

Modal Logic and Relativity

Abstract. We argue that modal logic is the natural logic to use to reason about relativity theory. We define a complete modal axiomatisation of the kinematics of special relativity theory.

Relativity Theory, in its most general sense, rejects the notion of absolute space. In Relativity Theory statements such as “this rod is one meter long” are frowned upon, rather we should say “this rod is one meter long when measured in this frame of reference”. When devising a logic to reason about relativity theory, it is therefore natural to adopt a modal logic where every statement has an implicit ‘point of view’. In that sense, a modal logic might be more true to the subject matter of relativity theory. We do not claim that such an approach will in itself provide new technical results in relativity theory, but apparent paradoxes and other conceptual difficulties with relativity theory might be more easily avoided in a modal setting. A second motivation for modal logic is that the complexity of reasoning in a modal logic can often be lower than with first-order logic. Thirdly, the current article is only an initial step and we believe that modal logic should be used to reason about general relativity where it is even more important to work in a local framework. In this article we will consider how a modal logic for special relativity theory might be devised.

A fundamental concept in relativity theory is that of the observation. For Einstein, and his group of followers in the *Vienna Circle*, the precise nature of an observation was of great importance. In Einstein’s original paper on Special Relativity (Einstein 1905), he takes care to clarify the meaning of words like “time”, “simultaneous”, “length”, etc. by replacing them by statements concerning observers and observations. Later, *Logical Positivists* formulated the *verification principle*, which stated that a proposition could be held to be true to the extent that it could be tested by experiments and observations. The critical thing about observations in relativity theory is that what is observed depends

not only on the event but on the observer too. Later, we will define a modal logic with a Kripke semantics in which each observer is a Kripke world.

One intriguing feature of observers is that they act both as subjects and as objects—they see, but they can also be seen. In Einstein’s original paper, he refers to an observer as “the man at the railway-carriage window”, suggesting a point-like body moving through space. In most presentations of relativity theory, an observer is a point-like body with its own world line in 4D spacetime. This tells us that when we see an observer, he looks one-dimensional. We see the observer’s time axis, but we do not see his space axes. However, from his point of view, an observer can see various events taking place at various spacetime points distributed throughout the four dimensions of spacetime. Indeed, when we think of an observer as a subject, it is better not to think of an individual point-like body, but a whole team of colleagues arranged in a grid in three-dimensional space, not moving relative to each other over time, who send messages to each other (or perhaps to a central control centre) about their immediate observations. But as we mentioned, only one member of the team of observers is directly visible to observers from other reference frames. These two aspects of observers are related in (Andréka & *al.* 2010) by an axiom that requires that all observers see themselves at the space origin of their reference frame, at all times, i.e. their world line is the time axis.

Here we consider two different applications of modal logic to special relativity theory: the first approach has to do with modal frame definability, and the second uses model definability. To take a simple and perhaps more familiar example of frame definability, the logic **S4** with axioms $\{\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q), \Box p \rightarrow p, \Box p \rightarrow \Box \Box p\}$ and inference rules modus ponens and necessitation defines the set of all validities over Kripke frames whose accessibility relation is reflexive and transitive, that is, **S4** defines the validities of the class of reflexive transitive frames. Furthermore, if these axioms are valid over any Kripke frame (W, R) , then R is transitive and reflexive over W , i.e. **S4** defines the class of reflexive transitive frames. For model definability, if we take the class of reflexive transitive frames as given, the formula $\Box(p \rightarrow \Diamond \neg p)$, which is not valid over any frame, defines those models where p is never ‘eventually true’.

For special relativity, the frame definability approach has already been considered. A Kripke frame may be defined whose Kripke worlds are the points of a four-dimensional Minkowski spacetime and where the accessibility relation is “can send a signal to” and the problem is to define a modal logic that derives all modal formulas valid over frames with this accessibility relation. If we require that signals travel at less than the speed of light and that a signal may be sent from a spacetime point to itself (reflexivity of the accessibility relation) then the set of modal validities is **S4.2** (Goldblatt 1980; Shehtman 1983), the logic obtained by adding the axiom $\Diamond \Box p \rightarrow \Box \Diamond p$ to the axioms of **S4**. However, **S4.2** does not define this class of four-dimensional models, as the same set of

validities hold over Minkowski spacetime of dimension $2, 3, \dots$, over Galilean models or even over models with discrete time. **S4.2** actually defines the class of all reflexive, transitive and *directed* frames. If you consider an irreflexive accessibility relation “can send a signal to a different point” then modal logic can be more discriminating, however, the purpose of all this is to axiomatise the validities of the given accessibility relation, it does not express the lengths of rods or how they may be transformed when viewed by another observer.

The main focus of the current article is about model definability. We seek a modal logic which is able to express the kinds of observations we might want to make in four-dimensional spacetime, and it should also be able to express the kinds of observations we might expect other observers to make. It can be considered as part of a line of research which seeks to provide simple logical axioms from which the main theorems of relativity theory can be derived. Many such logics have been devised, we make no attempt to survey them here, see (Andréka & *al.* 2006) for an extensive survey. In general, previous attempts to provide a logic for relativity adopt a first-order, or in many cases second-order (e.g. the axiom of continuity in (Schutz 1997)) logic. Consequently they adopt Tarskian semantics. Global variables may be used to denote bodies, field values etc. Instead of the Newtonian statement “Body b is at (x, y, z) at time t ”, the dependence of an observation on the observer is expressed by an observation predicate W , so that $W(o, b, x, y, z, t)$ expresses “observer o sees body b at (x, y, z, t) ”.

In a modal logic in which the role of Kripke world is taken by an inertial frame of reference, the dependence on the observer will be suppressed. The modal operator \Diamond will permit us to transfer from one inertial frame to another. We will take as given a frame in which the accessibility relation is the universal relation. But what language should be adopted to describe the observations made within a single reference frame? In order to keep the presentation fairly general, here we use a two sorted first-order logic with one sort for bodies (B) and the other for quantities (Q). Variables of sort Q will be used to record coordinate values. We may use subscripts b, q for constants, functions and predicates to indicate their sort. The predicate See_{q^4b} requires four coordinates and one body and tells us that this body is observed at the four coordinate values. We avoid a number of difficulties that sometimes arise with modal first-order languages by requiring that the first-order variables have the same domain at each world of a structure.

The Language

Variables:	b, c, \dots (type B), x, y, z, \dots (type Q)
Constants/functions:	$0, 1, +, \times$ (type Q)
Predicates:	$Ph_b, Obs_b, \leq_{qq}, \text{See}_{q^4b}$
Formulas:	$\phi ::= \text{Atom} \mid \neg\phi \mid (\phi_1 \vee \phi_2) \mid \exists \text{var} \phi \mid \Diamond\phi$

The following abbreviations will be useful later.

$$\begin{aligned} \text{vel}(b) = (v_0, v_1, v_2) \quad \text{means} \quad & \exists \bar{x} \forall \bar{y} \\ & [\text{See}(\bar{y}b) \leftrightarrow \exists \lambda \bar{y} = \bar{x} + \lambda \times (v_0, v_1, v_2, 1)] \\ |(x_0, x_1, x_2)| \quad \text{means} \quad & \sqrt{x_0^2 + x_1^2 + x_2^2} \end{aligned}$$

Structure As promised, our semantics will be based on Kripke-like structures, where the Kripke worlds are inertial frames of reference. Thus a structure will have the form

$$(W, \beta, F, I, W \times W)$$

where W is the set of Kripke worlds, β is the set of bodies, F is the set of quantities, I interprets variables as elements of β or F (depending on the sort of the variable), $0, 1, +, \times, \leq$ as functions/predicates on F , and I satisfies $I(Ph) \cup I(Obs) \subseteq \beta$, $I(See) \subseteq W \times F^4 \times \beta$. As we mentioned before, the sets β, F are the same for all worlds (constant domains). Further, all constants, functions and predicates in our language are rigidly designated, except for See_{q^4b} which is expected to vary from one world to another. Given such a structure, we may evaluate formulas in the obvious way. Let $\mathcal{S} = (W, \beta, F, I, W \times W)$ and $w \in W$.

$$\begin{aligned} \mathcal{S} \models Obs(b) & \iff I(b) \in I(Obs) \\ \mathcal{S} \models t \leq s & \iff (I(t), I(s)) \in I(\leq) \\ \mathcal{S}, w \models \text{See}(x, y, z, t, b) & \iff (w, I(x, y, z, t, b)) \in I(See) \\ \mathcal{S}, w \models \exists x \phi & \iff (W, \beta, F, I', W \times W), w \models \phi \\ & \quad \text{(some } I' \text{ that agrees with } I \text{ except per-} \\ & \quad \text{haps on } x) \\ \mathcal{S}, w \models \Diamond \phi & \iff \mathcal{S}, v \models \phi \text{ (some } v \in W) \end{aligned}$$

Axioms for defining class of structures \Box - \forall -closure of:

- (1) $(F, 0, 1, +, \times, \leq)$ is a Real Closed Field, i.e. an ordered field where every non-negative element has a square root (Euclidean) and every polynomial of odd degree has a root.

In many presentations of special relativity, the ordered field is only required to be Euclidean and this suffices for most of our results. Here we assume that the field is a Real Closed Field in order to obtain a decidability result. Note that all Real Closed Fields are elementarily equivalent to the real numbers (Tarski 1951).

- (2) Observers are inertial: $Obs(b) \rightarrow \exists v_0 v_1 v_2 (\text{vel}(b) = (v_0, v_1, v_2))$

- (3) Speed of light is constant: $Ph(b) \rightarrow |\text{vel}(b)| = 1$
 (4) There is an observer on every ‘slow line’, there is a photon on every ‘fast line’.

$$|(v_0, v_1, v_2)| < 1 \rightarrow \exists b(\text{Obs}(b) \wedge \text{See}(\bar{x}, t, b) \wedge \text{vel}(b) = (v_0, v_1, v_2))$$

$$|(v_0, v_1, v_2)| = 1 \rightarrow \exists b(Ph(b) \wedge \text{See}(\bar{x}, t, b) \wedge \text{vel}(b) = (v_0, v_1, v_2))$$

In view of the last axiom, for each observer and for all $x, y, z, t \in F$, there are bodies b_1, b_2, b_3 moving on distinct lines that meet uniquely (pairwise and jointly) at (x, y, z, t) , according to that observer. The triple $e = (b_1, b_2, b_3)$ is called an event, we may write $\text{See}(x, y, z, t, e)$ instead of $\bigwedge_{i=1,2,3} \text{See}(x, y, z, t, b_i)$.

- (5) All observers see the same events:

$$\text{See}(x, y, z, t, e) \rightarrow \Box \exists x, y, z, t \text{ See}(x, y, z, t, e).$$

- (6) Symmetry. Let e_0, e_1, e'_0, e'_1 be events.

$$\bigwedge_{i=0,1} (\text{See}(0, 0, 0, i, e_i) \wedge \text{See}(x'_i, y'_i, z'_i, t'_i, e'_i)) \rightarrow$$

$$\Box (\bigwedge_{i=0,1} (\text{See}(0, 0, 0, i, e'_i) \wedge \text{See}(x_i, y_i, z_i, t_i, e_i)) \rightarrow (t'_1 - t'_0 = t_1 - t_0))$$

- (7) Isotropy: $(\bigwedge_{i,j < 4} (|\bar{x}_i - \bar{x}_j| = |\bar{x}'_i - \bar{x}'_j|) \wedge \bigwedge_{i < 4} \text{See}(\bar{x}_i, t, b_i))$

$$\rightarrow$$

$$\Diamond (\bigwedge_{i < 4} \text{See}(\bar{x}'_i, t', b_i))$$

where \bar{x}_i is a triple of three spatial coordinates, for $i < 4$.

The symmetry axiom implies that any two observers see each other’s clocks slow at the same rate. This usefully rules out a situation where one observer measures in seconds while another observer measures in years, it also rules out the situation where one observer measures time going forward while the other measures time going backwards. The isotropy axiom relates to the difference between the one dimensional appearance of observers and the four dimensions that an observer sees. Recall that when we see an observer, we see only his time axis, we do not see the orientation of his spatial axes. The isotropy axiom states that if I can see four bodies at $\bar{x}_0, \dots, \bar{x}_3$ at time t , and if the spatial Euclidean distances between the \bar{x}_i are identical to the distances between the \bar{x}'_i , then another observer can see the same four bodies at $\bar{x}'_0, \dots, \bar{x}'_3$ at time t' . I can transform myself to the second observer by performing an isometry of the spatial coordinates followed by a time translation. Let Ax be the set of six axioms, just defined.

Having defined the semantics of our language and the axioms for our logic, we now briefly evaluate our system by three criteria: how expressive is the language? are the axioms complete over an appropriate class of structures? what is the complexity of the satisfiability problem for formulas in our language,

over Minkowski structures? As far as expressive power is concerned, it seems that this language is capable of expressing most of the technical statements you find in a textbook on special relativity. For example, given a velocity vector $\bar{v} = (v_0, v_1, v_2)$ and any formula ψ we may write $\Diamond_{\bar{v}}\psi$ for $\exists b(\text{vel}(b) = \bar{v} \wedge \Diamond(\text{vel}(b) = 0 \wedge \psi))$, which means “there is a frame moving with velocity \bar{v} and ψ is true in that frame”. Our language should be able to express the purely kinematic properties of special relativity. However, our language can only express kinematic statements, we are not able to express properties relating to mass, energy or electric charge, for example.

Next, we assess the deductive strength of our axioms.

Lemma 1. *Let $\mathcal{S} = (W, \beta, F, I, W \times W) \models \text{Axioms 1--6}$ and let $w, v \in W$. There is a Poincaré map $p : F^4 \rightarrow F^4$ (an isometry with respect to Minkowski distance in F^4) such that*

$$(1) \quad \mathcal{S}, w \models \text{See}(x, y, z, t, b) \iff \mathcal{S}, v \models \text{See}(p(x, y, z, t))$$

PROOF:

A map $p : F^4 \rightarrow F^4$ satisfying (1) is uniquely defined, since $\mathcal{S} \models 4, 5$. We have to show that p is a Poincaré map. By axioms 2 and 3, for any $b \in I(Ob_s) \cup I(Ph)$, the set $\{(x, y, z, t) : \mathcal{S}, w \models \text{See}(x, y, z, t, b)\}$ is a line of F^4 and by axiom 4 each light line of F^4 is the trace of a photon and each slow line is the trace of an observer. Thus p maps lines to lines and maps light lines to light lines. By the Alexandrov-Zeeman theorem, p is a Poincaré transformation followed by a dilation and a field automorphism induced transformation. By axiom 6, the dilation and field induced transformations must be the identity transformations. \square

The isotropy axiom, axiom 7, is needed to complete the proof of the next lemma.

Lemma 2. *Let $\mathcal{S} \models Ax$, $\mathcal{S}' \models Ax$ be two models of our axioms, where $\mathcal{S} = (W, \beta, F, I, W \times W)$ and $\mathcal{S}' = (W', \beta', I', F', W' \times W')$. There are maps $i : W \rightarrow W'$, $j : \beta \rightarrow \beta'$ and $k : F \rightarrow F'$ such that k is an ordered field embedding, and for all $w \in W$, $b \in \beta$,*

$$\mathcal{S}, w \models \text{See}(x, y, z, t, b) \iff \mathcal{S}', i(w) \models \text{See}(k(x), k(y), k(z), k(t), j(b))$$

Next, we define a standard model \mathcal{M} . The set of worlds of \mathcal{M} is the set of Poincaré transformations of \mathbb{R}^4 . If p is a Poincaré transformation and l is a line then we write $p(l)$ for the image of l under p , note that $p(l)$ is always itself a line. The set of bodies of \mathcal{M} is L , the set of lines of \mathbb{R}^4 of gradient at most one, observers are the slow lines of gradient strictly less than one and photons are the lines of gradient one. The interpretation I_M in the standard model is given by

$$(p, x, y, z, t, b) \in I_M(\text{See}) \iff (x, y, z, t) \in p(b)$$

Theorem 1.

- (1) \mathcal{M} is a model of Ax .
- (2) For any formula ϕ of our language we have

$$\mathcal{M} \models \phi \iff Ax \vdash \phi$$

The first part can easily be proved, simply by verifying that each of our axioms holds in \mathcal{M} . The second part follows from lemma 2. Thus our axioms are complete for the validities of the standard model.

We now consider the decidability of the following decision problem: Is modal formula ϕ satisfiable over models of Ax ? If we allow arbitrary bodies to exist in our models, then this problem is undecidable. We may reduce the tiling problem to this satisfiability problem by considering each tile as a body that may be observed at positions with integer coordinates and where the adjacencies are expressed by universally quantified modal formulas. This undecidability arises purely from the first-order part of our language and can be proved without using the modalities. However, if we restrict to bodies whose paths are described by polynomial equations, then the problem becomes decidable. Since we have a universal modality, an arbitrary modal formula ϕ may be converted to a disjunction of clauses of the form $\Box\psi \wedge \bigwedge_i \Diamond\psi_i$, where ψ, ψ_i are non-modal. Such a clause is satisfiable iff $\psi \wedge \psi_i$ is satisfiable, for each i . By our assumption about the trajectories of bodies, such a non-modal formula may be translated into a first-order formula in a language with a binary predicate \leq and functions $+$, \times . Tarski showed (Tarski 1951; Canny 1988), by elimination of quantifiers, that the satisfiability problem for this language over real closed fields is decidable, although the complexity is rather high (at least double exponential). By imposing restrictions on the use of the first-order connectives in our language, satisfiability problems with lower complexities may be obtained.

Questions

- As an alternative to the current exposition, consider a modal logic for special relativity theory where an observer sees a body not merely as a line in four-dimensional space but as a line with a spatial orientation, so at an instance we see a point and three spatial unit vectors.
- Can we define models of special relativity without variables? One approach would be to use a propositional modal logic with **S5** modalities for moving in the directions of each of the spatial unit vectors and a temporal modality for moving in time, along with the already included modality for changing reference frame. It is known that this class of frames cannot be finitely axiomatised and the equational theory is undecidable (Hirsch & al. 2002), but here we are more interested in model definability.

- Can we use a similar framework to define a modal logic of general relativity? A Kripke world could be considered as an open neighbourhood in a coordinate system and the whole Kripke frame would correspond to a manifold. An extra modality to transfer to an adjacent neighbourhood would be needed. See (Shapirovski and Shehtman 2005) for useful results on modal logics of regions in Minkowski spacetime.

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