Jiří Raclavský

Partiality and Tichý's Transparent Intensional Logic

Abstract. The paper focuses on treating partiality within Tichý's logical system. Tichý's logic is two-valued and type-theoretic. His simple theory of types (and the deduction system for it) accepts both total and partial functions. Tichý's late framework is explicitly ramified. So-called constructions (roughly: algorithms) construct, e.g., values of functions at arguments; in some cases, however, they do not construct anything at all. This special partiality phenomenon is discussed in the second part of the paper.

1 INTRODUCTION

It will be convenient to begin with a sketch of the history of Pavel Tichý's transparent intensional logic (briefly *TIL*). One can find roots of TIL already in 1960s when Tichý construed intensions (functions from possible worlds) as classes of algorithms-procedures which were considered as meanings of (empirical) expressions; see Intensions in Terms of Turing Machines (Tichý 1969) reprinted in (Tichý 2004; hereafter *CP*). In the very beginning of the 1970's, Tichý modified Church's typed lambda calculus—accepting not only individuals and truth-values but also possible worlds; see An Approach to Intensional Analysis (1971) in CP. The system differs significantly from that developed by Montague (and Montagovians); the lack of space prevents me to give here a comparison (*cf.* Tichý's own remarks in CP 132-137 and the paper Two kinds of intensional logic, CP 307-325).

A new era of TIL (now explicitly designated by this name) is embodied in the large monograph (Tichý 1976) which remained unpublished. In this book, λ-terms and constructions recorded by them are explicitly distinguished (we will return to this issue later). Secondly, partial functions are admitted. Thirdly, natural deduction for the system is exposed. Selected parts of the book were

published as papers in the second half of 1970s; an exception is (Tichý 1982), which is a condensed paper on deduction.

In 1978 (*cf.* CP 269-270), Tichý added the temporal parameter; intensions are thus considered as functions from ⟨possible worlds, time-moment⟩ couples. Interesting logical analyses of temporal discourse (tenses, etc.) and episodic verbs were published by Tichý in 1980 (see CP). A more important modification of TIL is suggested in (Tichý 1988); Tichý exposed there a remarkable type theory which combines, in fact, simple and ramified type theory.

2 ADOPTING PARTIALITY

Let us begin with the recognition that there are both total and partial functions. Since many phenomena are to be modelled by partial functions (e.g., the chronology of American presidents in the actual world, or the individual concept "the king of France"), it is natural to accept them.[1]

Sometimes it is held that three-valued logic (3V-logic) captures partiality and so it is identical with two-valued logic which adopts partiality (2VP-logic). This is, however, a questionable matter. For 3V-logic—recognizing T (true), F (false), and U (unknown, undecided, …)—is a logic with total functions only. On the other hand, 2V-logic recognizes T and F and accepts also a lack of a value for some function(s). Thus domains of truth-values of 3V-logic and 2VP-logic do differ.[2] For instance, there are 27 unary 3V-truth-functions but there are just 9 total and partial unary 2VP-truth-functions:

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9
T	T	T	T	F	F	F			
F	T	F		T	F		T	F	

Clearly, the function f_4 is classical negation (often denoted by '¬'); it is, however, entirely missing in 3V-logic (this is why we should say that 3V-logic is not a classical logic). Of course, the 3V-function T→F, F→T, U→U looks like a counterpart of ¬. For an obvious reason, however, plenty of 3V-functions cannot be counterparts of any 2VP-functions. (I will return to the problem of representation at the end of the paper.)

Once partial functions are admitted, strange phenomena appear. For instance, Schönfinkel's reduction does not work because one multi-argument

[1] The reader knows that Imre Ruzsa stressed the importance of partiality (*cf.*, e.g., 1.2 in Ruzsa 1991).

[2] Of course, the acceptance of U or "gap" (as we may call it) is governed by the same intuition.

(m-ary; $m > 1$) partial function corresponds to more than one 1-argument function (Tichý 1982, 59-60); thus multi-argument functions are irreducible entities. Before we proceed further, let me introduce some notions.

3 TICHÝ'S SIMPLE THEORY OF TYPES

Tichý's simple theory of types—e.g., (Tichý 1982, 60)—treats both total and partial functions. It is quite general, since it has an unspecified basis B:

Let B consist of mutually non-overlapping collections of objects.

a) Any member of B is a *type over B*.

b) If $\zeta, \xi_1, \ldots, \xi_m$ are (not necessarily distinct) types over B, then $(\zeta\xi_1\ldots\xi_m)$, which is a collection of all total and partial functions from $\xi_1, \ldots, \xi_m$ into ζ, is a *type over B*.

The (specific) basis of TIL comprises ι (individuals), ο (truth-values T and F), ω (possible worlds), and τ (time-moments/real numbers). *Intensions* are functions from ω to (total or partial) chronologies of ξ-objects (a chronology is a function of type $(\xi\tau)$). Briefly speaking, intensions are functions from ⟨possible world, time-moment⟩ couples. '$((\xi\tau)\omega)$' will be abbreviated to '$\xi_{\tau\omega}$'. *Propositions* are of type $\text{ο}_{\tau\omega}$; *properties of individuals* are of type $(\text{οι})_{\tau\omega}$; individual offices (Tichý's term) are of type $\iota_{\tau\omega}$; etc. Objects which are not intensions may be called extensions. For instance, classical unary ($\neg$) or binary ($\wedge, \vee, \rightarrow, \leftrightarrow$) truth-functions are of types (οο) and (ooo), respectively; classical quantifiers ($\forall^\xi, \exists^\xi$) are of type $(\text{o}(\text{o}\xi))$; $=^\xi$ is of type $(\text{o}\xi\xi)$ ('$^\xi$' will be suppressed).[3]

4 CONSTRUCTIONS

To introduce the idea of constructions, consider the function:

$1 \rightarrow -2$
$2 \rightarrow 1$
$3 \rightarrow 6$
⋮ ⋮

This function can be reached by (infinitely) many different (mathematical) procedures. For instance, it is induced by multiplying an integer by itself and subtracting three from the result (i.e. by $(n \times n) - 3$) or by adding an integer to its square and subtracting what one gets by adding three to the integer from

[3] Ruzsa's type theory (*cf.* Ruzsa 1989, 3) does not allow some (types of) intensions which are admitted by Montagovians and Tichý. It should be added here that Tichý accepts not only functions of type $\xi_{\tau\omega}$ but also of type ξ_τ or ξ_ω (such functions are not called intensions in the present text).

the result (i.e. by $(n^2 + n) - (n + 3)$). To every such intuitive procedure, there corresponds a certain Tichý (numerical) construction. Tichý used λ-terms to record constructions and one may view constructions as so-called intensional (i.e. not extensional) senses of λ-terms. To get another analogy, recall hyperintensions ("structured meanings") often urged within the logical analysis of natural language. It seems also that Frege's *Sinn* or Russell's (structured) propositional functions are predecessors of constructions.

Unfortunately, a rigorous definition of constructions cannot be expounded here, see (Tichý 1988, 56-65) for that purpose. Omitting here so-called single and double execution, there are four kinds of constructions; their brief characterization is as follows. Let X be any object (a construction or non-construction) and C any construction; let v be any valuation (it is a field that consists of sequences of objects of given types):

1. *Trivialization* ${}^0X$ v-constructs X (i.e. ${}^0X$ takes X and leave it as it is).
2. *Variable* x_k v-constructs the kth member of the sequence of objects of a given type.
3. *Composition* $[CC_1 \ldots C_m]$ v-constructs the value of the function constructed by C at the string of objects (i.e. the argument for that function) which are constructed by $C_1, \ldots, C_m$; if C or C_1 (etc.) does not v-construct such object(s) or the function is undefined for that argument, $[CC_1 \ldots C_m]$ is v-*improper*—it does not v-construct anything at all.[4]
4. *Closure* $\lambda x[\ldots x \ldots]$ v-constructs, in a nutshell, the function which takes particular values of x to the objects v-constructed by $[\ldots x \ldots]$ on the respective valuations (e.g., $\lambda n[[n^0 \times n]^0 - {}^0 3]$ v-constructs the function sketched above).

One may thus say that these four kinds of constructions are objectual correlates of constants, variables (as letters), applications, and abstractions of λ-calculi. Realize, however, that constructions are not expressions—they are language-independent entities (the proper subject of Tichý's approach are constructions, not expressions of some formal language). For instance, the term '$\lambda n[[n^0 \times n]^0 - {}^0 3]$' denotes (stands for) the construction $\lambda n[[n^0 \times n]^0 - {}^0 3]$. Realize also that constructions are not set-theoretical entities. Note that the term '$\lambda n[[n^0 \times n]^0 - {}^0 3]$' denotes the procedure as such, not the aforementioned function constructed by $\lambda n[[n^0 \times n]^0 - {}^0 3]$ (analogously, '$[{}^0 8^0 \div {}^0 2]$' denotes the construction $[{}^0 8^0 \div {}^0 2]$, not its result—the number 4).

[4] The usual argument for the adoption of hyperintensions is this. Intensional semanticist suggests that all true mathematical sentences denote one and the same proposition (which is true in all possible worlds). Consequently, 'Xenia believes that 3+4=7', '49÷7=7' ∴ 'Xenia believes that 3+4=49÷7' is rendered as a valid inference which is obviously not. Hence more fine-grained entities than intensions are needed to be explications of meanings. For another reason consider 'Xenia calculates 3÷0'; the sentence surely describes the agent as related to a certain calculation, not to its (non-existing) result.

For non-circularity conditions, Tichý introduced a ramified theory of types. Its definition in (Tichý 1988, 66) has three parts: (a) types of ("classical") set-theoretic objects (*cf.* the simple-type theoretic part above), (b) types of constructions (some constructions are first-order constructions, belonging to the type $*_1$, other constructions are second-, third-, …, *n*-order constructions), (c) types of functions from/to constructions.

In the mid-1970s, Tichý already suggested that constructions are explications of (natural-language) meanings—having thus the following semantic scheme:

an expression *E*

expresses (means) in *L*

the construction, which is the *meaning* (or logical analysis) of *E* in *L*

constructs

an intension / non-intension / nothing (*cf.* '3÷0'), which is the *denotatum* of *E* in *L*

The value of an intension in a possible world w, time-moment t is the *referent* of an empirical expression *E* (such as 'dog', 'the king of France', 'It rains in London'); the denotatum and referent of a non-empirical expression are understood as identical.

For example (let w and t be variables v-constructing possible worlds and time-moments, respectively):

'The king of France is bald'	an expression *E*
$\lambda w \lambda t[{}^0\mathrm{Bald}_{wt}\ {}^0\mathrm{KF}_{wt}]$	the construction expressed by *E*[5]
$\langle w_1, t_1\rangle \rightarrow \mathrm{T}$	the proposition denoted by *E*
$\langle w_2, t_2\rangle \rightarrow$	(i.e. gap)
$\langle w_3, t_3\rangle \rightarrow \mathrm{F}$	
etc.	
T	the referent of *E* in w_1, t_1

It is not difficult to show that this semantic theory is capable to deal with puzzles created by "intensional" and "hyperintensional" contexts.

5 PARTIALITY AND FAILURE OF CLASSICAL LAWS

From the objectual viewpoint, logical laws are not strings of letters but constructions. It is clear that (let o be a variable v-constructing truth-values):

$$[{}^0\forall \lambda o[o^0 \vee [{}^0\neg o]]$$

[5] Trivializations of well-known mathematical or logical functions will be written in the infix manner (e.g., '$[{}^0 8^0 \div {}^0 2]$' instead of '$[{}^0 \div {}^0 8^0 2]$').

is tautological (the variable *o* is always a *v*-proper construction). However, this law is scarcely remarkable, one would rather declare that for any proposition, it obtains in w, t or it does not obtain in w, t (the excluded middle). Let p be a variable *v*-constructing propositions (i.e. objects of type $o_{\tau\omega}$). Then the following construction (which can be closed by $[^0\forall\lambda w[^0\forall[\lambda t$; similarly below) is contradictory:

$$[^0\forall\lambda p[p_{wt}{}^0\vee[^0\neg p_{wt}]]]$$

Since if some proposition is undefined in w, t, then $\lambda p[p_{wt}{}^0\vee[^0\neg p_{wt}]]$ *v*-constructs a partial class (partial characteristic function) which is empty, thus $\forall$ takes it to the truth-value F.

We get an analogous failure for the carelessly formulated De Morgan law for exchange of quantifiers. Let P be any construction of a proposition where P contains x as its free variable (e.g., $\lambda w\lambda t[x^0={}^0\mathrm{KF}_{wt}]$):

$$[[^0\neg[^0\exists\lambda xP_{wt}]]^0\leftrightarrow[^0\forall\lambda x[^0\neg P_{wt}]]]$$

The construction $\lambda w\lambda t[x^0={}^0\mathrm{KF}_{wt}]_{wt}$ (which is reducible to $\lambda x[x^0={}^0\mathrm{KF}_{wt}]$) can *v*-construct a partial class which is empty thus $\forall$ takes it to F, not to T as we wish.

To avoid the destructive power of partiality, formulating thus the correct versions of the laws, I suggest utilizing a "totalizer" overcoming the trouble. In Tichý's framework, there are three kinds of properties "be true" due to their applicability to (a) propositions, (b) constructions, (c) expressions (relatively to a given language *L*). Each kind has several variants; the (a)-kind has only two. The *"partial" truth property* of propositions (i.e. an object of type $(oo_{\tau\omega})_{\tau\omega}$) can be defined as:[6]

$$[^0\mathrm{True}^{\pi\mathrm{P}}{}_{wt}p]\equiv p_{wt}$$

thus certain propositions are not in the extension or the anti-extension of that property (in w, t). The *"total" truth property* of propositions can be defined as:

$$[^0\mathrm{True}^{\pi\mathrm{T}}{}_{wt}p]\equiv[^0\exists\lambda o[[p_{wt}{}^0=o]^0\wedge[o^0={}^0\mathrm{T}]]]$$

A partial proposition having no value in w, t belongs to the anti-extension of the property—it is not true in w, t.

Using $^0\mathrm{True}^{\pi\mathrm{T}}$ for "totalizing", the correct law of excluded middle is:

[6] 'C_{wt}' abbreviates '$[[Cw]t]$'. Of course, ^{0}KF is a simplification. The procedure consists in taking (a) the property "popular", (b) applying it to w and t (values of w and t), getting thus the extension of "popular", and then (c) taking "the king of France", (d) applying it to w and t, getting thus the individual who fills that office, and (e) asking whether that individual (if any) is in that extension—yielding thus T or F (analogously for other w's and t's).

$$[^0\forall\lambda p[[^0\mathrm{True}^{\pi\mathrm{T}}_{w\not t}p]^0\vee[^0\neg[^0\mathrm{True}^{\pi\mathrm{T}}_{w\not t}p]]]]$$

and the correct De Morgan law is:

$$[[^0\neg[^0\exists\lambda xP_{wt}]]^0\leftrightarrow[^0\forall\lambda x[^0\neg[^0\mathrm{True}^{\pi\mathrm{T}}_{wt}P]]]]$$

So it is clear that partiality affects the rules for substitutivity (e.g., whether o can be substituted by p_{wt}), which lead Tichý to the sophisticated theory exposed in (Tichý 1982) and in Indiscernibility of Identicals (Tichý 1986), where he paid closer attention to constructions involving identity.

6 BETA-REDUCTION, ETA-REDUCTION AND PARTIALITY

But there are more complications with partiality—even classical β-reduction fails (β-reduction rule says that $[\lambda x[\ldots x\ldots]C]$ is equivalent to $[\ldots C\ldots]$). Consider:

1. $\lambda w\lambda t[^0\neg[^0\mathrm{True}^{\pi\mathrm{T}}_{wt}\lambda w\lambda t[^0\mathrm{Bald}_{wt}\,^0\mathrm{KF}_{wt}]]]$
(the analysis of 'It is not true that the King of France is bald')

2. $\lambda w\lambda t[\,\lambda x[^0\neg[^0\mathrm{True}^{\pi\mathrm{T}}_{wt}\lambda w\lambda t[^0\mathrm{Bald}_{wt}x]]]\,^0\mathrm{KF}_{wt}]$
(the analysis of 'The King of France is such that it is not true that he is bald')

The two constructions v-construct distinct propositions because 1. v-constructs a total proposition whereas 2. v-constructs a partial proposition (if there is no king of France in w, t, $^0\mathrm{KF}_{wt}$ is v-improper, so the proposition is gappy). Thus 2. is not β-reducible to 1.[7]

In (Tichý 1982, 67), β-reduction and β-expansion are explained as deduction rules (The Rule of Contraction/Expansion). Tichý's rule of β-reduction contains an explicit condition that the construction C, which is substituted, is not v-improper. So conditioned, β-reduction preserves equivalence of constructions.[8]

Moreover, η-reduction (that $\lambda x[Cx]$ is reducible-equivalent to C) fails as well (Raclavský 2009, 283). Consider $[^0Fy]$, where ^0F v-constructs a function of type $((\theta\zeta)\xi)$ which is undefined for the ξ-object assigned to y by v. Thus $[^0Fy]$ is v-improper, it v-constructs nothing at all. But $\lambda x[[^0Fy]x]$ does v-construct an object, namely a function of type $(\theta\zeta)$ which is undefined for the ζ-object assigned to x by v. Hence $\lambda x[[^0Fy]x]$ cannot be equivalent to $[^0Fy]$. The remedy (*ibid.*) is the same as Tichý's conditioning of β-reduction.

[7] Here ≡ means inter-derivability of two constructions (*cf.* $\leftrightarrow_t$ in CP 489); the two constructions flanking ≡ v-construct one and the same object (a truth-value in this case) or they both v-construct nothing at all. The construction $[^0\mathrm{True}^{\varpi\mathrm{P}}_{w\not t}p]$ can be closed by $\lambda w\lambda t[\lambda p$ and then η-reduced to $^0\mathrm{True}^{\varpi\mathrm{P}}$ (which is used below). See (Raclavský 2008) for more.

[8] *Cf.* (Duží 2003) for more.

7 ANOTHER WAY OF REPAIRING PARTIALITY

When defining various concepts one sometimes needs to overcome partiality of some function. For the sake of illustration, imagine that you sum salaries of various people—including the king of France ("…+the salary of(KF)+…"). Since there is no king of France, "the salary of(KF)" returns no number. But you need a certain number (zero in this case) because you do not want the final sum ("…+…+…") to be undermined by the "local" partiality failure.

In Tichý's logic, the delivering of a "dummy value" (e.g. zero) can be easily managed in the following way (see Raclavský 2009, 243). Consider a partial function F from (type-theoretically appropriate) x's to (type-theoretically appropriate) y's. Since F is partial, $[^0Fx]$ can v-construct nothing; in such case, however, you need something—a dummy value. I suggest replacing $[^0Fx]$ in a construction C (which is affected by partiality of F) by the following construction which fulfils our demand:

$$\begin{array}{ll} [^0\mathrm{Sng}\lambda z[^0\mathrm{If}\_ & [^0\exists\lambda y[y^0=[^0Fx]]] \\ \quad \_\mathrm{Then}\_ & [^0\exists\lambda o[[o^0=[z^0=[^0Fx]]]^0\wedge[o^0={}^0\mathrm{T}]]] \\ \quad \_\mathrm{Else} & [z^0={}^0\mathrm{DummyValue}]]] \end{array}$$

(the singularization function, Sng^{ξ}, takes one-membered classes of ξ-objects to their sole members, it is undefined otherwise; if_then_else is the well-known ternary truth-function, its trivialization is written in parts; note that there is an analogue of "it is true$^{\pi\mathrm{T}}$ that $y=F(x)$" in the second line).

8 THREE-VALUED FUNCTIONS REPRESENTED BY PROCEDURES

To conclude this short paper, 2VP-logic incorporating procedures (constructions) is capable to capture the intuition which underlies 3V-logic, if 3V-functions are modelled not by (partial) functions but by procedures. For instance, the 3V-function T→F, F→T, U→U can be modelled by $[^0\neg p_{wt}]$ because this construction behaves in an analogous way as that 3V-function: it returns T when the proposition p has (in w, t) the truth-value F (and *vice versa*) but it returns nothing if the proposition p is undefined (in w, t). To define procedures representing other 3V-functions is usually more involved—one must utilize $^0\mathrm{True}^{\pi\mathrm{T}}$, often together with the dummy-value construction.

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[9] It seems that 2.4.2.5 in (Ruzsa 1989) captures an analogous restriction of β-reduction.