states that this phenomenon holds in general.[2]

Summing up: in extensional contexts, a semantic value gap is a special "illness" for which the treatment is: not allowing it to disappear. A truth value gap is really a *gap*, arising from clashes of language and reality; it is impossible for the "real world" to have gaps.

> My *credo* is simply this: A sentence may or may not have a truth value. If it has one then it expresses a statement which is either true or false. The lack of a truth value is not a third truth value. (Ruzsa 1991, 11.)

2 ABSOLUTELY UNDECIDABLE SENTENCES

In the epistemic conception the semantic value gaps are due to the gaps in our logic. By contrast, Ruzsa's approach is ontological. As he puts it:

> In (. . .) informal reasonings, the sources of value gaps are located in the realm of facts, in the formal semantics they [are] located in the interpretations of the (formal) language. (Ruzsa 1991, 12.)

The facts of which Ruzsa speaks are facts of the world *and* facts of our language, gaps only arise when something is mistaken in our expressions. Are there sources of truth value gaps "in the world" in which our language plays no significant role? In other words: are there gaps in reality? Before trying to answer this question, let's go back once more to the epistemic conception. According to our knowledge of it, every (well-formed, unambiguous) sentence p must fall in one of the following seven cases:

(1) p is true, and we know (proved, verified) that it is true
(2) p is true; we do not know that yet, but we will
(3) p is true, but we will never know that it is true
(4) p is absolutely undecidable (even God cannot determine its truth value)
(5) p is false, but we will never know that it is false
(6) p is false; we do not know that yet, but we will
(7) p is false, and we know that it is false

In cases (1), (2), (6) and (7) there are no difficulties. Moreover, we have good candidates that are of case (3) or of case (5). For example, let p be the sentence

> The value of the digit in the $10^{10^{10^{10}}}$th place of the decimal expansion of $\pi - 3$ equal to zero. [3]

[2] If A (of type α) is an extensional component of B (of type β), then $|A|_{vi} = \Theta(\alpha) \Rightarrow |B|_{vi} = \Theta(\beta)$, for the details see (Ruzsa 1991, 33.)

[3] Cf. (Feferman 2006).

The truth value of this sentence can be determined in principle by a mechanical check—but this check is far beyond our computational powers. Nevertheless, we can say that this sentence has a determinate truth value and God knows what it is.

What about case (4)? Does the existence of absolutely undecidable sentences threaten God's omniscience? We can say with Michael Dummett: not at all. God knows the answer to every question that has answer, and He knows of every question whether it has an answer. If there really are questions which has no answer even for Him then the divine logic must be three-valued, and instead of a truth value gap, there will be a genuine third truth value.[4]

In comparison with God, in this respect (too) we are in a more uncomfortable position: we cannot in principle distinguish cases (4), (5), and (6). The reason lies in what can be called the "Hauptsatz" of undecidable propositions: if p is an absolutely undecidable statement, then we cannot prove "constructively" that it is really the case. The proof (due to Martin-Löf[5]) relies heavily on the constructivist (verificationist) conception of negation: proving $\neg p$ amounts to showing that any attempt to prove p will eventually be blocked (in mathematics, by a contradiction; in general, by some serious difficulty). Proving that p is undecidable amounts to a proof that any attempt to prove p, as well as any attempt to prove $\neg p$, will eventually be blocked. But it is nothing but a proof of $\neg p$, and $\neg\neg p$, respectively. We arrive at a contradiction (a serious difficulty).

So we have to rely on our intuitions.

3 FINITE KNOWLEDGE OF THE INFINITE

We cannot prove of a sentence that it is absolutely undecidable, but we can perhaps imagine what such a sentence could be. Feferman's (in fact, the intuitionists') example about the decimal expansion of π gives a clue. If we—with our limited means—cannot end a process that is too long, it is conceivable that an infinite process is such that even an infinite mind cannot go through it.

But we must be careful. Even we, finite beings know very much about the natural numbers, we can prove for example that for every natural umber n, 7^n is divisible with 6, that there are infinitely many primes and so on. We have methods which make the infinite finite, that is, methods (first of all, mathematical induction) by which we can prove, say, that for some property F,

[4]See e.g. (Dummett 2006, 108—109).

[5](Martin-Löf 1995). Martin-Löf actually "proves" that there are no undecidable propositions. His proof relies on the intuitionist conceptions of proposition, truth, falsity, and knowledge. For someone not in the intuitionist camp, his conclusion can be formulated as follows: there may be absolutely undecidable propositions, but we cannot produce one about which we can prove that it is really absolutely undecidable. Cf. (Feferman 2006, 147.).

there are infinitely many numbers for which it holds. Augustine and many of his followers believe that for God it is so with *every* property F.

> As for their other assertion, that God's knowledge cannot comprehend things infinite, it only remains for them to affirm, in order that they may sound the depths of their impiety, that God does not know all numbers. For it is very certain that they are infinite. . . Does God, therefore, not know numbers on account of this infinity; and does His knowledge extend only to a certain height in numbers, while of the rest He is ignorant? Who is so left to himself as to say so?
>
> . . . if everything which is comprehended is defined or made finite by the comprehension of him who knows it, then all infinity is in some ineffable way made finite to God, for it is comprehensible by His knowledge.(Augustine 1993, Book XII., Chapter 18.))

According to this conception, the property 'n is one member of a twin prime-pair' is for God as simple as for us the property 'n is a prime number': He can decide whether it holds for infinitely many numbers or not. He can perhaps "see" some higher-order structure which decide the matter (as in the case of Fermat's last theorem there are structures revealed by the theory of analytic functions that decide that a property holds for all natural numbers bigger than 2). But is it really the case? Is every property of natural numbers such that it is in principle possible "making it finite"—deciding in finite steps, whether it holds for infinitely many numbers or not? If there is a property F for which even God has no other choice in order to decide whether it holds for infinitely many numbers than to check "all" the numbers one after another, then the sentence 'there are infinitely many numbers n for which $F(n)$ holds' is a good candidate for being absolutely undecidable.

We can argue that—*pace* Augustine—in this case even God cannot determine the truth value of this sentence (but He would know *that*). Such a sentence would then be per definitionem absolutely undecidable, and as such, it would signal the presence of a real gap "in the world".

4 INFINITE TASKS

What makes it impossible even for God to run through an infinite series of computations? The strongest argument can be extracted from the paradoxes of super-tasks. The classic example of these paradoxes is Thomson's lamp.

> There are certain reading-lamps that have a button in the base. If the lamp is off and you press the button the lamp goes on, and if the lamp is on and you press the button the lamp goes off.
>
> Suppose now that the lamp is off, and I succeed in pressing the button an infinite number of times, perhaps making one jab in one minute, another

jab in the next half-minute, and so on. . . After I have completed the whole infinite sequence of jabs, i.e., at the end of two minutes, is the lamp on or off? It seems impossible to answer this question. It cannot be on, because I did not ever turn it on without at once turning it off. It cannot be off, because I did in the first place turn it on, and thereafter I never turned it off without at once turning it on. But the lamp must be either on or off. This is a contradiction.[6]

One lesson from the paradox is simply this: carrying out a super-task is *conceptually* impossible. For "at the end" of the infinite series of tasks, there could be a discontinuity which cannot be explained. Russell famously called it only "medically impossible" running through the whole expansion of π, see (Russell 1953, 143.). By contrast, Michael Dummett argues that "the reason why we cannot survey an infinite totality is not the deficiency of human capabilities: it is that it is *senseless* to imagine an infinite task completed" (Dummett 2006, 70–71; the italic is Dummett's).

Arithmetic can be a natural realm of super-tasks. If there is a property F of natural numbers for which the truth value of $F(n)$ for each number n can be determined by finite computation but there are "absolutely" no general method for determining whether $F(n)$ holds for infinitely many numbers or not then even a "Divine Arithmetician" who can carry out a computation "infinitely quickly" cannot determine the truth value of the statement *there are infinitely many n for which F holds* or, of the statement *there are infinitely many n for which F does not hold*. For to decide these statements, She has to check every number n, that is, by running through an infinity of tasks. And this is impossible—even for the Divine Arithmetician. The reason is that the same kind "discontinuity" would arise as in the case of the Thomson's lamp. For suppose the "prover" has a white paper. After checking $F(0)$, she paint it black; then after checking $F(1)$ she paint it again white; and so on. (If she can decide whether $F(n)$ is true or false then manipulating the paper is only a simple extra.) What color will be the paper after checking all of the natural numbers? There is no answer—the super-task cannot be carried out.[7]

[6]Thomson (1954), cited in (Sainsbury 2009, 12.). The paradox resembles that of the staccato run, a variant of Zeno's Racetrack paradox. In the staccato version the runner - say, Achilles - runs for half a minute, then pauses for half a minute, then runs for a quarter of a minute, then pauses for a quarter of a minute, and so on ad infinitum. At the end of two minutes he will have stopped and started in this way infinitely many times. Each time he pauses he could perform a task of some kind. Then at the end of two minutes he will have performed infinitely many of these tasks. On the staccato run and other paradoxes of the infinite, see (Moore 1990).

[7]It is a (super-)task for the philosophers of time to explain exactly what makes it impossible for (even) the Divine Arithmetician to run through an infinite series of tasks. For if the continuum of time has the structure—say—of the real numbers than for "someone" who could count with no speed limit, it may be possible to determine whether $F(0)$ holds or not in a minute, $F(1)$ in half a minute, $F(2)$ in a quarter of a minute and so on. . .

If p is such a sentence then it may happen that God does not know whether it is true or false. (But even in this case He knows *that*.) And in this case we can call 2 a *real* truth value.

5 WHAT WOULD RUZSA SAY

Without any doubt this argument would not affect Ruzsa's philosophical position. If there was a knock-down argument against his—Platonist—view from the standpoint of the constructivists and the intuitionists, it would not be a philosophical argument like the preceding one. And I can imagine Imre Ruzsa stamping his foot saying: "after all, there are infinitely many n for which $F(n)$ is true or there are only finitely many such n, there is no third possibility".*

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